ALGEBRAIC TORI AS DEGENERATIONS OF ABELIAN VARIETIES

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Abstract. We first show that every algebraic torus over any field, not necessarily split, can be realized as the special fiber of a semi-abelian scheme whose generic fiber is an absolutely simple abelian variety. Then we investigate which algebraic tori can be thus obtained, when we require the generic fiber of the semi-abelian scheme to carry nontrivial endomorphism structures.

Contents

1. Introduction

We start with a well-known example, the Dwork family of elliptic curves

$$X^3 + Y^3 + Z^3 - 3tXYZ = 0$$

over the projective line $\mathbb{P}^1_{\mathbb{Q}}$ over $\mathbb{Q}$, where t is the affine coordinate, with the identity section given by the point at infinity $[X : Y : Z] = [1 : -1 : 0]$. It also doubles as the modular curve of level 3.

It has bad, semistable reduction at $t = \infty$ and the third roots of unity. The corresponding Néron model has the split torus as the identity component of the

2010 *Mathematics Subject Classification.* Primary 11G10, 14D06; Secondary 11G15, 11G18.
Key words and phrases. tori, degeneration of abelian varieties, polarizations, endomorphisms.
The first author's research is partially supported by the NSF Grant DMS-1352216 and by an Alfred P. Sloan Research Fellowship. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of these organizations.
Please refer to Mathematika 64 (2018), no. 2, pp. 303–329, doi:10.1112/S0025579317000420, for the official version.

fiber at $t = \infty$ (because the equation becomes $XYZ = 0$), but at $t = 1$ we get a nonsplit torus T that sits in the short exact sequence

$$1 \longrightarrow T \longrightarrow \mathrm{Res}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}} \mathbb{G}_{\mathrm{m}} \xrightarrow{\mathrm{Norm}} \mathbb{G}_{\mathrm{m}} \longrightarrow 1 ,$$

where Norm denotes the norm homomorphism of the quadratic extension $\mathbb{Q}(\sqrt{-3})$ over $\mathbb{Q}$. This leads to the following natural question:

Question 1.1. *To which algebraic tori can abelian varieties degenerate?*

However, this question turns out to be a bit naive, in the sense that it allows an answer that is somewhat shallow. Indeed, as we shall explain in the main text with more details, the question can be rather easily reduced to the case of the *split* one-dimensional torus $\mathbb{G}_{\mathrm{m}}$ by finite étale descent, where an affirmative answer is provided by the classical theory of Tate curves. The generic abelian variety obtained this way will be geometrically the n-fold product of an elliptic curve without CM (complex multiplication), where n is the dimension of the torus.

A better and deeper question is the following:

Question 1.2. *To which algebraic tori can* **absolutely simple** *abelian varieties degenerate?*

Here, as usual, an abelian variety is called *absolutely simple* if its base change to an algebraic closure of the base field is not isogenous to a nontrivial product of abelian varieties of smaller dimensions.

For the distinction between Questions 1.1 and 1.2, an apt analogy can be made with complex moduli of abelian varieties. To a simple algebra B with involution of types I–IV and a fixed complex representation of B, one attaches the moduli of polarized abelian varieties with endomorphisms of the given type. One expects, and can prove in most cases, that a very general member in the moduli has an endomorphism algebra no bigger than B; the rough idea is that those abelian varieties with more endomorphisms than prescribed should form a "thin" or nowhere dense subset. However, a careful proof is necessary, as there are exceptional cases where a very general member is not even simple (see [20, Sec. 4, Thm. 5 and Prop. 14, 15, 17, 18, and 19] or [2, Thm. 9.9.1]).

Our main theorem answers Question 1.2, in the affirmative, for all algebraic tori over all base fields:

Theorem 1.3. *Let T be any algebraic torus over any base field k. Then there exists a semi-abelian scheme G over a noetherian normal base scheme S with a k-valued closed point $s \in S(k)$ generalizing to a generic point η such that the following hold:*

(1) the special fiber G_s is isomorphic to T over k; and
(2) the generic fiber G_η is an **absolutely simple** *abelian variety.*

More precisely, we may take S to be of one of the following three kinds of Dedekind schemes:

(a) the spectrum of any complete discrete valuation ring, which can be freely prescribed (independent of T), with residue field k;
(b) a certain connected smooth curve over k (depending on T) with a k-valued point s; and
(c) (when k is a finite field) a certain "arithmetic curve" (depending on T), i.e., an open subset of the spectrum of some number ring, with a closed point s corresponding to an unramified prime ideal with residue field k.

In case T has dimension one, we can exhibit the desired semi-abelian schemes G over Dedekind S for each nonsplit torus T explicitly: Such tori are classified by the separable quadratic extensions of k, and we can write down suitable Weierstraß equations with special fibre T. (The generic absolute simpleness is automatic in dimension one.)

However, in dimensions $n \geq 2$ (already for $n = 2$), we quickly lose explicit control on both sides: The isomorphism classes of n-dimensional tori over k are in one-to-one correspondence with the isomorphism classes of continuous representations

$$\rho : \mathrm{Gal}(k^s/k) \to \mathrm{GL}_n(\mathbb{Z}),$$

where k^s denotes any separable closure of k, on the discrete module $\mathbb{Z}^n$ (corresponding to the Galois actions on the character groups of the tori), and it seems hopeless to classify them completely. On the other side, we do not have a good way of writing down families of general abelian varieties using explicit equations. In order to solve the problem, we use the theory of degeneration of *polarized* semi-abelian schemes (as in [18], [7, Ch. II and III], and [14, Ch. 4]).

After the main theorem, it is natural to ask a similar question with the *additional* condition that the (absolutely simple) generic abelian variety should be equipped with nontrivial endomorphisms. (Such endomorphism structures then necessarily and uniquely extend to the whole semi-abelian scheme; see [19, IX, 1.4], [7, Ch. I, Prop. 2.7], or [14, Prop. 3.3.1.5].) We will also answer this question, with an affirmative answer similar to Theorem 1.3 (except in two special cases).

We note, however, that Question 1.2 and its analogue as in the last paragraph are *not* about the fibers of the tautological semi-abelian schemes carried by the toroidal compactifications of the Siegel moduli stacks or more general PEL moduli problems constructed in [7, Ch. IV] and [14, Ch. 6], because only split tori are used (and needed in the valuative criterion for properness) in the constructions there. Nevertheless, our use of the theory of degeneration is inspired by such constructions.

Here is an outline of the article. In §2, we construct explicit families for one-dimensional tori. In §3, we review some basic facts about the descent data for tori, and give the above-mentioned explanation why Question 1.1 is a bit too naive. To attack Question 1.2 in general, we recall relevant facts from the theory of degeneration in §4. Using this theory, we give the proof of Theorem 1.3 in §5 and §6. Finally, in §7, we turn to the conditions for tori to be realizable as the degeneration of absolutely simple abelian varieties with nontrivial endomorphisms.

2. One-dimensional case

Let T be an algebraic torus of dimension one over a field k. Let us fix the choice of a separable closure k^s of k. Then the action of the Galois group $\mathrm{Gal}(k^s/k)$ on the character group $X^*(T) \cong \mathbb{Z}$ of T is either trivial (in which case T is split), or surjects onto $\{\pm 1\}$, corresponding to a separable quadratic extension $\widetilde{k}$ of k in k^s.

Let R be any Dedekind domain and $\mathfrak{P} = (\pi)$ a principal maximal ideal of R with residue field k. (For example, $R = k[t]$ and $\mathfrak{P} = (t)$.)

