BOUNDARY STRATA OF CONNECTED COMPONENTS IN POSITIVE CHARACTERISTICS

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Abstract. Under the assumption that the PEL datum involves no factor of type D and that the integral model has good reduction, we show that all boundary strata of the toroidal or minimal compactifications of the integral model (constructed in earlier works of the author) have nonempty pullbacks to connected components of geometric fibers, even in positive characteristics.

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A.1. **Introduction.** Toroidal and minimal compactifications of Shimura varieties and their integral models have played important roles in the study of arithmetic properties of cohomological automorphic representations. While all known models of them are equipped with natural stratifications, they often suffer from some imprecisions or redundancies due to their constructions. The situation is especially subtle in positive or mixed characteristics, or when we need purely algebraic constructions even in characteristic zero (for example, when we study the degeneration of abelian varieties), where the constructions are much less direct than algebraizing complex manifolds created by unions of explicit double coset spaces.

For example, integral models of Shimura varieties defined by moduli problems of PEL structures suffer from the so-called *failure of Hasse's principle*, because there is no known way to tell the difference between two moduli problems associated with algebraic groups which are everywhere locally isomorphic to each other. Similarly,

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when their toroidal and minimal compactifications are constructed using the theory of degeneration, the data for describing them are also local in nature. Unlike in the complex analytic construction, one cannot just express all the boundary points as the disjoint unions of some double coset spaces labeled by certain standard maximal (rational) parabolic subgroups. (Even the nonemptiness of the *whole boundaries* in positive characteristics was not straightforward—see the introduction to [9].) As we shall see (in Example A.7.2), when factors of type D are allowed, it is unrealistic to expect that the boundary stratifications in the algebraic and complex analytic constructions match with each other.

Our goal here is a simple-minded one—to show that the strata of good reduction integral models of toroidal and minimal compactifications constructed as in [11] have nonempty pullbacks to each connected component of each geometric fiber, under the assumption that the data defining them involve no factors of type D (in a sense we will make precise). We will also answer the analogous question for the integral models constructed by normalization in [12], allowing arbitrarily deep levels and ramifications (that is, *bad reductions* in general).

This goal is motivated by the study of p-adic families of Eisenstein series, for which it is crucial to know that the strata on connected components of the characteristic p fibers are all nonempty. For example, this is useful for the consideration of algebraic Fourier–Jacobi expansions. We expect it to play foundational roles in other applications of a similar nature.

A.2. **Main result.** We shall formulate our results in the notation system of [11], which we shall briefly review. (We shall follow [11, Notation and Conventions] unless otherwise specified. While for practical reasons we cannot explain everything we need from [11], we recommend the reader to make use of the reasonably detailed index and table of contents there, when looking for the numerous definitions.)

Let $(\mathcal{O}, \star, L, \langle \cdot\,, \cdot \rangle, h_0)$ be an *integral PEL datum*, where $\mathcal{O}$, $\star$, and $(L, \langle \cdot\,, \cdot \rangle, h_0)$ are as in [11, Def. 1.2.1.3], satisfying [11, Cond. 1.4.3.10], which defines a group functor G over $\mathbb{Z}$ as in [11, Def. 1.2.1.6], and the *reflex field* F_0 (as a subfield of $\mathbb{C}$) as in [11, Def. 1.2.5.4], with ring of integers $\mathcal{O}_{F_0}$. Let p be any *good prime* as in [11, Def. 1.4.1.1]. Let $\mathcal{H}^p$ be any open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^p)$ that is *neat* as in [11, Def. 1.4.1.8]. Then we have a moduli problem $\mathsf{M}_{\mathcal{H}^p}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ as in [11, Def. 1.4.1.4], which is representable by a scheme quasi-projective and smooth over S_0 by [11, Thm. 1.4.1.11 and Cor. 7.2.3.10]. By [11, Thm. 7.2.4.1 and Prop. 7.2.4.3], we have the *minimal compactification* $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p}$ of $\mathsf{M}_{\mathcal{H}^p}$, which is a scheme projective and flat over S_0, with geometrically normal fibers. Moreover, for each compatible collection Σ^p of cone decompositions for $\mathsf{M}_{\mathcal{H}^p}$ as in [11, Def. 6.3.3.4], we also have the *toroidal compactification* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ of $\mathsf{M}_{\mathcal{H}^p}$, which is an algebraic space proper and smooth over S_0, by [11, Thm. 6.4.1.1], which is representable by a scheme projective over M_0 when Σ^p is *projective* as in [11, Def. 7.3.1.3], by [11, Thm. 7.3.3.4]. Any such $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ admits a canonical surjection $\oint_{\mathcal{H}^p} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p}$, which is constructed by Stein factorization as in [11, Sec. 7.2.3], whose fibers are all geometrically connected. (The superscript "p" indicates that the objects are defined using level structures "away from p". We will also encounter their variants without the superscript "p", which also involve level structures "at p".)

By [11, Thm. 7.2.4.1(4)], there is a stratification of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p}$ by locally closed subschemes $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$, where $[(\Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]$ runs through the (finite) set of *cusp labels* for $\mathsf{M}_{\mathcal{H}^p}$ (see [11, Def. 5.4.2.4]). The open dense subscheme $\mathsf{M}_{\mathcal{H}^p}$ is the stratum labeled by $[(0,0)]$; we call all the other strata the *cusps* of $\mathsf{M}_{\mathcal{H}^p}$. Similarly, by [11, Thm. 6.4.1.1(2)], there is a stratification of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ by locally closed subschemes $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$, where $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$ runs through equivalence classes as in [11, Def. 6.2.6.1] with $\sigma^p \subset \mathbf{P}^+_{\Phi_{\mathcal{H}^p}}$ and $\sigma^p \in \Sigma_{\Phi_{\mathcal{H}^p}} \in \Sigma^p$. By [11, Thm. 7.2.4.1(5)], the surjection $\oint_{\mathcal{H}^p}$ induces a surjection from the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ to the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p}$.

Let $s \to \mathsf{S}_0$ be any geometric point with residue field $k(s)$, and let U be any connected component of the fiber $\mathsf{M}_{\mathcal{H}^p} \underset{\mathsf{S}_0}{\times} s$. Since $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p} \to \mathsf{S}_0$ is proper and has geometrically normal fibers, the closure U^{min} of U in $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p} \underset{\mathsf{S}_0}{\times} s$ is a connected component of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p} \underset{\mathsf{S}_0}{\times} s$. Similarly, since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \to \mathsf{S}_0$ is proper and smooth, the closure U^{tor} of U in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathsf{S}_0}{\times} s$ is a connected component of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathsf{S}_0}{\times} s$. (In these cases the connected components are also the irreducible components of the ambient spaces.)

The stratifications of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ induce stratifications of U^{min} and U^{tor}, respectively, by pullback. We shall denote the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ to U^{min} by $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$, and call it the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$-stratum of U^{min}. Similarly, we shall denote the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$ to U^{tor} by $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$, and call it the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$-stratum of U^{tor}. By construction, the surjection $\oint_{\mathcal{H}^p}$ induces a surjection $U^{\mathrm{tor}} \to U^{\mathrm{min}}$, which maps the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$-stratum $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$ of U^{tor} surjectively onto the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$-stratum $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ of U^{min}. It is natural to ask whether a particular stratum of U^{min} or U^{tor} is nonempty.

From now on, we shall assume the following:

Assumption A.2.1. *The semisimple algebra $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ over $\mathbb{Q}$ involves no factor of type* D (*in the sense of* [11, Def. 1.2.1.15]).

Our main result is the following:

Theorem A.2.2. *With the setting as above, all strata of U^{min} are nonempty.*

An immediate consequence is the following:

Corollary A.2.3. *With the setting as above, all strata of U^{tor} are nonempty.*

Proof. Since the canonical morphism $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]} \to U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ is surjective for each equivalence class $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$ with underlying cusp label $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$ as above, the nonemptiness of $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ implies that of $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$. □

Remark A.2.4. Each stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\mathrm{z}_{\mathcal{H}^p})]}$ (resp. $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\mathrm{z}_{\mathcal{H}^p},\sigma^p)]}$) is nonempty by [11, Thm. 7.2.4.1 (4) and (5), Cor. 6.4.1.2, and the explanation of the existence of complex points as in Rem. 1.4.3.14]. The question is whether its pullback to U^{min} (resp. U^{tor}) is still nonempty for every U as above.

Remark A.2.5. It easily follows from Theorem A.2.2 and Corollary A.2.3 that their analogues are also true when the geometric point $s \to \mathsf{S}_0$ is replaced with morphisms from general schemes, although we shall omit their statements. In particular, we can talk about connected components of fibers rather than geometric fibers.

