# MORE ON BIJECTIONS

**THM** LET $f: A \to B$ BE A BIJECTION. CONSIDER THE INVERSE MAP $f^{-1}: B \to A$ DEFINED BY: LET $b \in B$. THEN $b = f(a)$ FOR ONE AND ONLY ONE $a$. DEFINE $f^{-1}(b) = a$. THEN $f^{-1}$ IS A BIJECTION

**PROOF** $f^{-1}$ 1-TO-1. SUPPOSE $f^{-1}(b) = f^{-1}(b_1) = a$. THEN $f(a) = b$ AND $f(a) = b_1$. SINCE $f$ IS A FUNCTION, $b = b_1$. $f^{-1}$ ONTO: PICK $a \in A$. THEN $f(a) = b \in B$ HENCE $f^{-1}(b) = a$ AND $f^{-1}(B) = A$. ∎

THIS RESULT SHOWS THAT WE CAN GO BACK AND FORTH AS WE CHOOSE BETWEEN THE 2 SETS WHEN WE HAVE A BIJECTION. WE THEN HAVE SOME UNUSUAL RESULT

**THM** $\mathbb{N}$ CAN BE WRITTEN AS A COUNTABLY INFINITE UNION OF COUNTABLY INFINITE SETS $\{A_n \mid n \geq 1\}$, WITH $A_i \cap A_j = \phi$ IF $i \neq j$ (PAIRWISE DISJOINT)

**PF** WE HAVE A BIJECTION $g : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ PREVIOUSLY DESCRIBED. HENCE $g^{-1} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ IS A BIJECTION. NOW $\forall n \in \mathbb{N}$, LET $B_n = \{(n, i) \mid i \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N}$. THEN $B_n$ IS COUNTABLY INFINITE: USE $f_n : \mathbb{N} \to B_n$, $f_n(i) = (n, i)$ AND PAIRWISE DISJOINT: IF $n \neq n_1$ & $(m, p) \in B_n \cap B_{n_1}$, THEN $m = n$ AND $m = n_1$, A CONTRADICTION. LET $A_n = f^{-1}(B_n)$. THEN $A_n$ COUNTABLY INFINITE AND $n \neq n_1 \Rightarrow A_n \cap A_{n_1} = \phi$: IF $k \in A_n \cap A_{n_1}$ THEN $g(k) = (i_k, j_k) \in B_n \cap B_{n_1} \Rightarrow i_k = n$ AND $i_k = n_1$, A CONTRADICTION

**THM** LET $A$ FINITE, $A = \{a_1, \ldots, a_n\}$. THEN THE FOLLOWING ARE TRUE

1) $f: A \to A$ 1-TO-1 $\Rightarrow f$ ONTO
2) $f: A \to A$ ONTO $\Rightarrow f$ BIJECTION
3) $f: A \to A$ BIJECTION $\Rightarrow f$ 1-TO-1

**PF** 1) SUPPOSE $f: A \to A$ 1-TO-1. THEN $a_i \neq a_j \Rightarrow f(a_i) \neq f(a_j)$.
SINCE $A$ HAS $n$ ELEMENTS, $f(A) = \{f(a_1), f(a_2), \ldots, f(a_n)\} \subseteq A$ ALSO HAS $n$ ELEMENTS. THIS MEANS THAT $f(A) = A$, AND $f$ ONTO

2) SUPPOSE $f(A) = A$ AND $f$ NOT 1-TO-1. THEN $\exists\, i, j$ WITH $i \neq j$ AND $f(a_i) = f(a_j)$. THUS $f(A)$

$$= \underbrace{\{f(a_1) \ldots, f(a_i), \ldots f(a_i), \ldots f(a_n)\}}_{n-1 \text{ ELEMS}}$$

$\qquad 1 \qquad\quad i \qquad\qquad j$

SO $f(A)$ CAN HAVE AT MOST

FINALLY, SINCE $\mathbb{N}\times\mathbb{N} = \bigcup_{n=1}^{\infty} B_n$

$$\mathbb{N} = g^{-1}(\mathbb{N}\times\mathbb{N}) = \bigcup_{n=1}^{\infty} g^{-1}(B_n) = \bigcup_{n=1}^{\infty} A_n$$