Proposition 2.1. *Suppose that* $\mathrm{char}(k) \neq 2$, *and write* $\widetilde{k} = k(\sqrt{b})$, *where* $b \in k$ *is a nonsquare element. Then for any lift* $\mathfrak{b} \in R$ *of* b, *the Weierstraß equation*

$$Y^2 Z = X^3 + \mathfrak{b} X^2 Z + \pi Z^3$$

over Spec(R) *defines an elliptic curve over the fraction field of* R*, has semistable reduction at* $\mathfrak{P}$*, and the fiber of the Néron model at* $\mathfrak{P}$ *is the* k*-torus on whose character group* $\mathrm{Gal}(k^s/k)$ *acts via the quadratic character of* $\widetilde{k}/k$*.*

Proof. The discriminant of the Weierstraß equation (see [21, p. 180]) is given by

$$\Delta = (-16\pi)(4\mathfrak{b}^3 + 27\pi).$$

This is nonzero, and is in fact a uniformizer, in $R_{\mathfrak{P}}$, since 2 is a unit in $R_{\mathfrak{P}}$; hence the first assertion.

Let C denote the singular curve at $\mathfrak{P}$:

$$Y^2Z = X^3 + bX^2Z.$$

The complement of the unique singular point $P = [0 : 0 : 1]$ in C is the fiber of the Néron model. We know that it is a one-dimensional torus, and that it splits over $\widetilde{k}$. Therefore, it is enough to show that it is not split over k.

Let $\pi : \widetilde{C} \to C$ be the blowup of C at P, which induces an isomorphism $C\setminus\{P\} \cong \widetilde{C} \setminus \pi^{-1}(P)$. Because the slopes of the tangent cone

$$y^2 = bx^2$$

to C at P are $\pm\sqrt{b}$, which are not contained in k, the inverse image $\pi^{-1}(P)$ consists of one point of degree 2 on $\widetilde{C}$. If $C\setminus\{P\}$ were split over k, this would not happen. □

Proposition 2.2. *Suppose that* $\mathrm{char}(k) = 2$*, and* $\widetilde{k}$ *be a separable quadratic extension of* k*. Write* $\widetilde{k}$ *as the Artin–Schreier extension* $\widetilde{k} = k(\alpha)$*, where*

$$\alpha^2 + \alpha + b = 0,$$

and $b \in k$*. Then for any lift* $\mathfrak{b} \in R$ *of* b*, the Weierstraß equation*

$$Y^2Z + XYZ = X^3 + \mathfrak{b}X^2Z + \pi Z^3$$

over Spec(R) *defines an elliptic curve over the fraction field of* R*, has semistable reduction at* $\mathfrak{P}$*, and the fiber of the Néron model at* $\mathfrak{P}$ *is the* k*-torus on whose character group* $\mathrm{Gal}(k^s/k)$ *acts via the quadratic character of* $\widetilde{k}/k$*.*

Proof. The proof is parallel. We only note that the discriminant

$$\Delta = -(1 + 4\mathfrak{b})^3\pi - 16 \cdot 27\pi^2$$

is again a uniformizer in $R_{\mathfrak{P}}$, this time because 2 is in $\mathfrak{P}$. □

3. Descent data for tori

In this section, we review some basic facts about the descent data for tori. At the end of this section, we explain that the rather naive Question 1.1 can be easily reduced to the case of $\mathbb{G}_{\mathrm{m}}$, which can then be answered in the affirmative by the classical theory of Tate curves. In fact, the argument will also show that the construction of degenerating abelian varieties can be made functorial (and compatible with base field extensions) in the tori.

We start with the following basic fact:

Lemma 3.1. *Let* R *be a complete noetherian local ring with residue field* k*. Then pulling back from* R *to* k *induces an equivalence from the category of* **isotrivial** *tori over* R *to the category of tori over* k*. (Recall that a torus is isotrivial if it splits over a finite étale cover of its base scheme; see* [6, IX, 1.1]*.)*

Proof. This follows from [6, X, 3.3]. (Indeed, by [6, X, 2.1], all tori over $s = \mathrm{Spec}(k)$ are isotrivial, and so the lemma is a consequence of [6, X, 3.2].) □

Now, suppose R is any complete discrete valuation ring with residue field k and fraction field K. Let $S := \mathrm{Spec}(R)$, $\eta := \mathrm{Spec}(K)$, and $s := \mathrm{Spec}(k)$. We shall denote the pullbacks to η or s by subscripts η or s, respectively.

Suppose T is an isotrivial torus over S. Then there exists a finite étale extension $\widetilde{R}$ of R, which we may and we shall assume to be a complete discrete valuation ring as well, such that $T_{\widetilde{S}} := T \underset{R}{\otimes} \widetilde{R}$ is a split torus over $\widetilde{S} := \mathrm{Spec}(\widetilde{R})$. Let $\widetilde{K}$ and $\widetilde{k}$ denote the fraction and residue fields of $\widetilde{R}$, respectively, and let $\widetilde{\eta} := \mathrm{Spec}(\widetilde{K})$ and $\widetilde{s} := \mathrm{Spec}(\widetilde{k})$. By [8, V, 7, and 4 g)], up to replacing $\widetilde{R}$ with a further finite étale extension, we may and we shall assume that $\widetilde{K}$ is a Galois extension of K. Again, we shall denote the pullbacks to $\widetilde{\eta}$ or $\widetilde{s}$ by subscripts $\widetilde{\eta}$ or $\widetilde{s}$, respectively.

Let $\underline{X}$ denote the character group of T, which is an étale sheaf of free abelian groups of finite ranks over S. Then $\underline{X}$ is the unique étale sheaf lifting $\underline{X}_s$, the character group of T_s. Let n denote the relative dimension of T over S. Then there exists some isomorphism $\xi : \mathbb{Z}^n \xrightarrow{\sim} \underline{X}_{\widetilde{\eta}}$, where we abusively denote by $\mathbb{Z}^n$ the associated constant étale sheaf over $\widetilde{\eta}$, which induces a representation $\rho : \mathrm{Gal}(\widetilde{K}/K) \to \mathrm{GL}_n(\mathbb{Z})$ giving the descent data for the étale sheaf $\underline{X}_{\widetilde{\eta}}$. Since $\widetilde{R}$ is finite étale over R, we have $\mathrm{Gal}(\widetilde{K}/K) \cong \mathrm{Gal}(\widetilde{k}/k)$, and so ρ determines and is determined by a representation $\mathrm{Gal}(\widetilde{k}/k) \to \mathrm{GL}_n(\mathbb{Z})$. Since the automorphism group of any torus is canonically isomorphic to the automorphism group of the associated character group, the representation ρ also defines the data for T_S (resp. T_s) to descend from $T_{\widetilde{S}} \cong \mathbb{G}^n_{\mathrm{m},\widetilde{S}}$ (resp. $T_{\widetilde{s}} \cong \mathbb{G}^n_{\mathrm{m},\widetilde{s}}$). (See, for example, [3, Ch. 6, Sec. 6.2, Ex. B] for an explanation that descent data for schemes with respect to the finite étale base extension $R \to \widetilde{R}$ as above are equivalent to actions of the Galois group $\mathrm{Gal}(\widetilde{K}/K)$ on the schemes.)

Conversely, any torus T_s with character group $\underline{X}_s$ over s canonically lifts to an isotrivial torus T with character group $\underline{X}$ over S, by Lemma 3.1. If T_s is of dimension n, then we have trivializations $\xi : \mathbb{Z}^n \xrightarrow{\sim} \underline{X}_{\widetilde{\eta}}$ and representations $\rho : \mathrm{Gal}(\widetilde{K}/K) \cong \mathrm{Gal}(\widetilde{k}/k) \to \mathrm{GL}_n(\mathbb{Z})$ as above, for suitable choices of $\widetilde{S}$, etc.