The proof of Theorem A.2.2 will be carried out in Sections A.3, A.4, and A.5. In Sections A.5 and A.6, we will also state and prove analogues of Theorem A.2.2 in zero and arbitrarily ramified characteristics, respectively (see Theorems A.5.1

and A.6.1). We will give some examples in Section A.7, including one (see Example A.7.2) showing that we cannot expect Theorem A.2.2 to be true without the requirement (in Assumption A.2.1) that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves no factor of type D.

A.3. **Reduction to the case of characteristic zero.** The goal of this section is to prove the following:

Proposition A.3.1. *Suppose Theorem* A.2.2 *is true when* $\mathrm{char}(k(s)) = 0$. *Then it is also true when* $\mathrm{char}(k(s)) = p > 0$.

Remark A.3.2. Proposition A.3.1 holds regardless of Assumption A.2.1.

Remark A.3.3. It might seem that everything in characteristic zero is well known and straightforward. But Proposition A.3.1, which is insensitive to the crucial Assumption A.2.1, shows that the key difficulty is in fact in characteristic zero.

By [11, Thm. 7.2.4.1(4)], each $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ is isomorphic to a boundary moduli problem $\mathsf{M}_{\mathcal{H}^p}^{\mathsf{Z}_{\mathcal{H}^p}}$ defined in the same way as $\mathsf{M}_{\mathcal{H}^p}$ (but with certain integral PEL datum associated with $\mathsf{Z}_{\mathcal{H}^p}$). Then it makes sense to consider the minimal compactification $\mathsf{Z}^{\min}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$, which is proper flat and has geometrically normal fibers over $\mathsf{M}_{\mathcal{H}}$, as in [11, Thm. 7.2.4.1 and Prop. 7.2.4.3]. (So the connected components of the geometric fibers of $\mathsf{Z}^{\min}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \to \mathsf{S}_0$ are closures of those of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \to \mathsf{S}_0$.) By considering the Stein factorizations of the structural morphisms $\mathsf{Z}^{\min}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \to \mathsf{S}_0$ (see [7, III-1, 4.3.3 and 4.3.4]), we obtain the following:

Lemma A.3.4 (cf. [11, Cor. 6.4.1.2] and [5, Thm. 4.17])**.** *Suppose* $\mathrm{char}(k(s)) = p > 0$. *Then there exists some discrete valuation ring R flat over $\mathcal{O}_{F_0,(p)}$, with fraction field K and residue field $k(s)$, the latter lifting the structural homomorphism $\mathcal{O}_{F_0,(p)} \to k(s)$, such that, for each cusp label $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$, and for each connected component V of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$, the induced flat morphism $V \to \mathrm{Spec}(R)$ has connected special fiber over* $\mathrm{Spec}(k(s))$.

Proof of Proposition A.3.1. Let R be as in Lemma A.3.4. Let $\tilde{U}$ denote the connected component of $\mathsf{M}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R = \mathsf{Z}_{[(0,0)]} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$ such that $\tilde{U} \underset{R}{\otimes} k(s) = U$ as subsets of $\mathsf{M}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} k(s) = \mathsf{M}_{\mathcal{H}^p} \underset{\mathsf{S}_0}{\times} s$, and let $\tilde{U}^{\min}$ denote its closure in $\mathsf{M}^{\min}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$, which is a connected component of $\mathsf{M}^{\min}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$ because $\mathsf{M}^{\min}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$ is normal by [11, Prop. 7.2.4.3(4)]. For each cusp label $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$, let $\tilde{U}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ denote the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ to $\tilde{U}^{\min}$. Then $\tilde{U}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ is an open and closed subscheme of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$ such that $\tilde{U}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{R}{\otimes} k(s) = U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ as subsets of $\mathsf{M}^{\min}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} k(s)$. By Lemma A.3.4, it suffices to show that $\tilde{U}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{R}{\otimes} \bar{K} \neq \emptyset$ for some algebraic closure $\bar{K}$ of K. Also by Lemma A.3.4, $\tilde{U} \underset{R}{\otimes} \bar{K} \neq \emptyset$, and so $\tilde{U}^{\min} \underset{R}{\otimes} \bar{K}$ contains at least one connected component of $\mathsf{M}^{\min}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \bar{K}$. Thus, $\tilde{U}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{R}{\otimes} \bar{K} \neq \emptyset$ under the assumption of the proposition, as desired. $\square$

A.4. **Comparison of cusp labels.** Let $\mathcal{H}_p := \mathrm{G}(\mathbb{Z}_p)$ and $\mathcal{H} := \mathcal{H}^p\mathcal{H}_p$, the latter being a neat open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}})$. By the same references to [11] as in Section A.2, we have the moduli problem $\mathsf{M}_{\mathcal{H}}$ and its minimal compactification $\mathsf{M}_{\mathcal{H}}^{\min}$ over $\mathsf{S}_{0,\mathbb{Q}} := \mathsf{S}_0 \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong \mathrm{Spec}(F_0)$. For each compatible collection Σ' of cone decompositions for $\mathsf{M}_{\mathcal{H}}$, we also have a toroidal compactification $\mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}}$, together with a canonical morphism $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$, over $\mathsf{S}_{0,\mathbb{Q}}$. (Here Σ' does not have to be related to the Σ^p above.)

Each cusp label $[(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$ (where $\mathsf{Z}_{\mathcal{H}}$ has been suppressed in the notation for simplicity) can be described as an equivalence class of the $\mathcal{H}$-orbit $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of some triple $(\mathsf{Z}, \Phi, \delta)$, where:

(1) $\mathsf{Z} = \{\mathsf{Z}_{-i}\}_{i\in\mathbb{Z}}$ is an admissible filtration on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ that is fully symplectic as in [11, Def. 5.2.7.1]. In particular, $\mathsf{Z}_{-i} = (\mathsf{Z}_{-i} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \cap (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$; the symplectic filtration $\mathsf{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty$ extends to a symplectic filtration $\mathsf{Z}_{\mathbb{A}}$ on $\mathsf{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{A}$; and each graded piece of Z or $\mathsf{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is integrable as in [11, Def. 1.2.1.23], that is, it is the base extension of some $\mathcal{O}$-lattice.

(2) $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ is a torus argument as in [11, Def. 5.4.1.3], where $\phi : Y \hookrightarrow X$ is an embedding of $\mathcal{O}$-lattices with finite cokernel, and where $\varphi_{-2} : \mathrm{Gr}_{-2}^{\mathsf{Z}} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1))$ and $\varphi_0 : \mathrm{Gr}_0^{\mathsf{Z}} \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ are isomorphisms matching the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{20} : \mathrm{Gr}_{-2}^{\mathsf{Z}} \times \mathrm{Gr}_0^{\mathsf{Z}} \to \hat{\mathbb{Z}}(1)$ induced by $\langle\,\cdot\,,\,\cdot\,\rangle$ with the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\phi : \mathrm{Hom}_{\hat{\mathbb{Z}}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}) \to \hat{\mathbb{Z}}(1)$ induced by ϕ.

(3) $\delta : \mathrm{Gr}^{\mathsf{Z}} \xrightarrow{\sim} L$ is an $\mathcal{O}$-equivariant splitting of the filtration Z.

(4) Two triples $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent (as in [11, Def. 5.4.2.2]) if $\mathsf{Z}_{\mathcal{H}} = \mathsf{Z}'_{\mathcal{H}}$ and if there exists a pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ matching $\Phi_{\mathcal{H}}$ with $\Phi'_{\mathcal{H}}$.

Since $\mathcal{H} = \mathcal{H}^p\mathcal{H}_p$, it makes sense to consider the p-part of $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, which is the $\mathcal{H}_p$-orbit of some triple $(\mathsf{Z}_{\mathbb{Z}_p}, (\varphi_{-2,\mathbb{Z}_p}, \varphi_{0,\mathbb{Z}_p}), \delta_{\mathbb{Z}_p})$, where:

(1) $\mathsf{Z}_{\mathbb{Z}_p} = \{\mathsf{Z}_{\mathbb{Z}_p,-i}\}_{i\in\mathbb{Z}}$ is a symplectic admissible filtration on $L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$, which determines and is determined by a symplectic admissible filtration $\mathsf{Z}_{\mathbb{Q}_p} = \{\mathsf{Z}_{\mathbb{Q}_p,-i}\}_{i\in\mathbb{Z}}$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ by $\mathsf{Z}_{\mathbb{Q}_p,-i} = \mathsf{Z}_{\mathbb{Z}_p,-i} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and $\mathsf{Z}_{\mathbb{Z}_p,-i} = \mathsf{Z}_{\mathbb{Q}_p,-i} \cap (L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p)$, for all $i \in \mathbb{Z}$.