SINCE WE KNOW $g$, WE CAN DETERMINE SOME MEMBERS OF $A_n$'s

$A_1 = \{1, 3, 4, 10, 11, \ldots\}$

$A_2 = \{2, 5, 9, 12, \ldots\}$

$A_3 = \{6, 8, 13, \ldots\}$

$A_4 = \{7, 14, 18, \ldots\}$

$A_5 = \{15, 17, \ldots\}$

$A_6 = \{16, \ldots\}$

IN $A_1$, EXCEPT FOR 1, MEMBERS WILL ALWAYS BE IN PAIRS $K, K+1$. HOWEVER, $K$ GETS BIG RAPIDLY. IF $K, K+1 \in A_1$, THEN $K+2 \in A_2$.

BUT $f(A) = A \Rightarrow f(A)$ HAS $n$ ELEMENTS
THIS IS A CONTRADICTION. SO
$f$ IS 1-TO-1.

3) CLEAR: $f$ BIJECTION $\Rightarrow f$ 1-TO-1

<u>COROLLARY</u> THE FOLLOWING STATEMENTS ARE PAIRWISE EQUIVALENT FOR A FINITE SET $A$

1) $f: A \to A$ 1-TO-1
2) $f: A \to A$ ONTO
3) $f: A \to A$ BIJECTION

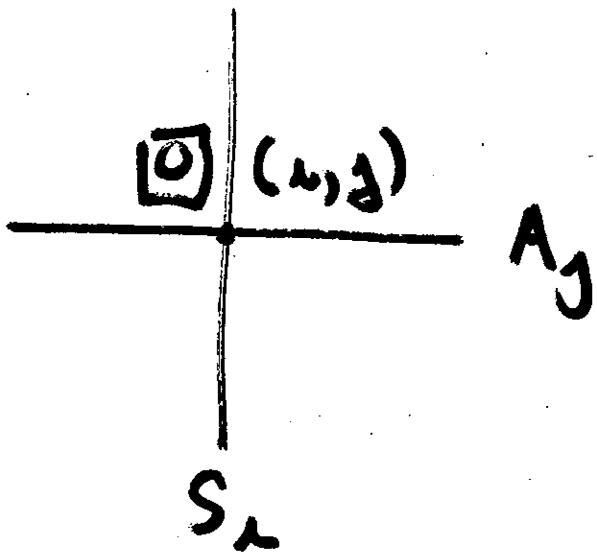
EXAMPLE (PIGEONHOLE PRINCIPLE)
SUPPOSE HAVE A FINITE # OF PIGEONS AND A FINITE # OF PIGEONHOLES. YOU WANT TO ASSIGN A UNIQUE HOLE TO EACH PIGEON. ON $1^{ST}$ TRY YOU FIND THAT AFTER ALL HOLES ARE FILLED, THERE ARE PIGEONS LEFT OVER. IS THERE [redacted] ANY OTHER WAY TO ASSIGN HOLES ST. EACH PIGEON HAS A UNIQUE HOLE

<u>ANSWER</u> SUPPOSE THERE IS A WAY WHICH WORKS. THEN HAVE $f: \{p_1, \dots, p_n\} \longrightarrow \{H_1, \dots, H_m\}$ BIJECTION : EVERY HOLE FILLED BY UNIQUE PIGEON. THEN HAVE $m=n$ AND HAVE $\hat{f}: \{1, \dots, n\} \longrightarrow \{1, \dots, m\}$, $\hat{f}(i) = j$ IF $f(p_i) = H_j$