Proposition 3.2. *Suppose G_0 is a semi-abelian scheme over S with special fiber $G_{0,s} \cong \mathbb{G}_{\mathrm{m},s}$ and with generic fiber $G_{0,\eta}$ an elliptic curve. Suppose T_s is a torus over s, and suppose $\widetilde{S} = \mathrm{Spec}(\widetilde{R})$, etc, are as above. Then there exists a semi-abelian scheme G over S such that $G_s \cong T_s$, such that G_η is an abelian variety, and such that $G_{\widetilde{\eta}} \cong G^n_{0,\widetilde{\eta}}$. In particular, Question* 1.1 *has an affirmative answer. Moreover, for a fixed choice of G_0, the assignment of G to T_s is functorial* (*and compatible with base field extensions*) *in the torus T_s.*

Proof. By [19, XI, 1.13] (see also [14, Rem. 3.3.3.9]), any semi-abelian scheme over the noetherian normal base scheme S is equipped with some relative ample invertible sheaf over S. By the theory of fpqc descent (as in [8, VIII, 7.8]), all descent data for such quasi-projective semi-abelian schemes are effective. Since the representation $\rho : \mathrm{Gal}(\widetilde{K}/K) \cong \mathrm{Gal}(\widetilde{k}/k) \to \mathrm{GL}_n(\mathbb{Z})$ defining the data for T_S to descend from $T_{\widetilde{S}} \cong \mathbb{G}^n_{\mathrm{m},\widetilde{S}}$ also defines some descent data for $G^n_{0,\widetilde{S}}$, which are effective as we have just explained, $G^n_{0,\widetilde{S}}$ descends to a uniquely determined semi-abelian scheme G over S, whose generic fiber G_η is an abelian scheme because

$G_{\widetilde{\eta}} \cong G^n_{0,\widetilde{\eta}}$ is; and whose special fiber G_s is canonically isomorphic to T_s because their descent data are both given by ρ. Such an assignment of G to T_s is functorial (and compatible with base field extensions) in the torus T_s because the construction of G by descent depends only on the descent data for T_s and on the choice of G_0. □

Thus, we have reduced the more naive Question 1.1 to the case of the one-dimensional split torus $\mathbb{G}_{\mathrm{m}}$ (over any base field), which can then be answered by the classical theory of Tate curves (see, for example, the rather universal construction over $\mathbb{Z}[[q]]$ in [5, Ch. VII]). However, any G given by Proposition 3.2 (or any assignment that is functorial and compatible with base field extensions in T_s) must satisfy $G_{\widetilde{\eta}} \cong G^n_{0,\widetilde{\eta}}$ over some extension $\widetilde{\eta}$ of η, which cannot be absolutely simple when $n > 1$, and hence cannot be used to answer Question 1.2. We need a theory more general than that of Tate curves, which we shall review in §4 below.

4. Theory of degeneration

Let R be a complete discrete valuation ring with residue field k and fraction field K. Let $S := \mathrm{Spec}(R)$, $\eta := \mathrm{Spec}(K)$, and $s := \mathrm{Spec}(k)$. We shall denote the pullbacks to η or s by subscripts η or s, respectively.

Definition 4.1 (cf. [14, Def. 4.4.2])**.** *With the setting as above, the category* $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{pol}}(R)$ *has objects consisting of pairs* (G, λ) *over* $S = \mathrm{Spec}(R)$*, where:*

(1) G *is a semi-abelian scheme over* S*;*
(2) G_η *is an abelian scheme over* η*, in which case there is a unique semi-abelian scheme* $G^\vee$ *(up to unique isomorphism) over* S*, called the dual semi-abelian scheme of* G*, such that* $G^\vee_\eta$ *is the dual abelian scheme of* G_η *(see* [14, Sec. 3.4.3]*);*
(3) G_s *is a torus over* s *(in which case* $G^\vee_s$ *also is); and*
(4) $\lambda : G \to G^\vee$ *is a group homomorphism that induces by restriction a polarization* λ_η *of* G_η*.*

Remark 4.2. The definition of pairs (G, λ) as in Definition 4.1 extends verbatim to the case where S is a noetherian normal local scheme, but we stated the definition this way only because the theory of degeneration below requires R to be complete, and we only need the special case over Dedekind domains.

Remark 4.3. By [19, XI, 1.13, and IX, 1.4] (see also [14, Rem. 3.3.3.9 and 4.4.3, and Prop. 3.3.1.5]), any semi-abelian scheme G over a noetherian normal domain R that is generically an abelian variety is automatically equipped with a homomorphism $\lambda : G \to G^\vee$ that is generically a polarization (of an abelian variety). In particular, any semi-abelian scheme G over S satisfying the first two conditions in Definition 4.1 is automatically equipped with some homomorphism $\lambda : G \to G^\vee$ satisfying the last condition in Definition 4.1.

By the theory of degeneration data (see [18]; see also [7, Ch. II and III] and [14, Ch. 4; see, in particular, Cor. 4.5.4.31], with all *abelian parts* in the degenerations suppressed; or see, for example, [16, Sec. 2.7], for the corresponding rigid analytic theory, which is in some sense simpler), there is an equivalence of categories

$$\mathrm{M}^{\mathrm{tor}}_{\mathrm{pol}}(R) :\ \mathrm{DD}^{\mathrm{tor}}_{\mathrm{pol}}(R) \to \mathrm{DEG}^{\mathrm{tor}}_{\mathrm{pol}}(R) : (\underline{X}, \underline{Y}, \phi, \tau) \mapsto (G, \lambda)$$

realizing (G, λ) (up to isomorphism) as the image of an object in $\mathrm{DD}^{\mathrm{tor}}_{\mathrm{pol}}(R)$ given by the following data:

(1) Two étale sheaves $\underline{X}$ and $\underline{Y}$ of free abelian groups of finite ranks over S, together with an embedding $\phi : \underline{Y} \to \underline{X}$ with finite cokernel. (Here $\phi : \underline{Y} \to \underline{X}$ corresponds to the isogeny of tori $T \to T^\vee$ uniquely lifting $G_s \to G_s^\vee$, by Lemma 3.1. Beware that the notation for objects related to the degeneration of $G_\eta^\vee$, such as $T^\vee$, is only symbolic in nature.)
(2) A bimultiplicative homomorphism $\tau : \underline{Y}_\eta \times \underline{X}_\eta \to \mathbb{G}_{\mathrm{m},\eta}$ of étale sheaves over η with *symmetric pullback* under $\mathrm{Id}_{\underline{Y}} \times \phi : \underline{Y} \times \underline{Y} \to \underline{Y} \times \underline{X}$, satisfying the following *positivity condition* for some (and hence every) finite extension $\widetilde{K}$ of K as in §3 over which both $\underline{X}_{\widetilde{\eta}}$ and $\underline{Y}_{\widetilde{\eta}}$, where $\widetilde{\eta} := \mathrm{Spec}(\widetilde{K})$, are constant: We have $\widetilde{v}(\tau(y, \phi(y))) > 0$ for every nonzero section y of $\underline{Y}_{\widetilde{\eta}}$, where $\widetilde{v} : \widetilde{K}^\times \to \mathbb{Z}$ is any nontrivial discrete valuation of $\widetilde{K}$.

We say that $(\underline{X}, \underline{Y}, \phi, \tau)$ is the *degeneration data* of (G, λ).

Remark 4.4. The bimultiplicative homomorphism $\tau : \underline{Y}_\eta \times \underline{X}_\eta \to \mathbb{G}_{\mathrm{m},\eta}$ induces (by the definition of $\underline{X}$ as the character group of T) a homomorphism

$$\iota : \underline{Y}_\eta \to T_\eta, \tag{4.5}$$

which is a generalization of the familiar Tate periods in the case of relative dimension one (i.e., of Tate curves). Nevertheless, we will not need ι in what follows.

5. Conditions for degenerations

Let R, k, K, S, s, and η be as in the beginning of §4.

