

Some miscellaneous definitions, theorems and lemmas

Theorem: If u is a tempered distribution, in $\mathcal{S}'(\mathbf{R})$, and $u' = 0$, then $u = c$, for some constant c , in the sense that, for all f in $\mathcal{S}(\mathbf{R})$, $(u, f) = c \int f(x) dx$.

Proof: The proof will show more, namely, that every tempered distribution has a primitive, and that any two primitives differ by a constant. We begin by constructing a “near-primitive,” in \mathcal{S} , for functions f in \mathcal{S} . Let

$$Uf(x) := \int_{-\infty}^x f(t) dt - \left(\int_{-\infty}^x f(t) dt \right) \int_{-\infty}^x W(t) dt, \text{ where } W(x) := \frac{e^{-x^2}}{\sqrt{\pi}}.$$

The function W is sometimes called the **Weierstrass kernel**.

We have $\frac{d}{dx} Uf(x) = f(x) - \left(\int_{-\infty}^x f(t) dt \right) W(x)$, so $\frac{d}{dx} Uf(x) \in \mathcal{S}$.

Note that $Uf(x) \rightarrow 0$ as $|x| \rightarrow \infty$. For every positive integer n , $x^n Uf(x) = \frac{Uf(x)}{x^{-n}}$, for x not 0.

We can apply the rule of l'Hospital; the limit of this ratio is equal to that of

$$\frac{f(x) - \left(\int_{-\infty}^x f(t) dt \right) W(x)}{(-n)x^{-n-1}},$$

in case it exists. The limit does exist, as $|x| \rightarrow \infty$, and it is 0. It follows that Uf is in \mathcal{S} . We will also need to know that U is a continuous map on \mathcal{S} . Let us take that for granted for the moment.

Now let u be in \mathcal{S}' . Define Pu by $(Pu, f) := -(u, Uf)$. This is, modulo what we are taking for granted, a tempered distribution. Let us calculate the distribution derivative of Pu . By definition, $\left(\frac{d}{dx} Pu, f\right) = -\left(Pu, \frac{df}{dx}\right) = -\left(u, -U\frac{df}{dx}\right) = (u, f)$, because $\int \frac{df}{dx} dt = 0$. Thus $\frac{d}{dx} Pu = u$. That is, they agree for all f in \mathcal{S} . Suppose now that there is also a primitive V for u . Then, with

$w := Pu - V$, $\left(\frac{d}{dx} w, f\right) = 0$ for all f in \mathcal{S} . In particular, $\left(\frac{d}{dx} w, Uf\right) = 0$ for all f in \mathcal{S} . This implies that $\left(w, \frac{d}{dx} Uf\right) = 0$ for all f in \mathcal{S} , or $\left(w, f - \left(\int_{-\infty}^x f(t) dt\right) W\right) = 0$. Therefore

$(w, f) = \left(\int_{-\infty}^x f(t) dt\right) (w, W)$. But $C := (w, W)$ is a constant. Thus, $(w, f) = \int C f(t) dt$ for all f in \mathcal{S} , and this is what we mean by “ $w = C$.”

Now we need a **lemma**: To show that U is a continuous map on \mathcal{S} . Recall that we must find an estimate for $\|Uf\|_{\alpha\beta}$, in terms of a finite number of norms $\|f\|_{\gamma\delta}$, for each (α, β) in $\mathbf{N} \times \mathbf{N}$.

If $\beta > 0$, $\|Uf\|_{\alpha\beta} = \|f - cW\|_{\alpha, \beta-1}$, where $c = \left(\int_{-\infty}^x f(t) dt\right)$. This can be estimated in the required manner, as we have seen before. The estimate will involve estimates for $\|f\|_1$, and we have seen how to estimate this: $\|f\|_1 \leq \|f\|_{00} + \|f\|_{2,0}$. Thus $\|Uf\|_{00}$ causes no difficulties either. So all we have to do is estimate $\|Uf\|_{n0}$, when $n > 0$. If $|x| \leq 1$,

$|x^n Uf(x)| \leq |Uf(x)| \leq \|f\|_1(1 + \|W\|_1) = C\|f\|_1$. If $x < -1$, we use the estimates

$|t^{n+1} f(t)| \leq \|f\|_{n+1,0}$ and $|t^{n+1} W(t)| \leq \|W\|_{n+1,0}$ to see that

$$|x^n Uf(x)| \leq |x|^n \int_{-\infty}^x |f(t)| dt + \left(\int_{-\infty}^x |f(t)| dt \right) |x|^n \int_{-\infty}^x W(t) dt \leq$$

$$|x|^n \int_{-\infty}^x \frac{\|f\|_{n+1,0}}{|t|^{n+1}} dt + \|f\|_1 |x|^n \int_{-\infty}^x \frac{\|W\|_{n+1,0}}{|t|^{n+1}} dt = \frac{\|f\|_{n+1,0}}{n} + \|f\|_1 \frac{\|W\|_{n+1,0}}{n}.$$

This is a suitable estimate; the lemma follows, since the case $x > 1$ is similarly treated.

