SOLUTIONS - EXAM 2, 3283W

Be sure to answer a total of FIVE QUESTIONS: Check to see that you have all questions and pages. Answer the questions in the space provided on the question sheets. If you need extra space, write on the other side of the page, in this case please clearly indicate that your work is continued on the other side.

1. (20 points) Prove that $a_n \to 0$ IF AND ONLY IF $|a_n| \to 0$.

$\Rightarrow$ SUPPOSE $a_n \to 0$

SOLUTION 1 $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}. \ n \geq n_0 \Rightarrow |a_n - 0| < \varepsilon$

$|a_n - 0| = |a_n| = ||a_n|| = ||a_n| - 0|$

Thus $n_0 \geq n_0 \Rightarrow ||a_n| - 0| < \varepsilon$. So $|a_n| \to 0$

SOLUTION 2 $|a_n| + a_n = \begin{cases} 2a_n & \text{IF } a_n \geq 0 \\ 0 & \text{IF } a_n < 0 \end{cases}$

THUS, $0 \leq |a_n| + a_n \leq 2a_n$ IF $n \in \mathbb{N}$. SINCE $a_n \to 0$, $2a_n \to 0$. BY SQEEZE THEOREM, $|a_n| + a_n \to 0$. HENCE $|a_n| = (|a_n| + a_n) - a_n \to 0 - 0 = 0$

$\Leftarrow$ SOLUTION 1

IF $n \in \mathbb{N}$, $-|a_n| \leq a_n \leq |a_n|$. SINCE $|a_n| \to 0$, $-|a_n| \to -0 = 0$. BY SQUEEZE THEOREM, $a_n \to 0$.

SOLUTION 2

SINCE $|a_n| \to 0$, $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}. \ n \geq n_0 \Rightarrow ||a_n| - 0| < \varepsilon$

BUT $\forall n \in \mathbb{N}. \ ||a_n| - 0| = ||a_n|| = |a_n| = |a_n - 0|$

THUS, $n \geq n_0 \Rightarrow |a_n - 0| < \varepsilon$. So $a_n \to 0$.

2. (20 points) Consider the recursive sequence given by $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

(a) Prove that for all $n \geq 1$, $\sqrt{2} \leq a_n \leq 2$.

Note: $\sqrt{2+x}$ is an increasing function so $x \leq y \iff \sqrt{2+x} \leq \sqrt{2+y}$

By part (b) $\langle a_n \rangle$ is increasing so $\forall n \in \mathbb{N}$ $a_n \geq a_1 = \sqrt{2}$.

Claim: $a_n \leq 2$ $\forall n \in \mathbb{N}$.

Base Case: $a_1 = \sqrt{2} < 2$.

Now assume for some $n \in \mathbb{N}, n \geq 1$, that $a_n \leq 2$.

Then $a_{n+1} = \sqrt{2+a_n} \leq \sqrt{2+2} = 2$

By induction $\sqrt{2} \leq a_n \leq 2$ $\forall n \in \mathbb{N}$.

(b) Prove that $\langle a_n \rangle$ is increasing.

Proof by induction.

Base Case: We have $a_2 = \sqrt{2+a_1} = \sqrt{2+\sqrt{2}} \geq \sqrt{2} = a_1$.

Now assume for some $n \in \mathbb{N}, n \geq 2$ that $a_n \geq a_{n-1}$.

Then we have $a_{n+1} = \sqrt{2+a_n} \geq \sqrt{2+a_{n-1}} = a_n$.

By induction, $a_{n+1} \geq a_n$ $\forall n \in \mathbb{N}$.

$\therefore$ $\langle a_n \rangle$ is increasing.

(c) Why does $\langle a_n \rangle$ converge? Compute $\lim_{n\to\infty} a_n = L$.

$\langle a_n \rangle$ converges because it is monotonic (increasing) & bounded, by the monotone convergence theorem.

Let $L = \lim_{n\to\infty} a_n$. Of course, $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n}$.

Since $\sqrt{2+x}$ is continuous on $[\sqrt{2}, 2]$ we have

$L = \lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2 + \lim_{n\to\infty} a_n} = \sqrt{2+L}$. Solving for L

we find $L = \sqrt{2+L} \Rightarrow L^2 = 2 + L \Rightarrow L^2 - L - 2 = 0$

$\Rightarrow (L-2)(L+1) = 0$ so $L = 2$ or $L = -1$.

Since $\sqrt{2} \leq a_n \leq 2$ $\forall n \in \mathbb{N}$ we must have

$\underline{L = 2}$.

3. (20 points) Let $\langle a_n \rangle$ be a sequence.

(a) Carefully and precisely write the definition of "a_n converges to L" .

$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |a_n - L| < \varepsilon$.

For parts (b) and (c), assume that $a_n \to L$.

(b) Prove that if $L < 0$, then there exists $n \in \mathbb{N}$ such that $a_n < 0$.

Let $\varepsilon = |L| = -L$, which is positive. Then $\exists n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $|a_n - L| < \varepsilon$

$$\Leftrightarrow -\varepsilon < a_n - L < \varepsilon$$

$$\underset{\text{since } \varepsilon = -L}{\Leftrightarrow} 2L < a_n < 0$$; in particular, $a_{n_0} < 0$.

(c) Prove that if for every $n \in \mathbb{N}$, $a_n \geq 0$ then $L \geq 0$. (Hint: There is a one-line proof of this result.)

This is the contrapositive of (b).

4. (20 points) Determine if the following sequences converge or diverge. If they converge, find their limit. Give reasons, but not a formal proof, for your answers. (Note: there are 5 parts to this problem.)