Suppose that we have tori T_s and $T_s^\vee$ over $s = \mathrm{Spec}(k)$ with character groups $\underline{X}_s$ and $\underline{Y}_s$ that are étale sheaves of free abelian groups of finite ranks, together with an embedding $\phi_s : \underline{Y}_s \hookrightarrow \underline{X}_s$ with finite cokernel inducing an isogeny $\lambda_{T_s} : T_s \twoheadrightarrow T_s^\vee$. (Certainly, we allow ϕ_s and hence λ_{T_s} to be isomorphisms.) In this section, we shall find necessary and sufficient conditions for the existence of an object (G, λ) in $\mathrm{DEG}_{\mathrm{pol}}^{\mathrm{tor}}(R)$ such that $\lambda_s : G_s \to G_s^\vee$ can be identified with $\lambda_{T_s} : T_s \to T_s^\vee$ (via some isomorphisms $G_s \cong T_s$ and $G_s^\vee \cong T_s^\vee$ over s), and such that G_η and hence $G_\eta^\vee$ are *absolutely simple* abelian varieties.

By Lemma 3.1, the embedding $\phi_s : \underline{Y}_s \hookrightarrow \underline{X}_s$ lifts to an embedding $\phi : \underline{Y} \hookrightarrow \underline{X}$ with finite cokernel over S of étale sheaves of free abelian groups of finite ranks, inducing an isogeny $\lambda_T : T \twoheadrightarrow T^\vee$ (between isotrivial tori) lifting λ_{T_s}. Let $\widetilde{S}$, etc, be as in §3 such that both $\underline{X}_{\widetilde{\eta}}$ and $\underline{Y}_{\widetilde{\eta}}$ are constant of some common finite rank n, so that there are isomorphisms $\xi : \mathbb{Z}^n \xrightarrow{\sim} \underline{X}_{\widetilde{\eta}}$ and $\xi^\vee : \mathbb{Z}^n \xrightarrow{\sim} \underline{Y}_{\widetilde{\eta}}$ and representations $\rho : \mathrm{Gal}(\widetilde{K}/K) \to \mathrm{GL}_n(\mathbb{Z})$ and $\rho^\vee : \mathrm{Gal}(\widetilde{K}/K) \to \mathrm{GL}_n(\mathbb{Z})$ defining the descent data for the étale sheaves $\underline{X}_{\widetilde{\eta}}$ and $\underline{Y}_{\widetilde{\eta}}$. Consider

$$\phi_{\xi,\xi^\vee} := \xi^{-1}\, \phi_{\widetilde{\eta}}\, \xi^\vee : \mathbb{Z}^n \to \mathbb{Z}^n, \tag{5.1}$$

which defines an element of $\mathrm{End}(\mathbb{Z}^n) \cap \mathrm{Aut}(\mathbb{Q}^n) \cong \mathrm{M}_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$.

Lemma 5.2. *We have*

$$\phi_{\xi,\xi^\vee}(\rho^\vee(\gamma)z) = \rho(\gamma)(\phi_{\xi,\xi^\vee}(z))$$

for all $\gamma \in \mathrm{Gal}(\widetilde{K}/K)$ *and* $z \in \mathbb{Z}^n$.

Proof. This is because $\phi : \underline{Y} \to \underline{X}$ is a morphism defined over S. □

Proposition 5.3. *With the setting as above, with the fixed choices of some* ξ, $\xi^\vee$, ρ, *and* $\rho^\vee$, *the datum of an object* (G, λ) *of* $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{pol}}(R)$ *(as in Definition* 4.1*) such that* $\lambda_s : G_s \to G_s^\vee$ *can be identified with* $\lambda_{T_s} : T_s \to T_s^\vee$ *(via some isomorphisms* $G_s \cong T_s$ *and* $G_s^\vee \cong T_s^\vee$ *over* s*) is equivalent to the datum of a bimultiplicative pairing*

$$\langle \cdot\,,\,\cdot \rangle_{\tau,\xi,\xi^\vee} : \mathbb{Z}^n \times \mathbb{Z}^n \to \widetilde{K}^\times \tag{5.4}$$

satisfying the following conditions:

1. (**Galois equivariance**:) $\langle \rho^\vee(\gamma)z, \rho(\gamma)w\rangle_{\tau,\xi,\xi^\vee} = \gamma\langle z, w\rangle_{\tau,\xi,\xi^\vee}$ *for all* $z, w \in \mathbb{Z}^n$ *and* $\gamma \in \mathrm{Gal}(\widetilde{K}/K)$. *(Note that the action of* $\mathrm{Gal}(\widetilde{K}/K)$ *on* $\widetilde{K}^\times$ *here is the naive one, because* $\widetilde{K}^\times$ *is just the group of points of* $\mathbb{G}_{\mathrm{m}}$ *over* $\widetilde{\eta} = \mathrm{Spec}(\widetilde{K})$, *and* $\mathbb{G}_{\mathrm{m}}$ *is already defined over* $\eta = \mathrm{Spec}(K)$ *or rather* $\mathrm{Spec}(\mathbb{Z})$.*)*
2. (**Symmetry**:) $\langle z, \phi_{\xi,\xi^\vee}(w)\rangle_{\tau,\xi,\xi^\vee} = \langle w, \phi_{\xi,\xi^\vee}(z)\rangle_{\tau,\xi,\xi^\vee}$ *for all* $z, w \in \mathbb{Z}^n$.
3. (**Positivity**:) $\widetilde{v}(\langle z, \phi_{\xi,\xi^\vee}(z)\rangle_{\tau,\xi,\xi^\vee}) > 0$ *for every nonzero* $z \in \mathbb{Z}^n$, *where* $\widetilde{v} : \widetilde{K}^\times \to \mathbb{Z}$ *is any nontrivial discrete valuation of* $\widetilde{K}$.

Proof. This follows from the theory reviewed in §4, by base change to $\widetilde{\eta}$, by pulling back under $\xi^\vee \times \xi$, and by descent (as in the proof of Proposition 3.2). □

Proposition 5.5. *In the setting of Proposition* 5.3*, if the pairing* $\langle \cdot\,,\,\cdot \rangle_{\tau,\xi,\xi^\vee}$ *corresponding to* (G, λ) *satisfies the additional condition that*

$$\langle z, w\rangle_{\tau,\xi,\xi^\vee} \neq 1 \tag{5.6}$$

for all nonzero $z, w \in \mathbb{Z}^n$, *then* G_η *and hence* $G_\eta^\vee$ *must be absolutely simple.*

Proof. Suppose G_η is not absolutely simple. Then there exists a finite extension K' of $\widetilde{K}$ over which the base change of G_η is isogenous to a nontrivial product of two abelian varieties of smaller dimension. By the theory of Néron models (see, in particular, [3, Ch. 7, Sec. 7.4, Thm. 1]), up to replacing K' with a finite extension, we may assume that these two abelian varieties extend to two semi-abelian schemes G_1 and G_2, respectively, over the integral closure R' of R in K'. Since the theory of degeneration in §4 is an equivalence of categories, there exist some nonzero $z, w \in \mathbb{Z}^n$, which are the images of some elements of the character groups of the torus parts of $G_1^\vee$ and G_2, such that $\langle z, w\rangle_{\tau,\xi,\xi^\vee} = 1$ (in K', and hence also in $\widetilde{K}$). This contradicts the condition (5.6) in this proposition, as desired. □

In the remainder of this section, let us show that the conditions in Propositions 5.3 and 5.5 can indeed be achieved by some bimultiplicative pairing $\langle \cdot\,,\,\cdot \rangle_{\tau,\xi,\xi^\vee}$.

