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More About Sequences

Let $\langle a_n \rangle_{\mathbb{N}}$ be a sequence.

Fact: $a_n \longrightarrow 0$ iff $|a_n| \longrightarrow 0$

Proof: Pinching theorem, noting that for any $n \in \mathbb{N}$

① $-a_n \le |a_n| \le a_n$ when $a_n \ge 0$ & $a_n \le |a_n| \le -a_n$ when $a_n \le 0$,

hence $b_n = \min\{-a_n, a_n\} \le |a_n| \le \max\{a_n, -a_n\} = c_n$

② $-|a_n| \le a_n \le |a_n|$

Note: If $\lim |a_n| \ne 0$, then we cannot in general say any thing about $\lim a_n$. On the other hand, if $\lim a_n \ne 0$ we know $\lim |a_n| = |\lim a_n|$.

Question: Suppose $\langle a_n \rangle$ diverges & $\langle b_n \rangle$ converges, what can be said about $\langle \frac{a_n}{b_n} \rangle$?

Answer: $\langle \frac{a_n}{b_n} \rangle$ must diverge. (Proof by contradiction).

If $\langle \frac{a_n}{b_n} \rangle$ converges, then (since $\langle b_n \rangle$ converges) so too must $\langle (\frac{a_n}{b_n}) \cdot b_n \rangle = \langle a_n \rangle$. #

Monotonicity.

Defn: Let $\langle s_n \rangle_{\mathbb{N}}$ be a sequence. We say $\langle s_n \rangle$ is "(monotonically) increasing" if for all $n \in \mathbb{N}$ $s_n \le s_{n+1}$. We say $\langle s_n \rangle$ is "(monotonically) decreasing" if for all $n \in \mathbb{N}$ $s_n \ge s_{n+1}$. We call sequences which are either increasing or decreasing are called "Monotone".

Recall the following theorem

If a sequence $\langle a_n \rangle$ converges, then $\{a_n \mid n \in \mathbb{N}\}$ is bounded.

In general, the converse fails. However; we do have the following:

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$\boxed{\text{Monotone Convergence Theorem:}}$ Let $\langle s_n \rangle$ be a Monotone Sequence, then,

$\langle s_n \rangle$ converges iff $\{s_n \mid n \in \mathbb{N}\}$ is bounded.

Moreover, if $\langle s_n \rangle$ is increasing & converges, then $\lim s_n = \sup\{s_n \mid n \in \mathbb{N}\}$;

if $\langle s_n \rangle$ is decreasing & converges, then $\lim s_n = \inf\{s_n \mid n \in \mathbb{N}\}$.

Finally, if $\langle s_n \rangle$ is increasing & divergent, then $s_n \longrightarrow \infty$; if $\langle s_n \rangle$ is decreasing & divergent, then $s_n \longrightarrow -\infty$

Note that monotonicity is not only essential for the initial result, but also for the statements about the limits.

Examples: $\langle (-1)^n \rangle_{\mathbb{N}}$ is bounded but divergent (thus monotone is necessary for initial result)

$\langle (-1)^n \cdot n \rangle$ is divergent but neither tends to ∞ nor $-\infty$. (thus monotone necessary for final statement).

$\langle (-1)^n \frac{1}{n} \rangle$ is convergent, but $\lim (-1)^n \frac{1}{n} = 0 \neq \sup\{\frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$ or $\inf\{\frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$

Proof: Already know convergent $\Rightarrow$ bounded. So it suffices to show

Decreasing case proved in analogous way

(for monotone increasing case) [divergent $\Rightarrow$ ($s_n \longrightarrow \infty$)] & and [bounded $\Rightarrow \lim s_n = \sup\{s_n \mid n \in \mathbb{N}\}$]. Bounded $\Rightarrow$ sup exists. Use def of sup & def of limit of sequence. Next, deduce: [(monotone & divergent) $\Rightarrow$ (bounded bellow, but not bounded above)]. Use this to show that, in this case, $s_n \longrightarrow \infty$

Examples: $a_n = \frac{n-1}{n+1}$ is increasing; $\frac{2n+1}{2n-1}$ is decreasing

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Recursive Sequences.

Def: A sequence $\langle x_n \rangle$ is said to be "recursively defined", if x_{n+1} depends on the terms $x_1, \ldots, x_n$, for each $n \in \mathbb{N}$. In practice, we will usually have $x_{n+1} = f(x_n)$ for some $f: \mathbb{R} \longrightarrow \mathbb{R}$.

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Examples: LET $f(x)=4-x$.

(1) let $\langle a_n \rangle$ be the recursive sequence defined by

$a_1=1 \quad a_{n+1}=f(a_n) \quad \forall n\geq 1$

Then $a_n=2+(-1)^n$ (Prove this), hence noncovergent

(2) let $\langle a_n \rangle$ be the recursive sequence

$a_1=2 \quad a_{n+1}=f(a_n) \quad \forall n\geq 1$

Then $a_n=2 \quad \forall n$ (Prove this), hence convergent.

LESSON: RECURSIVE SEQUENCE DEPEND NOT ONLY RECURSION EQUATION (i.e, $a_{n+1}=f(a_n)$), but also starting point!

Example: LET $f(x)=\frac{1}{2}(x+6)$

$\langle a_n \rangle \quad a_1=4 \quad a_{n+1}=f(a_n)$

Prove a_n converges using the following steps:

(1) $\langle a_n \rangle$ monotonically increasing (induction)

(2) $\langle a_n \rangle$ bounded above (induction).

REVIEW FOR EXAM