Here is an alternate approach to proving the estimate for $\|Uf\|_{n0}$, when $n > 0$. According to the Cauchy Mean value theorem, with y either much larger than x , or much smaller than x ,

$$\frac{Uf(x)}{x^{-n}} - \frac{Uf(y)}{y^{-n}} = \frac{f(z) - \left(\int f(t) dt \right) W(z)}{(-n)z^{-n-1}}, \text{ for some } z \text{ strictly between } x \text{ and } y.$$

The right-hand side, being $z^{n+1} \frac{f(z) - \left(\int f(t) dt \right) W(z)}{(-n)}$, is |dominated| by

$(\|f\|_{n+1,0} + \|f\|_1 \|W\|_{n+1,0}) n^{-1}$. Now, with absolute values on the left, let y tend to either infinity. An estimate of the desired form follows from this one.

Definition: The support, **supp** u , of a distribution u is the *complement* of the largest open set Ω with this property: for every f in \mathcal{S} such that $\text{supp } f \subseteq \Omega$, $(u, f) = 0$.

Theorem: If $\text{supp } u = \{0\}$, then u is a linear combination of the Dirac δ -function and its derivatives.

Reminder: a linear combination is always finite!

Proof: Because u is a tempered distribution, there exist N in \mathbf{N} and constants $C_{\alpha\beta}$, for $|\alpha| \leq N$, $|\beta| \leq N$, such that for all f in \mathcal{S} , $|(u, f)| \leq \sum_{|\alpha| \leq N, |\beta| \leq N} C_{\alpha\beta} \|f\|_{\alpha\beta}$. We are going to express an arbitrary

function f in \mathcal{S} , near 0, as a polynomial times a “patch function,” plus a remainder. We will use Taylor’s Theorem in n variables. Then we’ll show that the remainder is in the kernel of u . The last step yields the linear combination, as in the proof of the previous theorem.

the “patch” function

We know that there is a C^∞ function $p(t)$ of one real variable such that $0 \leq p(t) \leq 1$ for all t , $p(t) = 1$ for all t with $|t| \leq 1/2$, and $p(t) = 0$ for $|t| > 1$. We define a one-parameter family of “patch functions,” $p_\epsilon(x) := p(|x|/\epsilon)$. Note that p_ϵ is C^∞ and has support contained in the ball of radius ϵ , and center 0.

Now, for all f in \mathcal{S} , and all $\epsilon > 0$, $h_\epsilon(x) := p_\epsilon(x) f(x)$ is also in \mathcal{S} , and $(u, h_\epsilon) = (u, f)$. This is true because the support of $f - h_\epsilon$ does not contain 0. We can also use this idea to show that, if f and g

are in \mathcal{S} , and $f(x) = g(x)$ for $|x| < \varepsilon$, then $(u, f) = (u, g)$. Taylor's Theorem in n variables gives us the representation, near 0, of f in \mathcal{S} as the sum of a polynomial $T(x)$ of degree at most N , say, and a remainder $R(x)$ such that $R^{(\beta)}(x) = O(|x|^{N+1-|\beta|})$ for each β with $|\beta| \leq N$. Moreover, $R^{(\beta)}(x)$ is the remainder for $f^{(\beta)}(x)$ near 0, for each β with $|\beta| \leq N$.

Taylor's Theorem in n variables is derived from the one-dimensional theorem by way of the chain rule and a counting argument. We define $g(t) := f(x + ty)$. Then it can be shown by induction that $g^{(m)}(t) = \sum_{|\beta|=m} \frac{m!}{\beta!} D^\beta f(x + ty) y^\beta$. The induction step is a little tricky. For each j

such that $\beta_j > 0$. We write $\beta = \beta - e_j + e_j$. Then we multiply and divide the term $\frac{m!}{(\beta - e_j)!}$ by β_j .

When we apply Taylor's Theorem to g , we get $g(t) = \sum_{m=0}^M \frac{g^{(m)}(0)}{m!} t^m + R_M(t)$, where M is in \mathbf{N} .