Lemma 5.7. *There exists* $\theta \in \widetilde{R}^\times$ *such that* $\{\gamma\theta\}_{\gamma\in\mathrm{Gal}(\widetilde{K}/K)}$ *form a free* R*-basis of* $\widetilde{R}$, *and such that the reductions* $\{\gamma\overline{\theta}\}_{\gamma\in\mathrm{Gal}(\widetilde{K}/K)}$ *in* $\widetilde{k}$ *form a* k*-basis of* $\widetilde{k}$.

Proof. Since $\mathrm{Gal}(\widetilde{K}/K)$ is a finite group, the group algebra $R[\mathrm{Gal}(\widetilde{K}/K)]$ over R is an R-order (see [14, Def. 1.1.1.3]) in the finite-dimensional group algebra $K[\mathrm{Gal}(\widetilde{K}/K)]$ over K. Since $\widetilde{R}$ is finite étale over R, we have $\mathrm{Gal}(\widetilde{K}/K) \cong \mathrm{Gal}(\widetilde{k}/k)$ as groups, and $R[\mathrm{Gal}(\widetilde{K}/K)] \underset{R}{\otimes} k \cong k[\mathrm{Gal}(\widetilde{k}/k)]$ as group algebras over k. By applying the normal basis theorem (see, for example, [15, Ch. VI, Sec. 13]) to the finite Galois field extension $\widetilde{k}/k$, we have $\widetilde{k} \cong k[\mathrm{Gal}(\widetilde{k}/k)]$ as left $k[\mathrm{Gal}(\widetilde{k}/k)]$-modules. Therefore, since R is local with residue field k, by [14, Lem. 1.1.3.1] (which is a consequence of the usual Nakayama's lemma for finitely generated R-modules), we have $\widetilde{R} \cong R[\mathrm{Gal}(\widetilde{K}/K)]$ as left $R[\mathrm{Gal}(\widetilde{K}/K)]$-modules, and the lemma follows. □

Lemma 5.8. *There exists $u_0 \in \widetilde{R}^\times$ such that the Galois conjugates $\{\gamma u_0\}_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)}$ are* **multiplicatively independent** *in the sense that, if*

$$\prod_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} (\gamma u_0)^{c_\gamma} = 1$$

in $\widetilde{R}^\times$ for some integers $\{c_\gamma\}_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)}$, then these integers are all zero.

Proof. Let I and $\widetilde{I}$ denote the maximal ideals of R and $\widetilde{R}$, respectively, which satisfy $\widetilde{I} = I \cdot \widetilde{R}$ because $\widetilde{R}$ is finite étale over R. If $\mathrm{char}(k) = 0$, we set $\delta = 0$. Otherwise, we have $\mathrm{char}(k) = p$ for some rational prime number $p > 0$, and we fix the choice of any integer $\delta \geq 0$ such that the multiplicative subgroup $1 + \widetilde{I}^{1+\delta}$ of $\widetilde{R}^\times$ does not contain any nontrivial p-th roots of unity. (Such an integer δ always exists, because there are only finitely many p-th root of unity in $\widetilde{K}$ or any field, and because $\widetilde{R}$ is $\widetilde{I}$-adically separated.)

Let $\theta \in \widetilde{R}^\times$ be as in Lemma 5.7, let ϖ be any uniformizer of R (which is then also a uniformizer of $\widetilde{R}$), let δ be as in the previous paragraph, and let

$$u_0 := 1 + \theta \varpi^{1+\delta} \in 1 + \widetilde{I}^{1+\delta} \subset \widetilde{R}^\times.$$

We would like to show that u_0 satisfies the requirement of this lemma; namely, if

$$\prod_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} (\gamma u_0)^{c_\gamma} = \prod_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} (1 + (\gamma\theta)\varpi^{1+\delta})^{c_\gamma} = 1 \tag{5.9}$$

in $1 + \widetilde{I}^{1+\delta} \subset \widetilde{R}^\times$ for some integers $\{c_\gamma\}_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)}$, then these integers are all zero.

Suppose, to the contrary, that the relation (5.9) holds for some integers $\{c_\gamma\}_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)}$ that are not all zero. First, suppose $\mathrm{char}(k) = 0$. Then $\mathbb{Q} \subset k$, and the relation (5.9) implies that

$$\sum_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} c_\gamma (\gamma \bar{\theta}) = 0$$

in k, by considering the images of the terms in (5.9) in $(1 + \widetilde{I}^{1+\delta})/(1 + \widetilde{I}^{2+\delta}) \cong k$ (where $\delta = 0$). This contradicts the linear independence of $\{\gamma\bar{\theta}\}_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)}$ over k. Otherwise, suppose $\mathrm{char}(k) = p > 0$. Then there exists some power q of p such that $q | c_\gamma$ for all $\gamma \in \mathrm{Gal}(\widetilde{K}/K)$, but $(pq) \nmid c_{\gamma_0}$ for at least one $\gamma_0 \in \mathrm{Gal}(\widetilde{K}/K)$, so that the reductions $\{\bar{c}_\gamma := (c_\gamma/q) \bmod p\}$ are not all zero in $\mathbb{F}_p \subset k$. By the choice of δ, the multiplicative subgroup $1 + \widetilde{I}^{1+\delta}$ of $\widetilde{R}^\times$ cannot contain any nontrivial p-th roots of unity, and consequently the relation (5.9) still holds with the c_γ replaced with c_γ/q. After such a replacement, again by considering the images of the terms in (5.9) in $(1 + \widetilde{I}^{1+\delta})/(1 + \widetilde{I}^{2+\delta}) \cong k$, we get that

$$\sum_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} \bar{c}_\gamma (\gamma \bar{\theta}) = 0$$

in k. This again contradicts the linear independence of $\{\gamma\bar{\theta}\}_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)}$ over k. Thus, the integers $\{c_\gamma\}_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)}$ must be all zero, and the lemma follows. □

Lemma 5.10. *For any integer $N \geq 1$, there exist elements $u_1, u_2, \ldots, u_N$ in $\widetilde{R}^\times$ that are* **multiplicatively Galois independent** *in the sense that, if*

$$\prod_{1 \leq i \leq N; \gamma \in \mathrm{Gal}(\widetilde{K}/K)} (\gamma u_i)^{c_{i,\gamma}} = 1$$

in $\widetilde{R}^\times$ for some integers $\{c_{i,\gamma}\}_{1 \leq i \leq N; \gamma \in \mathrm{Gal}(\widetilde{K}/K)}$, then these integers are all zero.

Proof. Let $u_0 \in \widetilde{R}^\times$ be as in Lemma 5.8, and let ϖ be a uniformizer of R (which does not have to be the same as the uniformizer in the proof of Lemma 5.8). Let us choose elements $t_1, t_2, \ldots, t_N$ in $K^\times$ that are algebraically independent over the subfield $\widetilde{K}_0$ of $\widetilde{K}$ generated by ϖ and the finite subset $\{\gamma u_0\}_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)}$ over its prime subfield. This is possible because the transcendence degree of $\widetilde{K}$ over its prime subfield is (uncountably) infinite. For each $1 \leq i \leq N$, up to replacing t_i with a suitable multiple by a power of ϖ, we may assume that $t_i \in R^\times$. Let

$$u_i := 1 + u_0 t_i \varpi \in \widetilde{R}^\times,$$

so that

$$\gamma u_i = 1 + (\gamma u_0) t_i \varpi \in \widetilde{R}^\times,$$

for each $1 \leq i \leq N$. We would like to show that these elements $\{u_i\}_{1 \leq i \leq N}$ satisfy the requirement of the lemma; namely, if

$$\prod_{1 \leq i \leq N; \gamma \in \mathrm{Gal}(\widetilde{K}/K)} (\gamma u_i)^{c_{i,\gamma}} = \prod_{1 \leq i \leq N; \gamma \in \mathrm{Gal}(\widetilde{K}/K)} (1 + (\gamma u_0) t_i \varpi)^{c_{i,\gamma}} = 1 \tag{5.11}$$

for some integers $\{c_{i,\gamma}\}_{1 \leq i \leq N; \gamma \in \mathrm{Gal}(\widetilde{K}/K)}$, then these integers are all zero.