(a) $\frac{3n^4+n^2+7}{7n^4-n^3+2n+1}$

$$\frac{3n^4+n^2+7}{7n^4-n^3+2n+1} = \frac{\frac{1}{n^4}}{\frac{1}{n^4}}\left(\frac{3n^4+n^2+7}{7n^4-n^3+2n+1}\right) = \frac{3+\frac{1}{n^2}+\frac{7}{n^4}}{7+\frac{1}{n}+\frac{2}{n^3}+1}, \quad 3+\frac{1}{n^2}+\frac{7}{n^4} \to 3$$

AND $7+\frac{1}{n}+\frac{2}{n^3}+1 \longrightarrow 7$, BY QUOTIENT LIMIT LAW, $\frac{3n^4+n^2+7}{7n^4 \cdots}$

(b) $\cos\left(\log\left(\left(\frac{1}{n}+e\right)^{\pi}\right)\right)$

$e+\frac{1}{n} \to e-0=e$, SINCE $\text{LOG} X$ IS CONTINUOUS,

$\text{LOG}\left(\frac{1}{n}+e\right)^{\pi} = \pi \,\text{LOG}\left(\frac{1}{n}+e\right) \longrightarrow \pi \,\text{LOG}\, e = \pi$, SINCE $\text{LOG}\, e = 1$

SINCE COS IS CONTINUOUS, $\cos\left(\text{LOG}\left(\frac{1}{n}+e\right)^{\pi}\right) \longrightarrow$

$\cos\pi = -1$.

(c) $\frac{\sin^2(n^n)}{n^2+1}$

FOR ALL n, $0 \le \sin^2(n^n) \le 1$

THUS, $0 \le \frac{\sin^2(n^n)}{n^2+1} \le \frac{1}{n^2+1}$ FOR ALL n

SINCE $\frac{1}{n^2+1} \longrightarrow 0$, $\frac{\sin^2(n^n)}{n^2+1} \longrightarrow 0$ BY

THE SQUEEZE THEOREM

(d) .1,.13,.135, … ,.1357…(2n + 1)

THIS IS AN INCREASING SEQUENCE SINCE $.1357\ldots(2n+1) < .1357\ldots(2n+1)(2n+3)$ WHICH IS BOUNDED ABOVE BY .14, BY THE MONOTONE CONVERGENCE (BOUNDED CONVERGENCE) THEOREM, THE SEQUENCE CONVERGES. ITS LIMIT IS THE INFINITE NON-REPEATING DECIMAL .135791113....

(e) $(a_n)^2 \sin(\frac{\pi}{2} \cdot \frac{a_n}{L})$, IF $a_n \to L > 0$.

SINCE $f(x) = x^2$ IS CONTINUOUS, $a_n^2 \longrightarrow L^2$

SINCE $L > 0$, $\frac{a_n}{L} \longrightarrow \frac{L}{L} = 1$ BY LIMIT LAWS.

SINCE $\sin x$ IS CONTINUOUS,

$\sin\left(\frac{\pi}{2} \cdot \frac{a_n}{L}\right) \longrightarrow \sin\left(\frac{\pi}{2}\right) = 1$. BY PRODUCT RULE, $a_n^2 \cdot \sin\left(\frac{\pi}{2} \cdot \frac{a_n}{L}\right) \longrightarrow L^2 \cdot 1 = L^2$

5. (20 points) Determine if the following are true or false. If it is true, state reasons for your answer. If false provide an example which shows that it is false.

(a) If $\langle a_n + b_n \rangle$ diverges, then at least one of $\langle a_n \rangle$ or $\langle b_n \rangle$ diverges.

True. This is the contrapositive of the limit law for sums.

(b) If $\langle a_n \rangle$ diverges and $\langle b_n \rangle$ converges to $L > 0$, then $\langle \frac{a_n}{b_n} \rangle$ converges.

False. This is always false, in fact, so any example with $\langle a_n \rangle$ divergent $\langle b_n \rangle$ convergent (to a non-zero limit) will work as a counter example

(c) If $\langle a_n b_n \rangle$ converges, then at least one of $\langle a_n \rangle$ or $\langle b_n \rangle$ converges.

False. Counter-examples

① $\langle a_n \rangle = \langle (-1)^n \rangle = \langle b_n \rangle$ $\left(\begin{array}{l} \langle a_n \rangle, \langle b_n \rangle \text{ divergent} \\ \langle a_n b_n \rangle = \langle 1 \rangle \text{ convergent} \end{array} \right.$

②

$$\langle a_n \rangle = \langle 1, 1, \tfrac{1}{2}, 2, \tfrac{1}{3}, 3, \tfrac{1}{4}, 4, \tfrac{1}{5}, 5, \tfrac{1}{6}, 6, \tfrac{1}{7}, 7, \dots, \underbrace{\tfrac{1}{n}}_{(2n-1)\text{-term}}, \underbrace{n}_{2n\text{-term}}, \dots \rangle \text{ — divergent}$$

$$\langle b_n \rangle = \langle 1, 1, 2, \tfrac{1}{2}, 3, \tfrac{1}{3}, 4, \tfrac{1}{4}, 5, \tfrac{1}{5}, 6, \tfrac{1}{6}, \dots, \underbrace{n}_{(2n-1)\text{ term}}, \underbrace{\tfrac{1}{n}}_{2n\text{-term}}, \dots \rangle \text{ — divergent}$$

$\langle a_n b_n \rangle = \langle 1 \rangle$ convergent