(2) $\varphi_{-2,\mathbb{Z}_p} : \mathrm{Gr}_{-2}^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \mathbb{Z}_p(1))$ and $\varphi_0 : \mathrm{Gr}_0^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ are isomorphisms matching the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{20,\mathbb{Z}_p} : \mathrm{Gr}_{-2}^{\mathsf{Z}_{\mathbb{Z}_p}} \times \mathrm{Gr}_0^{\mathsf{Z}_{\mathbb{Z}_p}} \to \mathbb{Z}_p(1)$ induced by $\langle\,\cdot\,,\,\cdot\,\rangle$ with the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\phi,\mathbb{Z}_p} : \mathrm{Hom}_{\mathbb{Z}_p}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \mathbb{Z}_p(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p) \to \mathbb{Z}_p(1)$ induced by ϕ.

(3) $\delta_{\mathbb{Z}_p} : \mathrm{Gr}^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is a splitting of the filtration $\mathsf{Z}_{\mathbb{Z}_p}$.

By forgetting its p-part, each representative $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ for $\mathsf{M}_{\mathcal{H}}$ induces a representative $(\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})$ for $\mathsf{M}_{\mathcal{H}^p}$, and this assignment is compatible with the formation of equivalence classes. Therefore, we have well-defined assignments

$$(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \mapsto (\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p}) \tag{A.4.1}$$

and

$$[(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})] \mapsto [(\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]. \tag{A.4.2}$$

By construction, these assignments are compatible with surjections on their both sides (see [11, Def. 5.4.2.12]). We would like to show that they are both bijective.

Lemma A.4.3. *Let k be any field over $\mathbb{Z}_{(p)}$. Consider the assignment to each flag W of totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$-submodules of $L \underset{\mathbb{Z}}{\otimes} k$ (with respect to $\langle \cdot, \cdot \rangle \underset{\mathbb{Z}}{\otimes} k$) its stabilizer subgroup P_W in $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k$. Then each such P_W is a parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k$, and the assignment is bijective. Moreover, given any minimal parabolic subgroup P_{W_0} of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k$, which is the stabilizer of some maximal flag W_0 of totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$-submodules of $L \underset{\mathbb{Z}}{\otimes} k$, every parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k$ is conjugate under the action of $\mathrm{G}(k)$ to some parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k$ containing P_{W_0}, which is the stabilizer of some subflag of W_0.*

Although the assertions in this lemma are well known, we provide a proof because we cannot find a convenient reference in the literature in the generality we need.

Proof of Lemma A.4.3. Let k^{sep} be a separable closure of k. Since the characteristic of k is either 0 or p, the latter being a good prime by assumption, it follows from [11, Prop. 1.2.3.11] that each of the simple factors of the adjoint quotient of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ is isomorphic to one of the groups of standard type listed in the proof of [11, Prop. 1.2.3.11]. Then we can make an explicit choice of a Borel subgroup B of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ stabilizing a flag of totally isotropic submodules, with a maximal torus T of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ contained in B which is isomorphic to the group of automorphisms of the graded pieces of this flag. By [16, Thm. 6.2.7 and Thm. 8.4.3(iv)], since all parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ are conjugate to one containing B, the parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ are exactly the stabilizers of flags of totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$-submodules of $L \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$. Then the analogous assertion over k follows, because the assignment of maximal parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ is compatible with the actions of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on the set of flags of totally isotropic submodules of $L \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ and on the set of parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$. The last assertion of the lemma follows from [16, Thm. 15.1.2(ii) and Thm. 15.4.6(i)]. □

Lemma A.4.4. *The assignment*

$$\mathsf{Z}_{\mathcal{H}} \mapsto \mathsf{Z}_{\mathcal{H}_p} \tag{A.4.5}$$

is bijective.

Proof. Let $\mathsf{Z}_{\mathbb{Z}_p} = \{\mathsf{Z}_{\mathbb{Z}_p,-i}\}_{i\in\mathbb{Z}}$ be a symplectic admissible filtration on $L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ as above, which determines and is determined by a symplectic filtration $\mathsf{Z}_{\mathbb{Q}_p} = \{\mathsf{Z}_{\mathbb{Q}_p,-i}\}_{i\in\mathbb{Z}}$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$. By Lemma A.4.3, the action of $\mathrm{G}(\mathbb{Q}_p)$ on the set of such filtrations $\mathsf{Z}_{\mathbb{Q}_p}$ is transitive, because the $\mathcal{O}$-multirank (see [11, Def. 1.2.1.25]) of the bottom piece $\mathsf{Z}_{\mathbb{Q}_p,-2}$ of any such $\mathsf{Z}_{\mathbb{Q}_p}$ is determined by the existence of some isomorphism $\varphi_{-2,\mathbb{Z}_p} : \mathrm{Gr}_{-2}^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \mathbb{Z}_p(1))$. Let P denote the parabolic subgroup of $\mathrm{G}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}_p$ stabilizing any such $\mathtt{Z}_{\mathbb{Q}_p}$ (see Lemma A.4.3). Since p is a good prime by assumption, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p$ is self-dual, and hence $\mathrm{G}(\mathbb{Z}_p)$ is a maximal open compact subgroup of $\mathrm{G}(\mathbb{Q}_p)$ by [3, Cor. 3.3.2]. Since $\mathrm{G}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}_p$ is connected under Assumption A.2.1 (because the kernel of the similitude character of $\mathrm{G}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}_p$ factorizes over an algebraic closure of $\mathbb{Q}_p$ as a product of connected groups, by the proof of [11, Prop. 1.2.3.11]), we have the Iwasawa decomposition $\mathrm{G}(\mathbb{Q}_p)=\mathrm{G}(\mathbb{Z}_p)\mathrm{P}(\mathbb{Q}_p)$, by [3, Prop. 4.4.3] (see also [4, (18) on p. 392] for a more explicit statement). Consequently, $\mathcal{H}_p=\mathrm{G}(\mathbb{Z}_p)$ acts transitively on the set of possible filtrations $\mathtt{Z}_{\mathbb{Z}_p}$ as above, and hence the assignment (A.4.5) is injective.

As for the surjectivity of (A.4.5), it suffices to show that, for some symplectic admissible filtration $\mathtt{Z}_{\mathbb{Z}_p}$, an isomorphism $\varphi_{-2,\mathbb{Z}_p}:\mathrm{Gr}_{-2}^{\mathtt{Z}_{\mathbb{Z}_p}}\overset{\sim}{\to}\mathrm{Hom}_{\mathbb{Z}_p}(X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p,\mathbb{Z}_p(1))$ exists. By [14, Thm. 18.10] and [11, Cor. 1.1.2.6], it suffices to show that there exists some symplectic filtration $\mathtt{Z}_{\mathbb{Q}_p}$ such that $\mathtt{Z}_{\mathbb{Q}_p,-2}$ and $\mathrm{Hom}_{\mathbb{Q}_p}(X\underset{\mathbb{Z}}{\otimes}\mathbb{Q}_p,\mathbb{Q}_p(1))$ have the same $\mathcal{O}$-multirank. Or rather, we just need to notice that the $\mathcal{O}$-multirank of a totally isotropic $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}_p$-submodule can be any $\mathcal{O}$-multirank below a maximal one (with respect to the natural partial order), by Assumption A.2.1 and by the classification in [11, Prop. 1.2.3.7 and Cor. 1.2.3.10]. □

Lemma A.4.6. *The assignment* (A.4.1) *is bijective.*

Proof. It is already explained in the proof of Lemma A.4.4 that an isomorphism $\varphi_{-2,\mathbb{Z}_p}:\mathrm{Gr}_{-2}^{\mathtt{Z}_{\mathbb{Z}_p}}\overset{\sim}{\to}\mathrm{Hom}_{\mathbb{Z}_p}(X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p,\mathbb{Z}_p(1))$ exists for any $\mathtt{Z}_{\mathbb{Z}_p}$ considered there. Since p is a good prime, which forces both $[L^{\#}:L]$ and $[X:\phi(Y)]$ to be prime to p, any choice of $\varphi_{-2,\mathbb{Z}_p}$ above uniquely determines an isomorphism $\varphi_0:\mathrm{Gr}_0^{\mathtt{Z}_{\mathbb{Z}_p}}\overset{\sim}{\to}Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p$. Also by the explicit classification in [11, Prop. 1.2.3.7 and Cor. 1.2.3.10] as in the proof of Lemma A.4.4, there exists a splitting $\delta_{\mathbb{Z}_p}:\mathrm{Gr}^{\mathtt{Z}_{\mathbb{Z}_p}}\overset{\sim}{\to}L\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p$, and the action of $\mathrm{G}(\mathbb{Z}_p)\cap\mathrm{P}(\mathbb{Q}_p)$ acts transitively on the set of possible triples $(\varphi_{-2,\mathbb{Z}_p},\varphi_{0,\mathbb{Z}_p},\delta_{\mathbb{Z}_p})$. Hence the assignment (A.4.1) is bijective, as desired. □