For all i and γ, set $c_{i,\gamma}^+ = c_{i,\gamma}$ and $c_{i,\gamma}^- = 0$ when $c_{i,\gamma} \geq 0$, and set $c_{i,\gamma}^+ = 0$ and $c_{i,\gamma}^- = -c_{i,\gamma}$ when $c_{i,\gamma} \leq 0$. Then the relation (5.11) implies that

$$\prod_{1 \leq i \leq N; \gamma \in \mathrm{Gal}(\widetilde{K}/K)} (1 + (\gamma u_0) t_i \varpi)^{c_{i,\gamma}^+} = \prod_{1 \leq i \leq N; \gamma \in \mathrm{Gal}(\widetilde{K}/K)} (1 + (\gamma u_0) t_i \varpi)^{c_{i,\gamma}^-}. \tag{5.12}$$

Since $t_1, t_2, \ldots, t_N$ are algebraically independent over the subfield $\widetilde{K}_0$ of $\widetilde{K}$, the relation (5.12) is possible only when the corresponding "polynomials" in $t_1, t_2, \ldots, t_N$ (with coefficients in $\widetilde{K}_0$) on the two sides match. In particular, for each $1 \leq i \leq N$, by comparing the nonzero top degree "monomials" purely in t_i (with coefficients in $\widetilde{K}_0$) on the two sides, we have (by comparing exponents)

$$\sum_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} c_{i,\gamma}^+ = \sum_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} c_{i,\gamma}^-$$

in $\mathbb{Z}_{\geq 0}$, and have (by comparing coefficients, after cancelling powers of ϖ)

$$\prod_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} (\gamma u_0)^{c_{i,\gamma}^+} = \prod_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} (\gamma u_0)^{c_{i,\gamma}^-} \tag{5.13}$$

in $\widetilde{R}^\times$. By the assumption on u_0 (satisfying the requirement in Lemma 5.8), and by the definition of $c_{i,\gamma}^+$ and $c_{i,\gamma}^-$, this last relation (5.13) forces all $c_{i,\gamma}^+$, $c_{i,\gamma}^-$, and $c_{i,\gamma}$ to be zero, for all $\gamma \in \mathrm{Gal}(\widetilde{K}/K)$, as desired. □

Proposition 5.14. *There exists a bimultiplicative pairing*

$$\langle \cdot\,, \cdot \rangle_{\tau,\xi,\xi^\vee} : \mathbb{Z}^n \times \mathbb{Z}^n \to \widetilde{K}^\times$$

as in (5.4) *satisfying all the conditions in Propositions* 5.3 *and* 5.5.

Proof. Let

$$\langle \cdot\,, \cdot \rangle_0 : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$$

be any positive definite (symmetric) bilinear pairing. Let $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{Z}^n$, let ϖ be any element of $K^\times$ of positive valuation, and let $\{u_{ij}\}_{1\leq i\leq j\leq n}$ be elements in $\widetilde{R}^\times$ that are multiplicatively Galois independent as in Lemma 5.10. Then we define a symmetric bimultiplicative pairing

$$\langle \cdot\,, \cdot \rangle_1 : \mathbb{Z}^n \times \mathbb{Z}^n \to \widetilde{K}^\times$$

by setting

$$\langle e_i, e_j \rangle_1 := u_{ij} \cdot \varpi^{\langle e_i, e_j \rangle_0}$$

for all $1 \leq i \leq j \leq n$, and by extending the values of the pairing to the whole domain $\mathbb{Z}^n \times \mathbb{Z}^n$ by symmetry and bimultiplicativity. Next, we define a symmetric bimultiplicative pairing

$$\langle \cdot\,, \cdot \rangle_2 : \mathbb{Z}^n \times \mathbb{Z}^n \to \widetilde{K}^\times$$

satisfying the Galois equivariance

$$\langle \rho(\gamma) z, \rho(\gamma) w \rangle_2 = \gamma \langle z, w \rangle_2 \tag{5.15}$$

for all $\gamma \in \mathrm{Gal}(\widetilde{K}/K)$ and $z, w \in \mathbb{Z}^n$, by setting

$$\langle z, w \rangle_2 := \prod_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} \gamma^{-1} \langle \rho(\gamma) z, \rho(\gamma) w \rangle_1 \tag{5.16}$$

for all $z, w \in \mathbb{Z}^n$. Finally, we define the desired bimultiplicative pairing

$$\langle \cdot\,, \cdot \rangle_{\tau,\xi,\xi^\vee} : \mathbb{Z}^n \times \mathbb{Z}^n \to \widetilde{K}^\times$$

by setting

$$\langle z, w \rangle_{\tau,\xi,\xi^\vee} := \langle \phi_{\xi,\xi^\vee}(z), w \rangle_2 \tag{5.17}$$

for all $z, w \in \mathbb{Z}^n$, where $\phi_{\xi,\xi^\vee}$ is as in (5.1). Then $\langle \cdot\,, \cdot \rangle_{\tau,\xi,\xi^\vee}$ satisfies the three conditions (1), (2), and (3) in Proposition 5.3 by the symmetry and positive definiteness of $\langle \cdot\,, \cdot \rangle_0$; by the definitions of the pairings $\langle \cdot\,, \cdot \rangle_1$ and $\langle \cdot\,, \cdot \rangle_2$; by the choices of ϖ (of positive valuation in $\widetilde{K}^\times$) and $\{u_{ij}\}_{1\leq i\leq j\leq n}$ (of zero valuation in $\widetilde{K}^\times$); and by Lemma 5.2 and the relations (5.15) and (5.17).

It remains to show that $\langle z, w \rangle_{\tau,\xi,\xi^\vee} \neq 1$ (as in (5.6)) for all nonzero $z, w \in \mathbb{Z}^n$. Since $\phi_{\xi,\xi^\vee}$ is an embedding, by the defining relation (5.17), it suffices to show that $\langle z, w \rangle_2 \neq 1$ for all nonzero $z, w \in \mathbb{Z}^n$. By the choice of $\{u_{ij}\}_{1\leq i\leq j\leq n}$, the terms in the product (5.16) indexed by different elements γ have multiplicatively independent values (up to powers of ϖ), and so it suffices to show that $\langle z, w \rangle_1 \neq 1$ for all nonzero $z, w \in \mathbb{Z}^n$. Suppose, to the contrary, that there are some $z = \sum_{1\leq i\leq n} a_i e_i$ and $w = \sum_{1\leq i\leq n} b_i e_i$, where $(a_i)_{1\leq i\leq n}$ and $(b_i)_{1\leq i\leq n}$ are nonzero n-tuples of integers, such that $\langle z, w \rangle_1 = 1$. Then we have

$$\left(\prod_{1\leq i\leq n} \langle e_i, e_i \rangle_1^{a_i b_i} \right) \cdot \left(\prod_{1\leq i<j\leq n} \langle e_i, e_j \rangle_1^{a_i b_j + b_i a_j} \right) = 1$$

in $\widetilde{K}^\times$, which implies that

$$\Big(\prod_{1\leq i\leq n} u_{ii}^{a_ib_i}\Big)\cdot\Big(\prod_{1\leq i<j\leq n} u_{ij}^{a_ib_j+b_ia_j}\Big)=1 \tag{5.18}$$

in $\widetilde{R}^\times$ (by pulling out all powers of ϖ). By the choices of $\{u_{ij}\}_{1\leq i\leq j\leq n}$ again, the identity (5.18) is possible only when

$$a_ib_j+b_ia_j=0 \tag{5.19}$$

in $\mathbb{Z}$ for all $1\leq i\leq j\leq n$. Let i_0 (resp. j_0) be the smallest index i (resp. j) such that $a_i\neq 0$ (resp. $b_j\neq 0$), which exists by assumption. But then $a_{i_0}b_{j_0}+a_{j_0}b_{i_0}\neq 0$. This contradicts the condition (5.19), as desired. □