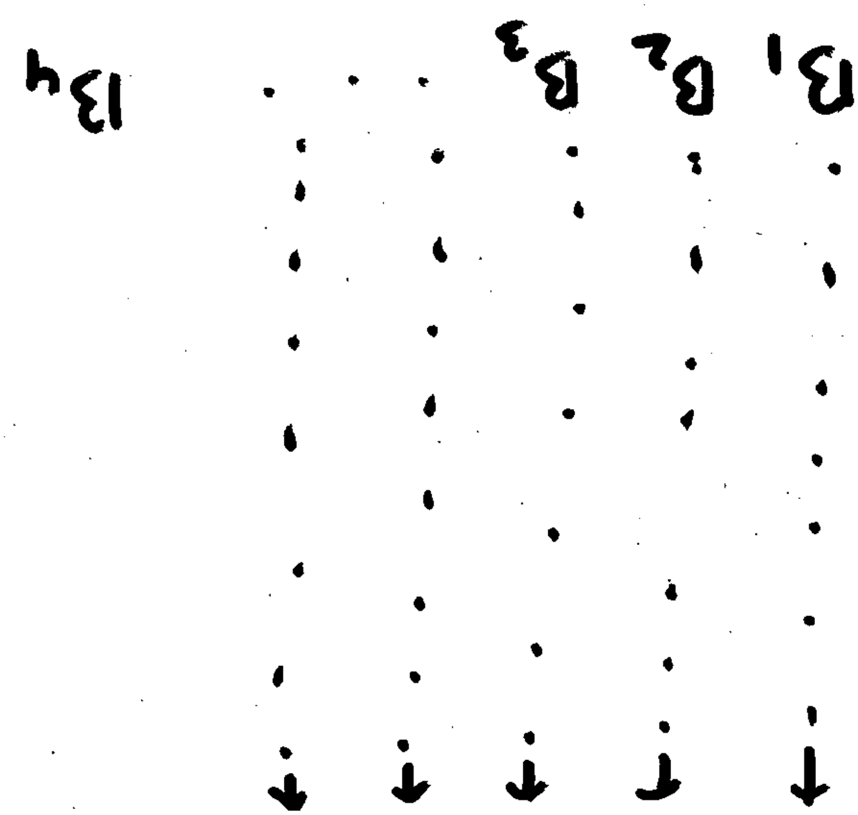
AGAIN, $\hat{f}$ BIJECTION. $m=n$ BUT OUR FIRST MAP $g: \{p_1, \dots, p_n\} \longrightarrow \{H_1, \dots, H_n\}$ SATISFIED $g: \{p_1, \dots, p_k\} = \{H_1, \dots, H_n\}$ AND $g$ 1-TO-1, THIS MEANS THAT $K = n$, WHERE $K < n$. A CONTRADICTION. SO IF ONE MAP WORKS, ALL MAPS WORK.

EXAMPLE - A VERY BIG CITY.
YOU ARE MANAGER OF (MEGA)$^{\text{MEGA}}$ TROPOLIS, THIS CITY IS SO BIG THAT THERE ARE A COUNTABLY INFINITE NO OF E-W STREET $\{S_n \mid n \geq 1\}$ AND A COUNTABLY INF NO OF N-S AVENUES $\{A_n \mid n \geq 1\}$. AT EACH CORNER, THERE IS A STOPLIGHT

1. HOW MANY STOPLIGHTS ARE THERE IN $M^M$TPLS?
2. THE CITY MANAGER WANTS TO CHECK OPS ON EACH STPLT. DEVISE A ROUTE WHICH WILL ENABLE HIM TO VISIT EACH STPLT.




$$f : \mathrm{INTER} \longrightarrow \mathbb{N} \times \mathbb{N}$$
$$S_i \cap A_j \longrightarrow (i, j)$$



LET $A = \mathbb{N}$. CONSIDER $B =$ EVENS

$C =$ ODDS AND $D = \{n^2 \mid n \in \mathbb{N}\}$

CORRESPONDING TO $B$ IS THE

SEQUENCE OF 0'S AND 1'S $S_B = \langle a_{\lambda}$

WHERE $a_{\lambda} = 1$ IF $\lambda \in B$ AND $a_{\lambda} = 0$ IF $\lambda \notin$

SO $\langle a_{\lambda} \rangle$: $\underset{\;\;2\;\;4\;\;6\;\;8}{01010101}$.... SIMILARLY $S_C$