Lemma A.4.7. *The assignment* (A.4.2) *is bijective.*

Proof. By Lemma A.4.6, it suffices to show that (A.4.2) is injective. Suppose two representatives $(\mathtt{Z}_{\mathcal{H}},\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ and $(\mathtt{Z}'_{\mathcal{H}},\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ with $\Phi_{\mathcal{H}}=(X,Y,\phi,\varphi_{-2,\mathcal{H}},\varphi_{0,\mathcal{H}})$ and $\Phi'_{\mathcal{H}}=(X',Y',\phi',\varphi'_{-2,\mathcal{H}},\varphi'_{0,\mathcal{H}})$ are such that the induced $(\mathtt{Z}_{\mathcal{H}^p},\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})$ and $(\mathtt{Z}'_{\mathcal{H}^p},\Phi'_{\mathcal{H}^p},\delta'_{\mathcal{H}^p})$ are equivalent to each other. By definition, $\mathtt{Z}_{\mathcal{H}^p}=\mathtt{Z}'_{\mathcal{H}^p}$, so that $\mathtt{Z}_{\mathcal{H}}=\mathtt{Z}'_{\mathcal{H}}$ by Lemma A.4.4; and there exists a pair $(\gamma_X:X'\overset{\sim}{\to}X,\gamma_Y:Y\overset{\sim}{\to}Y')$ matching $\Phi_{\mathcal{H}^p}$ with $\Phi'_{\mathcal{H}^p}$. Hence we may assume that $(X,Y,\phi)=(X',Y',\phi')$, take any $\mathtt{Z}$ in $\mathtt{Z}_{\mathcal{H}^p}=\mathtt{Z}'_{\mathcal{H}^p}$, and take any pairs $(\varphi_{-2}:\mathrm{Gr}_{-2}^{\mathtt{Z}}\overset{\sim}{\to}\mathrm{Hom}_{\hat{\mathbb{Z}}}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}},\hat{\mathbb{Z}}(1)),\varphi_0:\mathrm{Gr}_0^{\mathtt{Z}}\overset{\sim}{\to}Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}})$ and $(\varphi'_{-2}:\mathrm{Gr}_{-2}^{\mathtt{Z}}\overset{\sim}{\to}\mathrm{Hom}_{\hat{\mathbb{Z}}}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}},\hat{\mathbb{Z}}(1)),\varphi'_0:\mathrm{Gr}_0^{\mathtt{Z}}\overset{\sim}{\to}Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}})$ inducing $(\varphi_{-2,\mathcal{H}},\varphi_{0,\mathcal{H}})$ and $(\varphi'_{-2,\mathcal{H}},\varphi'_{0,\mathcal{H}})$, respectively, and inducing the same $(\varphi_{-2,\mathcal{H}^p},\varphi_{0,\mathcal{H}^p})$ and $(\varphi'_{-2,\mathcal{H}^p},\varphi'_{0,\mathcal{H}^p})$. Then the injectivity of (A.4.2) follows from that of (A.4.1). □

Lemma A.4.8. *If* $(\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})$ *is assigned to* $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *under* (A.4.1)*, then we have a canonical isomorphism*

$$\Gamma_{\Phi_{\mathcal{H}}} \xrightarrow{\sim} \Gamma_{\Phi_{\mathcal{H}^p}} \tag{A.4.9}$$

(*see* [11, Def. 6.2.4.1])*. Moreover, we have a canonical isomorphism*

$$\mathbf{S}_{\Phi_{\mathcal{H}^p}} \xrightarrow{\sim} \mathbf{S}_{\Phi_{\mathcal{H}}}, \tag{A.4.10}$$

which induces a canonical isomorphism

$$(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}} \xrightarrow{\sim} (\mathbf{S}_{\Phi_{\mathcal{H}^p}})^{\vee}_{\mathbb{R}} \tag{A.4.11}$$

matching $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *(resp.* $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$*) with* $\mathbf{P}_{\Phi_{\mathcal{H}^p}}$ *(resp.* $\mathbf{P}^+_{\Phi_{\mathcal{H}^p}}$*), both isomorphisms being equivariant with the actions of the two sides of* (A.4.9) *above.*

Proof. Since p is a good prime, with $\mathcal{H}_p = \mathrm{G}(\mathbb{Z}_p)$, the levels at p are not needed in the constructions of $\Gamma_{\Phi_{\mathcal{H}}}$ and $\mathbf{S}_{\Phi_{\mathcal{H}}}$ in [11, Sec. 6.2.3–6.2.4], and hence we have the desired isomorphisms (A.4.9) and (A.4.10). The induced morphism (A.4.11) matches $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$) with $\mathbf{P}_{\Phi_{\mathcal{H}^p}}$ (resp. $\mathbf{P}^+_{\Phi_{\mathcal{H}^p}}$) because both sides of (A.4.11) can be canonically identified with the space of Hermitian forms over $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, as explained in the beginning of [11, Sec. 6.2.5], regardless of the levels $\mathcal{H}$ and $\mathcal{H}^p$. $\square$

Therefore, we also have assignments

$$(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma) \mapsto (\Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p}, \sigma^p) \tag{A.4.12}$$

and

$$[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)] \mapsto [(\Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p}, \sigma^p)] \tag{A.4.13}$$

(see [11, Def. 6.2.6.2]), which are compatible with (A.4.1) and (A.4.2), where we have suppressed $\mathsf{Z}_{\mathcal{H}}$ and $\mathsf{Z}_{\mathcal{H}^p}$ from the notation, where $\sigma \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$, and where $\sigma^p \subset (\mathbf{S}_{\Phi_{\mathcal{H}^p}})^{\vee}_{\mathbb{R}}$ is the image of σ under isomorphism (A.4.11).

Lemma A.4.14. *The assignment* (A.4.12) *is bijective.*

Proof. This follows from Lemma A.4.6 and the definition of (A.4.12) based on Lemma A.4.8. $\square$

Lemma A.4.15. *The assignment* (A.4.13) *is bijective.*

Proof. By [11, Def. 6.2.6.2], given any representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of a cusp label, the collection of the cones $\sigma \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ defining the same equivalence class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ form a $\Gamma_{\Phi_{\mathcal{H}}}$-orbit. Similarly, the collection of the cones $\sigma^p \subset (\mathbf{S}_{\Phi_{\mathcal{H}^p}})^{\vee}_{\mathbb{R}}$ defining the same equivalence class $[(\Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p}, \sigma^p)]$ form a $\Gamma_{\Phi_{\mathcal{H}^p}}$-orbit. Hence, given (A.4.9), the lemma follows from Lemma A.4.7. $\square$

Definition A.4.16. *We say that* Σ *is* **induced** *by* Σ^p *if, for each cusp label* $[(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ *of* $\mathsf{M}_{\mathcal{H}}$ *represented by some* $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$*, with assigned* $(\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})$ *as in* (A.4.1)*, the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *is the pullback of the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}^p}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}^p}}$ *under* (A.4.11)*.*

By forgetting the p-parts of level structures, we obtain a canonical isomorphism

$$\mathsf{M}_{\mathcal{H}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \tag{A.4.17}$$

over $\mathsf{S}_{0,\mathbb{Q}}$ (as in [11, (1.4.4.1)]), by [11, Prop. 1.4.4.3 and Rem. 1.4.4.4] and by Assumption A.2.1. Given any Σ^p for $\mathsf{M}_{\mathcal{H}^p}$, with induced Σ for $\mathsf{M}_{\mathcal{H}}$ as in Definition

A.4.16, by comparing the universal properties of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ as in [11, Thm. 6.4.1.1 (5) and (6)], the isomorphism (A.4.17) extends to a canonical isomorphism

$$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \xrightarrow{\sim} \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \tag{A.4.18}$$

over $\mathsf{S}_{0,\mathbb{Q}}$, mapping $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ isomorphically to $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ when $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$ is assigned to $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ under (A.4.13), such that the pullback of the tautological semi-abelian scheme over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is canonically isomorphic to the pullback of the tautological semi-abelian scheme over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. Consequently, by [11, Thm. 7.2.4.1 (3) and (4)], and by the fact that the pullback of the Hodge invertible sheaf over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is canonically isomorphic to the pullback of the Hodge invertible sheaf over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (because their definitions only use the tautological semi-abelian schemes), the canonical isomorphism (A.4.18) induces a canonical isomorphism

$$\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \xrightarrow{\sim} \mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \tag{A.4.19}$$

over $\mathsf{S}_{0,\mathbb{Q}}$, extending (A.4.17), compatible with (A.4.18) (under the canonical morphisms $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ and $\oint_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$), and mapping $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ isomorphically to $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ when $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$ is assigned to $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ under (A.4.2) (where we have suppressed $\mathsf{Z}_{\mathcal{H}}$ and $\mathsf{Z}_{\mathcal{H}^p}$ from the notation).