Remark 5.20. If we start with any element ϖ in $K^\times$ of positive valuation and any positive definite (symmetric) bilinear pairing $\langle\,\cdot\,,\,\cdot\,\rangle_0:\mathbb{Z}^n\times\mathbb{Z}^n\to\mathbb{Z}$, and consider the Galois invariant pairing $\langle\,\cdot\,,\,\cdot\,\rangle_1:\mathbb{Z}^n\times\mathbb{Z}^n\to\mathbb{Z}$ defined by the sum

$$\langle z,w\rangle_1:=\sum_{\gamma\in\mathrm{Gal}(\widetilde{K}/K)}\langle\rho(\gamma)z,\rho(\gamma)w\rangle_0,$$

then the bimultiplicative pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\xi,\xi^\vee}:\mathbb{Z}^n\times\mathbb{Z}^n\to\widetilde{K}^\times$ defined by

$$\langle z,w\rangle_{\tau,\xi,\xi^\vee}:=\varpi^{\langle\phi_{\xi,\xi^\vee}(z),w\rangle_1}\in K^\times$$

satisfies all the conditions in Proposition 5.3. However, by completion of squares, there exists $f\in\mathrm{GL}_n(\mathbb{Q})$ such that $(f\times f)^*\circ(\langle\,\cdot\,,\,\cdot\,\rangle_1\underset{\mathbb{Z}}{\otimes}\mathbb{Q}):\mathbb{Q}^n\times\mathbb{Q}^n\to\mathbb{Q}$ decomposes as an orthogonal direct sum of pairings over one-dimensional subspaces. Thus, when $n>1$, no pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\xi,\xi^\vee}$ defined as above can possibly satisfy the condition in Proposition 5.5. Furthermore, by Proposition 5.3, and by the functoriality in the theory of degeneration in §4, some nonzero multiple of f defines an isogeny between $G_{\widetilde{\eta}}$ and a product of one-dimensional abelian varieties (i.e., elliptic curves) over $\widetilde{\eta}$. In particular, when $n>1$, no G_η thus obtained can be absolutely simple.

Remark 5.21. If we consider only $K^\times$-valued pairings

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\xi,\xi^\vee}:\mathbb{Z}^n\times\mathbb{Z}^n\to K^\times$$

satisfying all the conditions in Proposition 5.3, then the Galois equivariance condition (1) there becomes the simpler Galois invariance condition

$$\langle\rho^\vee(\gamma)z,\rho(\gamma)w\rangle_{\tau,\xi,\xi^\vee}=\langle z,w\rangle_{\tau,\xi,\xi^\vee} \tag{5.22}$$

for all $z,w\in\mathbb{Z}^n$ and $\gamma\in\mathrm{Gal}(\widetilde{K}/K)$. Since $\mathrm{char}(\mathbb{Q})=0$, and since $\mathrm{Gal}(\widetilde{K}/K)$ is finite, the (finite-dimensional) $\mathbb{Q}$-subalgebra B of $\mathrm{End}(\mathbb{Q}^n)$ generated by the images $\{\rho(\gamma)\}_{\gamma\in\mathrm{Gal}(\widetilde{K}/K)}$ is *semisimple* (see, for example, [4, Thm. 15.6 and (25.8)]), and the $\mathbb{Z}$-subalgebra $\mathcal{O}$ of $\mathrm{End}(\mathbb{Z}^n)$ generated by the same images is an order in B. The anti-automorphism $\gamma\mapsto\gamma^{-1}$ of $\mathrm{Gal}(\widetilde{K}/K)$ induces an involution $\star:B\to B$ stabilizing $\mathcal{O}$, which is *positive* in the sense that $\mathrm{Tr}_{B/\mathbb{Q}}(bb^\star)>0$ for all nonzero $b\in B$. (Such orders and algebras with positive involutions are exactly the ones considered in the context of endomorphisms of abelian varieties; see §7 below.) Then the above condition (5.22) can be rewritten as

$$\langle bz,w\rangle_{\tau,\xi,\xi^\vee}=\langle z,b^\star w\rangle_{\tau,\xi,\xi^\vee}$$

for all $z, w \in \mathbb{Z}^n$ and $b \in \mathcal{O}$. By Proposition 5.3, and by the functoriality in the theory of degeneration in §4, we obtain an injective homomorphism $i_{\widetilde{S}} : \mathcal{O} \to \mathrm{End}_{\widetilde{S}}(G_{\widetilde{S}})$, whose restriction $i_{\widetilde{\eta}} : \mathcal{O} \to \mathrm{End}_{\widetilde{\eta}}(G_{\widetilde{\eta}})$ satisfies the Rosati condition defined by the polarization $\lambda_{\widetilde{\eta}}$ and by the involution $\star$. (See the theory with endomorphisms in §7 below. But beware that we cannot descend $i_{\widetilde{S}}$ to a homomorphism $i : \mathcal{O} \to \mathrm{End}_S(G)$ in general.) When $B = \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is not a division algebra (which is quite often the case), the injective homomorphism $i_{\widetilde{\eta}} : \mathcal{O} \to \mathrm{End}_{\widetilde{\eta}}(G_{\widetilde{\eta}})$ forces $G_{\widetilde{\eta}}$ to be isogenous to a nontrivial product of abelian varieties of smaller dimensions, in which case G_η cannot be absolutely simple.

Example 5.23. In the context of Remark 5.21, suppose $\widetilde{k}$ is a quadratic extension of k, and suppose $\rho : \mathrm{Gal}(\widetilde{K}/K) \cong \mathrm{Gal}(\widetilde{k}/k) \to \mathrm{GL}_2(\mathbb{Z})$ maps the nontrivial element of $\mathrm{Gal}(\widetilde{k}/k)$ to the automorphism ς of $\mathbb{Z}^2$ swapping the two factors. Then $\mathcal{O} = \mathbb{Z}[\varsigma]$, and $B = \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = \mathbb{Q}[\varsigma] \cong \mathbb{Q} \times \mathbb{Q}$ is a product of two fields. Consequently, every $K^\times$-valued pairing $\langle \cdot, \cdot \rangle_{\tau,\xi,\xi^\vee} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to K^\times$ satisfying all the conditions in Proposition 5.3 defines an object (G, λ) in $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{pol}}(R)$ such that $G_{\widetilde{\eta}}$ is isogenous to the product of two one-dimensional abelian varieties (i.e., elliptic curves) over $\widetilde{\eta}$, in which case G_η cannot be absolutely simple.

Remark 5.24. The proof of Proposition 5.14 avoided the issues in Remarks 5.20 and 5.21 by making crucial use of elements in $\widetilde{K}^\times$ that are multiplicatively Galois independent as in Lemma 5.10.

6. Proof of main theorem

Now we are ready to prove our main theorem. Let us first state and prove a finer statement for the case of degenerations over complete discrete valuation rings.