$= \langle b_{\lambda} \rangle$, $b_{\lambda}$: $\underset{1\;\;3\;\;5\;\;7}{1010101}$...., $S_D = \langle d_{\lambda} \rangle$,

$d_{\lambda}$: $\underset{1\;\;4\;\;\;\;\;\;9}{100100001}$....$\underset{16}{010}$....$\underset{25}{010}$..., CONVERSELY,

LET $\langle a_{\lambda} \rangle_{\lambda=1}^{\infty}$ BE ANY SEQUENCE OF

0'S AND 1'S. CONSIDER THE

SET $A = \{\lambda \mid a_{\lambda} = 1\} \subset \mathbb{N}$. IF WE

NOW FORM $S_A$ AS ABOVE, THEN

$S_A = \langle a_{\lambda} \rangle_{\lambda=1}^{\infty}$, THE GIVEN SEQUENCE

WE THUS HAVE SHOW

<u>THM</u> CONSIDER THE SET $\mathcal{P}(\mathbb{N})$

OF ALL SUBSETS OF $\mathbb{N}$. THEN

$\wp(\mathbb{N})$ IS DESCRIBED BY, LOOKING AT $2^{\mathbb{N}} = \{\langle a_\lambda \rangle_{\lambda=1}^{\infty} \mid a_\lambda = 0 \text{ OR } 1\}$, THE SET OF ALL SEQS OF 0's AND 1's.

<u>EXAMPLES</u>

① LET $A = \{1, \ldots, n\} \subseteq \mathbb{N}$. THEN $S_A = \langle a_\lambda \rangle : \underset{1}{1}, 1, 1, \ldots \underset{n}{1}, 0, 0, 0 \ldots$. EVERY SUBSET $B \subset A$ HAS 1's ONLY IN THE FIRST $n$ TERMS OF $S_B$. THUS, IF $S_B = \langle b_\lambda \rangle$, THEN $b_\lambda = 0$ IF $\lambda > n$. IN THIS WAY, $2^{\{1, \ldots, n\}}$ CORRESPONDS TO $\{\langle a_\lambda \rangle_{\lambda=1}^{\infty} \mid a_\lambda = 0 \text{ IF } \lambda > n\} \subset 2^{\mathbb{N}}$

SO $\#(2^{\mathbb{N}})$ IS "BIGGER" THAN $\# 2^{\{1, \ldots n\}} = 2^n$ FOR ANY $n \in \mathbb{N}$

② SOME SEQUENCES $\langle a_\lambda \rangle_{\lambda=1}^{\infty} \in 2^{\mathbb{N}}$ ARE HARD TO DESCRIBE. LET $A = \{p \in \mathbb{N} \mid p \text{ IS PRIME}\}$. THEN

$S_A = \langle a_\lambda \rangle : a_\lambda = 1$ IF $\lambda$ IS PRIME. SINCE WE DON'T KNOW EXACTLY ALL THE PRIME #'S (THERE ARE INFINITE #), WE CAN'T COMPLETELY DESCRIBE ALL TERMS OF $S_A$.

③ CONSIDER THE SET $M = \{m_n \mid n \geq 1\} \subset \mathbb{N}$ DEFINED RECURSIVELY BY $m_1 =$ $m_{n+1} = 2^{m_n}$. SO $m_1 = 1$, $m_2 = 2$, $m_3 = 2^2$ $m_4 = 2^{(2^2)} = 2^4$, $m_5 = 2^{2^4} = 2^{16} = 2^{(2^{2^2})}$ $m_{n+1} = 2^{\left(2^{2^{\cdot^{\cdot^{2}}}}\right)}$ : n "LEVELS" OF 2'S

$S_M$ HAS HUGE BLOCKS OF 0'S FOLLOWED BY A 1 AND THEN EVEN A BIGGER BLOCK OF 0'S. $m^5 - m^4$ $= 2^{16} - 2^4 = 2^4(2^{12} - 1) > 2^4 \cdot 2^{11} = 2^{15}$