A.5. **Complex analytic construction.** By Proposition A.3.1, in order to prove Theorem A.2.2, we may and we shall assume that $\mathrm{char}(k(s)) = 0$. Thanks to the isomorphisms (A.4.17) and (A.4.19), we shall identify U with a connected component of $\mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} k(s)$, identify U^{min} with the connected component of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{F_0}{\otimes} k(s)$ that is the closure of U, and identify $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ with $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$, the pullback of the stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ under the canonical morphism $U^{\mathrm{min}} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$, when $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$ is assigned to $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ under (A.4.2).

Now in characteristic zero we no longer need $\mathcal{H}$ to be of the form $\mathcal{H} = \mathcal{H}^p\mathcal{H}_p$ as in Section A.4. We shall allow $\mathcal{H}$ to be any neat open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}})$. Then $\mathsf{M}_{\mathcal{H}}$ and $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ are still defined over $\mathsf{M}_{0,\mathbb{Q}} = \mathrm{Spec}(F_0)$, with the stratification on the latter by locally closed subschemes $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ labeled by cusp labels $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$ (see the same references as in Section A.2). For any geometric point $s \to \mathsf{S}_{0,\mathbb{Q}}$ with residue field $k(s)$ and for any connected component U of the fiber $\mathsf{M}_{\mathcal{H}^p} \underset{\mathsf{S}_0}{\times} s$, we define U^{min} to be the closure of U in $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} s$, and define $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ to be the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ to of U^{min}, for each cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$. (These are consistent with what we have done before, when the settings overlap.)

Then we have the following analogue of Theorem A.2.2:

Theorem A.5.1. *With the setting as above, every stratum $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ is nonempty.*

Since $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ is projective over $\mathsf{S}_{0,\mathbb{Q}}$, we may and we shall assume that $k(s) \cong \mathbb{C}$. We shall denote base changes to $\mathbb{C}$ with a subscript, such as $\mathsf{M}_{\mathcal{H},\mathbb{C}} = \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} \mathbb{C}$.

Let X denote the $\mathrm{G}(\mathbb{R})$-orbit of h_0, which is a finite disjoint union of Hermitian symmetric domains, and let X_0 denote the connected component of X containing h_0. Let $\mathrm{G}(\mathbb{Q})_0$ denote the finite index subgroup of $\mathrm{G}(\mathbb{Q})$ stabilizing X_0. Let

$\mathsf{Sh}_{\mathcal{H}} := \mathrm{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$. By [10, Lem. 2.5.1], we have a canonical bijection $\mathrm{G}(\mathbb{Q})_0\backslash \mathsf{X}_0 \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} \to \mathrm{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$. Let $\{g_i\}_{i\in I}$ be any finite set of elements of $\mathrm{G}(\mathbb{A}^\infty)$ such that $\mathrm{G}(\mathbb{A}^\infty) = \coprod\limits_{i\in I} \mathrm{G}(\mathbb{Q})_0 h_i \mathcal{H}$, which exists because of [2, Thm. 5.1] and because $\mathrm{G}(\mathbb{Q})_0$ is of finite index in $\mathrm{G}(\mathbb{Q})$. Then we have

$$\mathsf{Sh}_{\mathcal{H}} = \mathrm{G}(\mathbb{Q})_0\backslash \mathsf{X}_0 \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} = \coprod_{i\in I} \Gamma^{(g_i)}\backslash \mathsf{X}_0, \tag{A.5.2}$$

where $\Gamma^{(g_i)} := (g_i \mathcal{H} g_i^{-1}) \cap \mathrm{G}(\mathbb{Q})_0$ for each $i \in I$. By applying [1, 10.11] to each $\Gamma^{(g_i)}\backslash \mathsf{X}_0$, we obtain the minimal compactification $\mathsf{Sh}_{\mathcal{H}}^{\min}$ of $\mathsf{Sh}_{\mathcal{H}}$, which is the complex analytification of a normal projective variety $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\min}$ over $\mathbb{C}$. Thus $\mathsf{Sh}_{\mathcal{H}}$ is the analytification of a quasi-projective variety $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$ (embedded in $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\min}$).

By [10, Lem. 3.1.1], the rational boundary components $\mathsf{X}_{\mathtt{V}}$ of X_0 (see [1, 3.5]) correspond to parabolic subgroups of $\mathrm{G} \otimes_{\mathbb{Z}} \mathbb{Q}$ stabilizing symplectic filtrations $\mathtt{V}$ on $L \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\mathtt{V}_{-3} = 0 \subset \mathtt{V}_{-2} \subset \mathtt{V}_{-1} = \mathtt{V}_{-2}^{\perp} \subset \mathtt{V}_0 = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider the rational boundary components of $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)$ as in [10, Def. 3.1.2], which are $\mathrm{G}(\mathbb{Q})$-orbits of pairs $(\mathtt{V}, g)$, where $\mathtt{V}$ are as above and $g \in \mathrm{G}(\mathbb{A}^\infty)$. Consider the boundary components $\mathrm{G}(\mathbb{Q})\backslash(\mathrm{G}(\mathbb{Q})\mathsf{X}_{\mathtt{V}}) \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} = \mathrm{G}(\mathbb{Q})_0\backslash(\mathrm{G}(\mathbb{Q})_0\mathsf{X}_{\mathtt{V}}) \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$ of $\mathsf{Sh}_{\mathcal{H}} = \mathrm{G}(\mathbb{Q})_0\backslash \mathsf{X}_0 \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$. By the construction in [1], each such component defines a nonempty locally closed subset and meets all connected components of $\mathsf{Sh}_{\mathcal{H}}^{\min}$, corresponding to a nonempty locally closed subscheme of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\min}$ which we call its $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H}$-stratum. Thus, we obtain the following:

Proposition A.5.3 (Satake, Baily–Borel)**.** *Each $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H}$-stratum as above meets every connected component of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\min}$.*

For each $g \in \mathrm{G}(\mathbb{A}^\infty)$, let $L^{(g)}$ denote the $\mathcal{O}$-lattice in $L \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $L^{(g)} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} = g(L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})$ in $L \otimes_{\mathbb{Z}} \mathbb{A}^\infty$. Let $r \in \mathbb{Q}_{>0}^\times$ be the unique element such that $\nu(g) = ru$ for some $u \in \hat{\mathbb{Z}}$, and let $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)} : L^{(g)} \times L^{(g)} \to \mathbb{Z}(1)$ denote the pairing induced by $r\langle\,\cdot\,,\,\cdot\,\rangle \otimes_{\mathbb{Z}} \mathbb{Q}$ (see [10, Sec. 2.4]; the key point being that $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)}$ is valued in $\mathbb{Z}(1)$).

Construction A.5.4. As explained in [10, Sec. 3.1], we have an assignment of a fully symplectic admissible filtration $\mathtt{Z}^{(g)}$ on $\mathtt{Z} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and a torus argument $\Phi^{(g)} = (X^{(g)}, Y^{(g)}, \phi^{(g)}, \varphi_{-2}^{(g)}, \varphi_0^{(g)})$ to $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)$, by setting:

(1) $\mathtt{F}^{(g)} := \{\mathtt{F}_{-i}^{(g)} := \mathtt{V}_{-i} \cap L^{(g)}\}_{i\in\mathbb{Z}}$.
(2) $\mathtt{Z}^{(g)} := \{\mathtt{Z}_{-i}^{(g)} := g^{-1}(\mathtt{F}_{-i}^{(g)} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})\}_{i\in\mathbb{Z}} = \{g^{-1}(\mathtt{V}_{-i} \otimes_{\mathbb{Q}} \mathbb{A}^\infty) \cap (L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})\}_{i\in\mathbb{Z}}$.
(3) $X^{(g)} := \mathrm{Hom}_{\mathbb{Z}}(\mathtt{F}_{-2}^{(g)}, \mathbb{Z}(1)) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Gr}_{-2}^{\mathtt{F}^{(g)}}, \mathbb{Z}(1))$.
(4) $Y^{(g)} := \mathrm{Gr}_0^{\mathtt{F}^{(g)}} = \mathtt{F}_0^{(g)}/\mathtt{F}_{-1}^{(g)}$.
(5) $\phi^{(g)} : Y^{(g)} \hookrightarrow X^{(g)}$ is equivalent to the nondegenerate pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_{20}^{(g)} : \mathrm{Gr}_{-2}^{\mathtt{F}^{(g)}} \times \mathrm{Gr}_0^{\mathtt{F}^{(g)}} \to \mathbb{Z}(1)$$

induced by $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)} : L^{(g)} \times L^{(g)} \to \mathbb{Z}(1)$.

