Homework #1, Solutions.

Part A

1. a) $\neg[P \vee Q \vee R]$

$\iff \neg[P \vee [Q \vee R]]$ (associativity of $\vee$)

$\iff \neg P \wedge \neg[Q \vee R]$ (Note 1.1 b)

$\iff \neg P \wedge \neg Q \wedge \neg P$ (Note 1.1b)

b) $\neg[P \wedge Q \wedge R]$

$\iff \neg[P \wedge [Q \wedge R]]$ (associativity of &)

$\iff \neg P \vee \neg[Q \wedge R]$ (Note 1.1 a)

$\iff \neg P \vee \neg Q \vee \neg R$ (Note 1.1 a)

c) $\neg[(P \Rightarrow R) \wedge Q]$

$\iff \neg(P \Rightarrow R) \vee \neg Q$ (Note 1.1 a)

$\iff [P \wedge \neg R] \vee \neg Q$ (Note 1.1c)

d) $\neg[P \vee (Q \wedge R)]$

$\iff \neg P \wedge \neg(Q \wedge R)$ (Note 1.1 b)

$\iff \neg P \wedge (\neg Q \vee \neg R)$ (Note 1.1 a)

1. continued

e) $\neg[(P \vee Q) \wedge R]$

$\Leftrightarrow \neg(P \vee Q) \vee \neg R$ (Note 1.1 a)

$\Leftrightarrow (\neg P \wedge \neg Q) \vee \neg R$ (Note 1.1 b)

2. Given $x, y \in \mathbb{R}$ let $P(x,y)$ be the statement $x+y>0$. Let $x \in \mathbb{R}$. If y is any number such that $y > -x$, then $x+y > x-x = 0$. Thus we have shown $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ P(x,y)$.

On the other hand, we claim that the statement $[\exists y\ \forall x\ P(x,y)]$ is false. To see this, suppose it were true. Then for some $y \in \mathbb{R}$ $P(x,y)$ would be true $\forall x \in \mathbb{R}$. But if we set $x = -y$ then $P(x,y)$ is the statement $-y+y>0$, which is clearly false. $\therefore$ the statement $[\exists y\ \forall x\ P(x,y)]$ is false

3. Using the defining truth tables on p. 3 we construct the following truth table showing

$\neg(P \wedge Q \wedge R) \equiv \neg P \vee \neg Q \vee \neg R$

P	Q	R	$\neg P$	$\neg Q$	$\neg R$	$P \wedge Q \wedge R$	$\neg(P \wedge Q \wedge R)$	$\neg P \vee \neg Q \vee \neg R$
T	T	T	F	F	F	T	F	F
T	T	F	F	F	T	F	T	T
T	F	T	F	T	F	F	T	T
T	F	F	F	T	T	F	T	T
F	T	T	T	F	F	F	T	T
F	T	F	T	F	T	F	T	T
F	F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T	T

Part B

1. For every $n \in \mathbb{N}$, n^3 is even $\Longleftrightarrow$ n is even.

Pf: $\Leftarrow$: Assume n is even. Then $\exists\, k \in \mathbb{N}$ s.t. $n = 2k$.

It follows that $n^3 = (2k)^3 = 8k^3 = 2(4k^3)$,

& hence n^3 is even.

$\Rightarrow$: To show that n^3 is even implies n is even we prove the contrapositive: n is odd $\Rightarrow$ n^3 is odd.

If n is odd then $\exists\, k \in \mathbb{N}$ s.t. $n = 2k-1$.

We have
$$n^3 = (2k-1)^3 = (2k)^3 - 3\cdot(2k)^2 + 3\cdot(2k) - 1$$
$$= 8k^3 - 12k^2 + 6k - 1$$
$$= 2(4k^3 - 6k^2 + 3k) - 1,$$

which is of the form $2j-1$. We conclude that n^3 is odd as desired.

2. For every $x \in (0,\infty)$ $\exists\, y \in (0,\infty)$ s.t. $x > y^3$.

Pf: Let $y = \frac{x^{1/3}}{2}$. Then since $x > 0$ we clearly have that $y > 0$. Moreover

$$y^3 = \left(\frac{x^{1/3}}{2}\right)^3 = \frac{x}{8} < x.$$

3\. a) $[\forall x \in (0,1)\ Q(x)] \& [\forall y \in [0,1]\ P(y)]$

b) $\exists x \in \mathbb{R}\ [P(x) \wedge \sim Q(x)]$

c) $\forall x \in (-\infty, 0) \sim P(x)$

d) $\forall x \in \mathbb{R}\ [Q(x) \Rightarrow P(x)]$

4\. Let p be a prime number. Then the statement "for all $n \in \mathbb{N}$, n^2+n+p is prime" is false.

Pf: We need to show that the negation of the statement is true. That is, given a prime p,

$\exists\, n \in \mathbb{N}$, n^2+n+p is not prime.

To prove existence of such an n, it suffices to provide a counterexample. Let $n=p$. Then

$n^2+n+p = p^2+p+p = p(p+2)$. Thus when $n=p$ n^2+n+p is divisible by p. Clearly n^2+n+p fails to be prime in that case.

(Note: $n=p-1$, yields $(p-1)^2+p-1+p = p^2$)