WE NOW COME TO THE MAJOR RESULT OF THIS SECTION

<u>THM</u> $\mathcal{P}(\mathbb{N})$ (OR $2^{\mathbb{N}}$) IS INFINITE BUT NOT COUNTABLY INFINITE

PRF THAT $\mathcal{P}(\mathbb{N})$ IS NOT FINITE IS CLEAR. $\forall n \in \mathbb{N}$, CONSIDER $A_n = \{n\}$. THEN $\forall n, m,\ n \neq m \Rightarrow A_n \neq A_m$. SO $\{A_n \mid n \in \mathbb{N}\}$ IS AN INFINITE SUBSET OF $\mathcal{P}(\mathbb{N})$.

TO SHOW THAT $\mathcal{P}(\mathbb{N})$ IS NOT COUNTABLE, WE USE $2^{\mathbb{N}}$ AND A PROOF BY CONTRADICTION. WE WILL ASSUME THAT $2^{\mathbb{N}}$ IS COUNTABLE AND THEN FIND A SEQUENCE THAT WE DID NOT COUNT. THE PROCESS OF CREATING THIS UNCOUNTED SEQUENCE IS CALLED (CANTOR) DIAGONALIZATION: WE FORM THE SEQUENCE BY CHANGING TERMS ON THE "DIAGONAL"

ASSUME $2^{\mathbb{N}}$ IS COUNTABLE. THE

WE HAVE $2^{\mathbb{N}} = \{ S_n \mid n \in \mathbb{N} \}$, EAC

$S_n = \langle a_{i,n} \rangle_{i=1}^{\infty}$, A SEQ OF 0's & 1's

WRITE AS AN "INFINITE MATRIX"

$$
\begin{array}{lccccc}
S_1: & a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} & \cdots \\
S_2: & a_{1,2} & a_{2,2} & a_{3,2} & a_{4,2} & \cdots \\
S_3 & a_{1,3} & a_{2,3} & a_{3,3} & a_{4,3} & \cdots \\
S_4 & a_{1,4} & a_{2,4} & a_{3,4} & a_{4,4} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

WE FORM A NEW SEQUENCE $S = \langle \hat{S}$

AS FOLLOWS:

CONSIDER $a_{1,1}$. IF $a_{1,1} = 0$, $\hat{S}_1 = 1$

$a_{1,1} = 1$, $\hat{S}_1 = 0$

FOR SIMPLICITY, SAY $\hat{S}_1$ IS THE <u>DUAL</u>

OF $a_{1,1}$

NEXT, USE $a_{2,2}$. SET $\hat{S}_2$ TO BE THE

DUAL OF $a_{2,2}$

CONTINUE THIS WAY: $\forall n \in \mathbb{N}$,

SET $\hat{S}_n$ TO BE THE DUAL OF $a_{n,n}$

WE NOW HAVE A SEQUENCE $S$.

CLAIM. $S \neq S_n$ FOR ANY $n$.

TO SEE THIS, WE HAVE THAT 2 SEQUENCES $\langle C_i \rangle$ AND $\langle D_i \rangle$ ARE EQUAL IF $\forall i \in \mathbb{N}$, $C_i = D_i$. HENCE, $\langle C_i \rangle \neq \langle D_i \rangle$ IF $\exists i \in \mathbb{N}$, $C_i \neq D_i$, I.E., THE SEQUENCES DIFFER AT SOME TERM.

PICK $n \in \mathbb{N}$. THE $n^{\text{TH}}$ TERM OF $S_n$ IS $a_{n,n}$ AND THE $n^{\text{TH}}$ TERM OF $S$ IS $\hat{S}_n$, THE DUAL OF $a_{n,n}$. HENCE $\hat{S}_n \neq a_{n,n}$ AND $S_n \neq S$.

BUT $S$ IS A SEQUENCE OF 0'S AND 1'S, AND THUS $S \in 2^{\mathbb{N}}$. THIS IS A CONTRADICTION. SO $2^{\mathbb{N}}$ AND $\wp(\mathbb{N})$ ARE NOT COUNTABLY INFINITE