(6) $\varphi^{(g)}_{-2} : \mathrm{Gr}^{\mathtt{Z}^{(g)}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}}(X^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1))$ is the composition

$$\mathrm{Gr}^{\mathtt{Z}^{(g)}}_{-2} \overset{\mathrm{Gr}_{-2}(g)}{\xrightarrow{\sim}} \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}}(X^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1)).$$

(7) $\varphi^{(g)}_{0} : \mathrm{Gr}^{\mathtt{Z}^{(g)}}_{0} \xrightarrow{\sim} Y^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ is the composition

$$\mathrm{Gr}^{\mathtt{Z}^{(g)}}_{0} \overset{\mathrm{Gr}_{0}(g)}{\xrightarrow{\sim}} \mathrm{Gr}^{\mathtt{F}^{(g)}}_{0} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} Y^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}.$$

By the assumption that our integral PEL datum satisfies [11, Cond. 1.4.3.10], and by the fact that maximal orders over Dedekind domains are *hereditary* (see [14, Thm. 21.4 and Cor. 21.5]), there exists some splitting $\varepsilon^{(g)} : \mathrm{Gr}^{\mathtt{F}^{(g)}} \xrightarrow{\sim} L^{(g)}$, whose base extension from $\mathbb{Z}$ to $\hat{\mathbb{Z}}$ defines by pre- and post- compositions with $\mathrm{Gr}(g)$ and g^{-1} a splitting $\delta^{(g)} : \mathrm{Gr}^{\mathtt{Z}^{(g)}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$. These define an assignment

$$\mathrm{G}(\mathbb{Q})(\mathtt{V}, g) \mapsto [(\mathtt{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)})], \tag{A.5.5}$$

which is compatible with the formation of $\mathcal{H}$-orbits and induces an assignment

$$\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H} \mapsto [(\mathtt{Z}^{(g)}_{\mathcal{H}}, \Phi^{(g)}_{\mathcal{H}}, \delta^{(g)}_{\mathcal{H}})]. \tag{A.5.6}$$

Definition A.5.7. *For each cusp label $[(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$, the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$-stratum of $\mathsf{Sh}^{\min}_{\mathcal{H},\mathrm{alg}}$ is the union of all the $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H}$-strata such that $[(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ is assigned to $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H}$ under* (A.5.6).

Proposition A.5.8. *Given the $\mathcal{H}$-orbit $\mathtt{Z}_{\mathcal{H}}$ of any $\mathtt{Z} = \{\mathtt{Z}_{-i}\}_{i\in\mathbb{Z}}$ as above, there exists some totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-submodule $\mathtt{V}_{-2}$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that $\mathtt{V}_{-2} \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\infty}$ lies in the $\mathcal{H}$-orbit of $\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.*

Proof. Up to replacing $\mathcal{H}$ with an open compact subgroup, which is harmless for proving this proposition, we may and we shall assume that $\mathcal{H} = \mathcal{H}^S\mathcal{H}_S$, where S is a finite set of primes containing all bad ones for the integral PEL datum (see [11, Def. 1.4.1.1]), such that $\mathcal{H}^S = \mathrm{G}(\hat{\mathbb{Z}}^S) = \prod_{\ell\notin S} \mathrm{G}(\mathbb{Z}_\ell)$ and $\mathcal{H}_S \subset \mathrm{G}(\hat{\mathbb{Z}}_S) = \prod_{\ell\in S} \mathrm{G}(\mathbb{Z}_\ell)$, where $\ell \notin S$ means that ℓ runs through all prime numbers not in S.

By Assumption A.2.1, by reduction to the case where $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a product of division algebras by Morita equivalence (see [11, Prop. 1.2.1.14]), and by the local-global principle for isotropy in [15, table on p. 347, and its references], it follows that, if $\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is nonzero and extends to some isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{A}$-submodule of $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}$ isomorphic to the base extension of some $\mathcal{O}$-lattice, then there exists some nonzero isotropic element in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. By induction on the $\mathcal{O}$-multirank of $\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$—by replacing $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (resp. $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty}$) with the orthogonal complement modulo the span of a nonzero isotropic element in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (resp. $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty}$)—there exists some totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-submodule $\mathtt{V}^0_{-2}$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that $\mathtt{V}^0_{-2} \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\infty}$ and $\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ have the same $\mathcal{O}$-multirank.

Let G' denote the derived subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (see [6, $\mathrm{VI_B}$, 7.2(vii) and 7.10]). Then the pullback to G' induces a bijection between the parabolic subgroups of

$\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and those of G' (see [6, XXII, 6.2.4 and 6.2.8] and [16, Thm. 15.1.2(ii) and Thm. 15.4.6(i)]), and they both are in bijection with the stabilizers of flags of totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-submodules as in Lemma A.4.3. Therefore, there exists some element $h = (h_\ell) \in \mathrm{G}'(\mathbb{A}^\infty)$, where the index ℓ runs through all prime numbers, such that $\mathtt{V}^0_{-2} \underset{\mathbb{Q}}{\otimes} \mathbb{A}^\infty = h(\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$.

Since G' is simply connected by Assumption A.2.1 (because the kernel of the similitude character of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ factorizes over an algebraic closure of $\mathbb{Q}$ as a product of groups with simply connected derived groups, by the proof of [11, Prop. 1.2.3.11]), by weak approximation (see [13, Thm. 7.8]), there exists $\gamma \in \mathrm{G}'(\mathbb{Q})$ such that $\gamma(h_\ell)_{\ell \in S} \in \mathcal{H}_S$. On the other hand, by using the Iwasawa decomposition at the places $\ell \in S$ as in the proof of Lemma A.4.4, up to replacing h_ℓ with a right multiple of h_ℓ by an element of $\mathrm{G}'(\mathbb{Q}_\ell)$ stabilizing $\mathtt{Z}_{-2} \underset{\hat{\mathbb{Z}}}{\otimes} \mathbb{Q}_\ell$, we may assume that $\gamma h_\ell \in \mathrm{G}(\mathbb{Z}_\ell)$ for all $\ell \notin S$. Thus, we can conclude by taking $\mathtt{V}_{-2} := \gamma(\mathtt{V}^0_{-2})$. □

Proposition A.5.9. *For each cusp label* $[(\mathtt{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$*, there exists some rational boundary component* $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)$ *of* $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)$ *such that* $[(\mathtt{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$ *is assigned to* $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H}$ *under* (A.5.6).

Proof. Let $(\mathtt{Z}, \Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0), \delta)$ be any triple whose $\mathcal{H}$-orbit induces $[(\mathtt{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$, and let $\mathtt{V}_{-2}$ be as in Proposition A.5.8. Up to replacing $(\mathtt{Z}, \Phi, \delta)$ with another such triple, we may and we shall assume that

$$\mathtt{Z}_{-2} = (\mathtt{V}_{-2} \underset{\mathbb{Q}}{\otimes} \mathbb{A}^\infty) \cap (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}) = \mathtt{Z}^{(1)}_{-2}, \tag{A.5.10}$$

where $\mathtt{F}^{(1)} = \{\mathtt{F}^{(1)}_{-i}\}_{i \in \mathbb{Z}}$, $\mathtt{Z}^{(1)} = \{\mathtt{Z}^{(1)}_{-i}\}_{i \in \mathbb{Z}}$, and $\Phi^{(1)} = (X^{(1)}, Y^{(1)}, \phi^{(1)}, \varphi^{(1)}_{-2}, \varphi^{(1)}_0)$ are assigned to $(\mathtt{V}, 1)$ as in Construction A.5.4, together with some noncanonical choices of $\varepsilon^{(1)}$ and $\delta^{(1)}$.

Let P denote the parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ stabilizing $\mathtt{V}_{-2}$ (see Lemma A.4.3). By (A.5.10), the elements of $\mathrm{P}(\mathbb{A}^\infty)$ also stabilize $\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. Therefore, for each $g \in \mathrm{P}(\mathbb{A}^\infty)$, the filtration $\mathtt{Z}^{(g)}$ defined as in Construction A.5.4 coincides with $\mathtt{Z}$.

Using (A.5.10) and the compatibility among the objects, both $\phi \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ and $\phi^{(1)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ can be identified (under $(\varphi_{-2}, \varphi_0)$ and $(\varphi^{(1)}_{-2}, \varphi^{(1)}_0)$) with the canonical morphism

$$\langle\,\cdot\,,\,\cdot\,\rangle^*_{20} : \mathrm{Gr}^{\mathtt{Z}}_0 \to \mathrm{Hom}_{\hat{\mathbb{Z}}}(\mathrm{Gr}^{\mathtt{Z}}_{-2}, \hat{\mathbb{Z}}(1)) \tag{A.5.11}$$

induced by the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$, which induce compatible isomorphisms

$${}^t(\varphi^{(1)}_{-2} \circ \varphi^{-1}_{-2}) : X^{(1)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \tag{A.5.12}$$

and

$$\varphi^{(1)}_0 \circ \varphi^{-1}_0 : Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} Y^{(1)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}. \tag{A.5.13}$$

By [11, Cond. 1.4.3.10], there exists some maximal order $\mathcal{O}'$ in $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, containing $\mathcal{O}$, such that the $\mathcal{O}$-action on L extends to an $\mathcal{O}'$-action; hence the $\mathcal{O}$-actions on Y and $Y^{(1)}$ also extend to $\mathcal{O}'$-actions. Using the local isomorphisms given by (A.5.13), by [14, Thm. 18.10] (which is applicable because we are now considering modules of the maximal order $\mathcal{O}'$) and [11, Cor.