The remainder, $R_M(t)$, is usually expressed in one of the two forms

$$R_M(t) = \frac{g^{(M+1)}(s)}{(M+1)!} t^m, \text{ where } s \text{ is strictly between } 0 \text{ and } t,$$

or

$$R_M(t) = \int_0^t \frac{(t-s)^M}{M!} g^{(M+1)}(s) ds.$$

Finally, we set $t = 1$, and express $g^{(m)}(0)$ in terms of f . This gives Taylor's Theorem for functions of n variables, in L. Schwartz' (?) notation:

$$f(x + y) = \sum_{|\beta| \leq M} \frac{D^\beta f(x)}{\beta!} y^\beta + R_M(y), \text{ where the remainder, } R_M(y), \text{ is usually expressed in one of the two forms}$$

$$R_M(y) = \sum_{|\beta|=M+1} \frac{D^\beta f(x + ty)}{\beta!} y^\beta, \text{ where } 0 < t < 1,$$

or

$$R_M(t) = \int_0^t (M+1) (t-s)^M \sum_{|\beta|=M+1} \frac{D^\beta f(x + sy)}{\beta!} y^\beta ds.$$

We now write $f(x) = T(x) + R(x)$, where T is the Taylor polynomial involving all the partial derivatives of f of order $\leq N$, and R is the remainder, and we let $f_\varepsilon(x) := p_1(x) T(x) + p_\varepsilon(x) R(x)$. Since $f_\varepsilon(x) = f(x)$ for all x with $|x| < \varepsilon$, $(u, f_\varepsilon) = (u, f)$. Let us examine $(u, p_\varepsilon R)$, and show it must be 0. Now it is time to use the continuity of u :

$$|(u, p_\varepsilon R)| \leq \sum_{|\alpha| \leq N, |\beta| \leq N} C_{\alpha\beta} \|p_\varepsilon R\|_{\alpha\beta}.$$

If $\beta \neq 0$, $D^\beta p_\varepsilon R$ is, by Leibniz' Rule, a sum of terms of the form

$$(D^{\beta-\gamma} p_\varepsilon) (D^\gamma R) = \varepsilon^{-|\beta|+|\gamma|} (D^{\beta-\gamma} p_1)(x/\varepsilon) O(|x|^{N+1-|\gamma|}) \\ \leq C_\varepsilon^{-|\beta|+|\gamma|+N+1-|\gamma|} = C_\varepsilon^{N+1-|\beta|} \leq C_\varepsilon.$$

Here, C can be taken to be the larger of: the largest of the constants involved in the estimates $O(|x|^{N+1-|\gamma|})$ for $R(x)$, and the largest of the quantities $\|p_1\|_{0,\gamma}$, for $\gamma \leq \beta$. If $\alpha \neq 0$, the estimate is only improved. As we have argued before, $(u, p_\varepsilon R)$ is independent of positive ε , so $(u, p_\varepsilon R)$ is indeed 0, as desired.

Now we have $(u, f) = (u, p_1 T) + (u, p_\varepsilon R) = (u, p_1 T) = \sum_{|\beta| \leq N} \frac{D^\beta f(0)}{\beta!} (u, x^\beta p_1 T)$.

Some abuse of notation has taken place; x^β really “should” be replaced by a symbol that does not involve the variable x . But it would be cumbersome to introduce notation, or, worse yet, represent x^β in terms of the multiplication operator P_α .

Definition: Suppose $u \in \mathcal{S}'$. Then, for any $h \in \mathbf{R}^n$, we define the **translate of u by h** by the equation $(\tau_h u, f) := (u, \tau_{-h} f)$, for all f in \mathcal{S} .

Definition: Suppose $u \in \mathcal{S}'$, and $p \in \mathbf{R}^n$. Then **u is periodic, of period p** , if $\tau_p u = u$.

Theorem: If $u \in \mathcal{S}'$ is periodic of period $p \in \mathbf{R}^n$, then the support of \hat{u} is contained in ?

Proof: We have $(\hat{u}, f) = (u, \hat{f}) = (\tau_p u, \hat{f}) = (u, \tau_{-p} \hat{f}) = (u, (e^{ip \cdot x} f(x))^\wedge) = (\hat{u}, e^{ip \cdot x} f(x))$.

That is, $(\hat{u}, f) = (\hat{u}, e^{ip \cdot x} f(x))$. The same is true for $-p$. This means that $(\hat{u}, (e^{ip \cdot x} - 2 + e^{-ip \cdot x})f) = -4 (\hat{u}, \sin^2((p \cdot x)/2) f) = 0$. The function $\sin^2((p \cdot x)/2)$ is 0 whenever $p \cdot x = 2\pi n$ for some n in \mathbf{Z} . This occurs in the union Π of a family of parallel hyperplanes. We will actually show that, if f is in \mathcal{S} , and $f(x)$ and all $f^{(\beta)}(x)$ are 0 whenever x is in Π , then $(\hat{u}, f) = 0$. This will be done by showing that, if f is an \mathcal{S} -function of this sort (said to **vanish to all orders on Π**), then $\frac{f(x)}{\sin^2 \frac{p \cdot x}{2}}$ is in \mathcal{S} . The ratio is defined to be 0 in Π .