1.1.2.6], there exists an element $g_0 \in \mathrm{GL}_{\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{A}^\infty}(\mathrm{Gr}_0^{\mathrm{Z}} \otimes_{\mathbb{Z}} \mathbb{Q})$ and an $\mathcal{O}$-equivariant embedding $h_0 : Y^{(1)} \hookrightarrow Y \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $(h_0(Y^{(1)})) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} = (\varphi_0 \otimes_{\mathbb{Z}} \mathbb{Q})(g_0(\mathrm{Gr}_0^{\mathrm{Z}}))$ in $Y \otimes_{\mathbb{Z}} \mathbb{A}^\infty$. Let $g_{-2} := {}^t g_0^{-1} \in \mathrm{GL}_{\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{A}^\infty}(\mathrm{Gr}_{-2}^{\mathrm{Z}} \otimes_{\mathbb{Z}} \mathbb{Q})$, where the transposition is induced by (A.5.11). Then there is a corresponding $\mathcal{O}$-equivariant embedding $h_{-2} : \mathrm{Hom}_{\mathbb{Z}}(X^{(1)}, \mathbb{Z}(1)) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}(1)) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $(h_{-2}(\mathrm{Hom}_{\mathbb{Z}}(X^{(1)}, \mathbb{Z}(1)))) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} = (\varphi_{-2} \otimes_{\mathbb{Z}} \mathbb{Q})(g_{-2}(\mathrm{Gr}_{-2}^{\mathrm{Z}}))$ in $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}(1)) \otimes_{\mathbb{Z}} \mathbb{A}^\infty$.

Take $g \in \mathrm{P}(\mathbb{A}^\infty)$ such that $\mathrm{Gr}_{-2}(g) = g_{-2}$, $\mathrm{Gr}_0(g) = g_0$, and $\nu(g) = 1$, which exists thanks to the splitting δ. Then $X^{(g)}$ and $Y^{(g)}$ are realized as the preimages of X and Y under ${}^t h_{-2} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $h_0^{-1} \otimes_{\mathbb{Z}} \mathbb{Q}$, respectively; and the induced pair $(\gamma_X : X^{(g)} \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y^{(g)})$ matches $\Phi^{(g)}$ with Φ. Such a $(\mathtt{V}, g)$ is what we want. □

As explained in [10, Sec. 2.5], there is a canonical open and closed immersion

$$\mathsf{Sh}_{\mathcal{H},\mathrm{alg}} \hookrightarrow \mathsf{M}_{\mathcal{H},\mathbb{C}}. \tag{A.5.14}$$

As explained in [8, §8, p. 399] (see also [11, Rem. 1.4.3.12]), $\mathsf{M}_{\mathcal{H},\mathbb{C}}$ is the disjoint union of the images of morphisms like (A.5.14), from certain $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{(j)}$ defined by some $(\mathcal{O}, \star, L^{(j)}, \langle\cdot,\cdot\rangle^{(j)}, h_0)$ such that $(L^{(j)}, \langle\cdot,\cdot\rangle^{(j)}) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \cong (L, \langle\cdot,\cdot\rangle) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and $(L^{(j)}, \langle\cdot,\cdot\rangle^{(j)}) \otimes_{\mathbb{Z}} \mathbb{R} \cong (L, \langle\cdot,\cdot\rangle) \otimes_{\mathbb{Z}} \mathbb{R}$, but not necessarily satisfying $(L^{(j)}, \langle\cdot,\cdot\rangle^{(j)}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (L, \langle\cdot,\cdot\rangle) \otimes_{\mathbb{Z}} \mathbb{Q}$, for all j in some index set J (whose precise description is not important for our purpose). (Each $(L^{(j)}, \langle\cdot,\cdot\rangle^{(j)})$ is determined by its rational version $(L^{(j)}, \langle\cdot,\cdot\rangle^{(j)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ by taking the intersection of the latter with $(L^{(j)}, \langle\cdot,\cdot\rangle^{(j)}) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \cong (L, \langle\cdot,\cdot\rangle) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ in $(L^{(j)}, \langle\cdot,\cdot\rangle^{(j)}) \otimes_{\mathbb{Z}} \mathbb{A}^\infty \cong (L, \langle\cdot,\cdot\rangle) \otimes_{\mathbb{Z}} \mathbb{A}^\infty$. Due to the *failure of Hasse's principle*, J might have more than one element.)

By [10, Thm. 5.1.1], (A.5.14) extends to a canonical open and closed immersion

$$\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\min} \hookrightarrow \mathsf{M}_{\mathcal{H},\mathbb{C}}^{\min}, \tag{A.5.15}$$

respecting the stratifications on both sides labeled by cusp labels (see Definition A.5.7). Again, $\mathsf{M}_{\mathcal{H},\mathbb{C}}^{\min}$ is the disjoint union of the images of morphisms like (A.5.15), from the minimal compactifications $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{(j),\min}$ of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{(j)}$, for all $j \in J$.

Everything we have proved remains true after replacing the objects defined by $(L, \langle\cdot,\cdot\rangle)$ with those defined by $(L^{(j)}, \langle\cdot,\cdot\rangle^{(j)})$, for each $j \in J$. Thus, in order to show that $U_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ is nonempty, it suffices to note that, by Propositions A.5.3 and A.5.9, the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$-stratum of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{(j),\min}$ meets every connected component of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{(j),\min}$, for all $j \in J$. The proof of Theorem A.5.1 is now complete. □

By Proposition A.3.1, and by the explanations in Section A.4 and in the beginning of this section, the proof of Theorem A.2.2 is also complete. □

A.6. **Extension to cases of ramified characteristics.** In this section, we shall no longer assume that p is a good prime for the integral PEL datum $(\mathcal{O}, \star, L, \langle\cdot,\cdot\rangle, h_0)$, but we shall assume that the image $\mathcal{H}^p$ of $\mathcal{H}$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}^p)$ is *neat*.

Even for such general $\mathcal{H}$ and p, for any collections of lattices stabilized by $\mathcal{H}$ as in [12, Sec. 2], we still have an integral model $\vec{\mathsf{M}}_{\mathcal{H}}$ of $\mathsf{M}_{\mathcal{H}}$ flat over S_0 constructed by "taking normalization" (see [12, Prop. 6.1; see also the introduction]). Moreover, we have an integral model $\vec{\mathsf{M}}_{\mathcal{H}}^{\min}$ of $\mathsf{M}_{\mathcal{H}}^{\min}$ projective and flat over S_0 (see [12, Prop. 6.4]), with a stratification by locally closed subschemes $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ labeled by cusp labels $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$, which extends the stratification of $\mathsf{M}_{\mathcal{H}}$ by the locally closed subschemes $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ (see [12, Thm. 12.1]). For certain (possibly nonsmooth) compatible collections Σ (not the same ones for which we can construct $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ over $\mathsf{M}_{0,\mathbb{Q}}$), we also have the toroidal compactifications $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ of $\vec{\mathsf{M}}_{\mathcal{H}}$ projective and flat over S_0 (see [12, Sec. 7]), with a stratification by locally closed subschemes $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ (see [12, Thm. 9.13]), and with a canonical surjection $\vec{\oint}_{\mathcal{H}}:\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}\to\vec{\mathsf{M}}_{\mathcal{H}}^{\min}$ with geometrically connected fibers (see [12, Lem. 12.9 and its proof]), inducing surjections $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}\to\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ (see [12, Thm. 12.16]).

As in Section A.2, consider a geometric point $s\to\mathsf{S}_0=\mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ with algebraically closed residue field $k(s)$, and consider a connected component $U^{\min}$ of the fiber $\vec{\mathsf{M}}_{\mathcal{H}}^{\min}\underset{\mathsf{S}_0}{\times}s$. For each cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$, we define $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ to be the pullback of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ to $U^{\min}$. Since the fibers of $\vec{\oint}_{\mathcal{H}}$ are geometrically connected, the preimage of $U^{\min}$ under $\vec{\oint}_{\mathcal{H}}\underset{\mathsf{S}_0}{\times}s$ is a connected component U^{tor} of $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}\underset{\mathsf{S}_0}{\times}s$. (In general neither $\vec{\mathsf{M}}_{\mathcal{H}}^{\min}\underset{\mathsf{S}_0}{\times}s$ nor $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}\underset{\mathsf{S}_0}{\times}s$ is normal.) For each equivalence class $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ defining a stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ of $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, we define $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ to be the pullback of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$. Then we also have a canonical surjection $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}\to U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ induced by $\vec{\oint}_{\mathcal{H}}$.

Theorem A.6.1. *With the setting as above, all strata of $U^{\min}$ are nonempty.*

By using the canonical surjection $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}\to U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ (as in the proof of Corollary A.2.3), Theorem A.6.1 implies the following:

Corollary A.6.2. *With the setting as above, all strata of U^{tor} are nonempty.*

As in Section A.3, it suffices to prove the following:

Proposition A.6.3. *Suppose Theorem* A.6.1 *is true when* $\mathrm{char}(k(s))=0$. *Then it is also true when* $\mathrm{char}(k(s))=p>0$.

Remark A.6.4. Since $\vec{\mathsf{M}}_{\mathcal{H}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}\cong\mathsf{M}_{\mathcal{H}}$ and $\vec{\mathsf{M}}_{\mathcal{H}}^{\min}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}\cong\mathsf{M}_{\mathcal{H}}^{\min}$ by construction, by Theorem A.5.1, the assumption in Proposition A.6.3 always holds. Nevertheless, the proof of Proposition A.6.3 will clarify that the deduction of Theorem A.6.1 from Theorem A.5.1 does not require Assumption A.2.1 (cf. Remark A.3.2).

The remainder of this section will be devoted to the proof of Proposition A.6.3. We shall assume that $\mathrm{char}(k(s))=p>0$.

While each $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ is isomorphic to some boundary moduli problem $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, each stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\vec{\mathsf{M}}_{\mathcal{H}}^{\min}$ is similarly isomorphic to some integral model $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ defined by taking normalization (see [12, Prop. 7.4, and Thm. 12.1 and 12.16]). Hence it also makes sense to consider the minimal compactification $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}^{\min}$ of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$, which is proper flat (with possibly non-normal geometric fibers) over S_0, and we obtain the following:

Lemma A.6.5 (cf. Lemma A.3.4 and [5, Thm. 4.17(ii)])**.** *There exists some discrete valuation ring R flat over $\mathcal{O}_{F_0,(p)}$, with fraction field K and residue field $k(s)$, the latter lifting the structural homomorphism $\mathcal{O}_{F_0,(p)} \to k(s)$, such that, for each cusp label $[(\Phi_\mathcal{H}, \delta_\mathcal{H})]$, and for each connected component V of $\vec{\mathsf{Z}}^{\min}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$, the induced flat morphism $V \to \operatorname{Spec}(R)$ has connected special fiber over* $\operatorname{Spec}(k(s))$.

Proof of Proposition A.6.3. By [12, Cor. 12.4], it suffices to show that $U_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]} \neq \emptyset$ when $[(\Phi_\mathcal{H}, \delta_\mathcal{H})]$ is *maximal* with respect to the surjection relations as in [11, Def. 5.4.2.13]. In this case, by [12, Thm. 12.1], $\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ is a closed stratum of $\vec{\mathsf{M}}^{\min}_\mathcal{H}$, and so $\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]} = \vec{\mathsf{Z}}^{\min}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$. Hence the lemma follows from Theorem A.5.1 and the same argument as in the proof of Proposition A.3.1, with the reference to Lemma A.3.4 replaced with an analogous reference to Lemma A.6.5. $\square$

As explained in Remark A.6.4, the proof of Theorem A.6.1 is now complete. $\square$

A.7. **Examples.**

Example A.7.1. Suppose $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a CM field F with maximal totally subfield F^+, with positive involution given by the complex conjugation of F over F^+. Suppose $L = \mathcal{O}_F^{\oplus\, a+b}$, where $a \geq b \geq 0$ are integers. Suppose $(2\pi\sqrt{-1})^{-1}\langle\,\cdot\,,\,\cdot\,\rangle$ is the skew-Hermitian pairing defined in block matrix form $\begin{pmatrix} & & 1_b \\ & S & \\ -1_b & & \end{pmatrix}$ where S is some $(a-b)\times(a-b)$ matrix over F such that $\sqrt{-1}S$ is Hermitian and either positive or negative definite. Then, for each $0 \leq r \leq b$, the $\mathcal{O}$-submodule $\mathtt{Z}^{(r)}_{-2}$ of $L = \mathcal{O}_F^{\oplus(a+b)}$ with the last $a+b-r$ entries zero is totally isotropic, and $\mathtt{V}^{(r)}_{-2} := \mathtt{F}^{(r)}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a totally isotropic F-submodule of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = F^{\oplus(a+b)}$, which is maximal when $r = b$. The stabilizer of $\mathtt{V}^{(r)}_{-2}$ either is the whole group (when $r = 0$) or defines a maximal (proper) parabolic subgroup $\mathrm{P}^{(r)}$ of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (when $r > 0$), and all maximal parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ are conjugate to one of these *standard* ones, by Lemma A.4.3. Similarly, $\mathtt{Z}^{(r)}_{-2} := \mathtt{F}^{(r)}_{-2} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ is a totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$-submodule of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$, and the left $\mathrm{G}(\mathbb{Q})$- and right $\mathcal{H}$- double orbits of $\mathtt{Z}^{(r)}_{-2}$, for $0 \leq r \leq b$, exhaust all the possible $\mathtt{Z}_\mathcal{H}$ appearing in cusp labels $[(\mathtt{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$ for $\mathsf{M}_\mathcal{H}$, by Proposition A.5.8. By Lemma A.4.7, by forgetting their p-parts, their left $\mathrm{G}(\mathbb{Q})$- and right $\mathcal{H}^p$- double orbits also exhaust all possible $\mathtt{Z}_{\mathcal{H}^p}$'s appearing in cusp labels $[(\mathtt{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]$ for $\mathsf{M}_{\mathcal{H}^p}$. Let us say that a cusp label $[(\mathtt{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$ for $\mathsf{M}_\mathcal{H}$ is *of rank* r if $\mathtt{Z}_\mathcal{H}$ is in the double orbit of $\mathtt{Z}^{(r)}_{-2}$, and that a cusp $[(\mathtt{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]$ for $\mathsf{M}_{\mathcal{H}^p}$ is *of rank* r if it is assigned to one of rank r under (A.4.1). (This is consistent with [11, Def. 5.4.1.12 and 5.4.2.7].) On the other hand, as a byproduct of the proof of Proposition A.5.9, any $\mathtt{Z}_\mathcal{H}$ in the double orbit of $\mathtt{Z}^{(r)}_{-2}$ does extend to some cusp label $[(\mathtt{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$ for $\mathsf{M}_\mathcal{H}$, inducing some cusp label $[(\mathtt{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]$ for $\mathsf{M}_{\mathcal{H}^p}$ under (A.4.1). Then Theorem A.2.2 shows that, in the boundary stratification of every connected component of every geometric fiber of $\mathsf{M}^{\min}_{\mathcal{H}^p} \to \mathsf{S}_0 = \operatorname{Spec}(\mathcal{O}_{F_0,(p)})$, there exist nonempty strata labeled by cusp labels for $\mathsf{M}_{\mathcal{H}^p}$ of all possible ranks $0 \leq r \leq b$. (The theorem shows the more refined nonemptiness for strata labeled by cusp labels, not just by ranks.)

The next example shows that we cannot expect Theorem A.2.2 to be true without the requirement (in Assumption A.2.1) that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves no factor of type D.

Example A.7.2. Suppose $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a central division algebra D over a totally real field F as in [11, Prop. 1.2.1.13] such that $D \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathbb{H}$, the real Hamiltonian quaternion algebra, for every embedding $\tau : F \to \mathbb{R}$, with $\star = \diamond$ given by $x \mapsto x^\diamond := \mathrm{Tr}_{D/F}(x) - x$. Suppose that D is nonsplit at strictly more than two places. Suppose L is chosen such that $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong D^{\oplus 2}$. By the Gram–Schmidt process as in [11, Sec. 1.2.4], and by [11, Cor. 1.1.2.6], there is up to isomorphism only one isotropic skew-Hermitian pairing over $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. But we do know the failure of Hasse's principle (see [8, §7, p. 393]) in this case (see [15, Rem. 10.4.6]), which means there exists a choice of $(L, \langle \cdot\,, \cdot \rangle)$ as above that is globally anisotropic but locally isotropic everywhere. Thus, even when $k(s) \cong \mathbb{C}$, there exists some connected component U of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$ and some nonzero cusp label $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$ such that $U_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]} = \emptyset$.

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