Theorem 6.1. *Let R, k, K, S, s, and η be as in the beginning of* §4. *Given any tori T_s and $T_s^\vee$ with character groups $\underline{X}_s$ and $\underline{Y}_s$, respectively, and an embedding $\phi_s : \underline{Y}_s \hookrightarrow \underline{X}_s$ inducing an isogeny $\lambda_{T_s} = \phi_s^* : T_s \to T_s^\vee$ (as in the beginning of* §5*), there exists an object (G, λ) in $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{pol}}(R)$ (see Definition* 4.1*) such that $\lambda_s : G_s \to G_s^\vee$ can be identified with $\lambda_{T_s} : T_s \to T_s^\vee$ (via some isomorphisms $G_s \cong T_s$ and $G_s^\vee \cong T_s^\vee$ over s), and such that G_η is absolute simple.*

Proof. This follows from the combination of Propositions 5.3, 5.5, and 5.14. □

Thus, the case (a) of Theorem 1.3 over complete discrete valuation rings with residue field k follows from the following:

Corollary 6.2. *Given any field k and any complete discrete valuation ring R with residue field k, any torus over k can be realized as the special fiber of some semi-abelian scheme over R whose generic fiber is an absolutely simple abelian variety* (*which can be principally polarized*) *over K.*

Proof. By Remark 4.3, it suffices to prove the assertion for *polarized* abelian varieties, and so it suffices to apply Theorem 6.1 to the given torus over k, with the isogeny given by the identity morphism from this torus to itself. □

It still remains to prove the cases (b) and (c) of Theorem 1.3. We shall deduce them from the case (a) of Theorem 1.3 over complete discrete valuation rings, by slightly modifying the argument in [7, Ch. IV, Sec. 4] based on Artin's approximation. Let us retain the setting of Theorem 6.1, but with the more specific choice that $R = k[[t]]$ (resp. the Witt vectors $R = W(k)$) in case (b) (resp. in case (c)). Let I denote the maximal ideal of R.

Let $R_0 := k[t]$ (resp. a number ring with an absolutely unramified prime ideal $\mathfrak{P}$ with residue field k, so that R_0 is embedded in R via $\mathfrak{P}$-adic completion), and let I_0 denote the ideal (t) (resp. $\mathfrak{P}$) of R_0. Let R_1 be the Henselization of R_0 at I_0, which is canonically a subring of R. (See, for example, [9, IV-4, 18.6].) Let I_1 denote the ideal of R_1 generated by I_0, so that the maximal ideal I of R is generated by I_1. For $0 \leq j \leq 1$, we shall denote the fraction field of R_j by K_j; define $S_j := \mathrm{Spec}(R_j)$ and $\eta_j := \mathrm{Spec}(K_j)$; and, by abuse of notation, still denote by s the closed points $\mathrm{Spec}(k) \to \mathrm{Spec}(R_j)$ defined by I_j.

Proposition 6.3. *Suppose R_2 is a subalgebra of R that is of finite type over R_1. Then, given any integer $m \geq 1$, the natural inclusion $R_1 \hookrightarrow R_2$ has some (homomorphic) section $R_2 \to R_1$ (depending on m) such that the composition $R_2 \to R_1 \hookrightarrow R$ coincides with the natural inclusion $R_2 \hookrightarrow R$ after reduction modulo I^m in R.*

Proof. This follows from Artin's approximation, as in [1, Thm. 1.10], because R_1 is the Henselization of the excellent Dedekind domain R_0 at the maximal ideal I_0. □

Theorem 6.4. *Given any (G, λ) in $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{pol}}(R)$ as in Theorem 6.1, there exists some (G_1, λ_1) in $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{pol}}(R_1)$ (see Remark 4.2) such that $(G_1, \lambda_1) \underset{R_1}{\otimes} k \cong (G, \lambda) \underset{R_1}{\otimes} k$, and such that the generic fiber G_{1,η_1} is absolutely simple. Consequently, the special fiber $\lambda_{1,s} : G_{1,s} \to G^\vee_{1,s}$ of λ_0 can also be identified with $\lambda_{T_s} : T_s \to T^\vee_s$ (via some isomorphisms $G_{1,s} \cong T_s$ and $G^\vee_{1,s} \cong T^\vee_s$ over s).*

These assertions remain true with $S_1 = \mathrm{Spec}(R_1)$ replaced with some connected affine étale neighborhood $s \to U \to S_0 = \mathrm{Spec}(R_0)$ of s.

Proof. Since $G \to S = \mathrm{Spec}(R)$ and $G^\vee \to S$ are both of finite presentation, and since R is the filtering direct limit (union) of its normal subalgebras R_2 of finite type over R_1, by [9, IV-3, 8.8.2], we may assume that there exist some such R_2, some semi-abelian schemes $G_2 \to S_2 := \mathrm{Spec}(R_2)$ and $G^\vee \to S_2$, and some homomorphism $\lambda_2 : G_2 \to G^\vee_2$ such that $(G_2, G^\vee_2, \lambda_2) \underset{R_2}{\otimes} R \cong (G, G^\vee, \lambda)$. Since the pullback of G_2 and $G^\vee_2$ to η are the absolutely simple abelian varieties G_η and $G^\vee_\eta$, respectively, by [7, Ch. I, Thm. 2.10] or [14, Ch. 3, Thm. 3.3.1.9]; and by [19, IX, 1.4], [7, Ch. I, Prop. 2.7], or [14, Prop. 3.3.1.5], there exists a nonempty open subset W of S_2 such that $G_2|_W$ and $G^\vee_2|_W$ are abelian schemes whose fibers are all absolutely simple, and such that $\lambda_2|_W$ is a polarization of abelian schemes. Suppose W is the complement of a closed subset of S_2 defined by some nonzero ideal J_2 of R_2. Since $R_2 \subset R$, and since R is noetherian and I-adically separated, there exists an integer $m \geq 1$ such that J_2 is not contained in I^m. By Proposition 6.3, there exists a section $R_2 \to R_1$ of $R_1 \hookrightarrow R_2$ such that the composition $R_2 \to R_1 \hookrightarrow R$ coincides with the natural inclusion $R_2 \hookrightarrow R$ after reduction modulo I^m in R. Let $(G_1, G^\vee_1, \lambda_1) := (G_2, G^\vee_2, \lambda_2) \underset{R_2}{\otimes} R_1$. Then $(G_1, G^\vee_1, \lambda_1) \underset{R_1}{\otimes} k \cong (G, G^\vee, \lambda) \underset{R}{\otimes} k$ because $m \geq 1$. Moreover, J_2 has nonzero image in R_1 under the section $R_2 \to R_1$ above, and therefore the induced morphism $S_1 = \mathrm{Spec}(R_1) \to S_2 = \mathrm{Spec}(R_2)$ maps the generic point η_1 of S_1 to the above open subset W of S_2. Hence, (G_1, λ_1)

defines an object in $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{pol}}(R_1)$, and the generic fiber G_{1,η_1} is absolutely simple. This proves the first paragraph of the theorem.

Since $G_1 \to S_1$ and $G_1^\vee \to S_1$ are both of finite presentation, and since R_1 is the filtering direct limit of the coordinate rings of all connected affine étale neighborhoods $s \to U \to S_0$ of s, by [9, IV-3, 8.8.2] again, the second paragraph of the theorem follows from the first, as desired. □

Now, since an étale neighborhood $s \to U \to S_0$ of s is necessarily a smooth curve over k (with a k-rational point lifting s, which can still be denoted s) in case (b) (resp. an open subset of the spectrum of a number ring in case (c)), the cases (b) and (c) of Theorem 1.3 follow from the following:

Corollary 6.5. *Given any field k and any complete discrete valuation ring R with residue field k, any torus over k can be realized as the special fiber of some semi-abelian scheme over some connected affine étale neighborhood $s \to U \to S_0$ of s that is generically an absolutely simple abelian variety* (*which can be principally polarized*).

Proof. As explained in the proof of Corollary 6.2, this follows from Theorem 6.4 just as Corollary 6.2 does from Theorem 6.1. (Alternatively, we could have approximated the semi-abelian scheme in Corollary 6.2 directly, using [9, IV-3, 8.8.2] and Proposition 6.3, as in the proof of Theorem 6.4.) □

Remark 6.6. By the theory of Néron models (see, in particular, [3, Ch. 7, Sec. 7.4, Prop. 3]), the semi-abelian schemes in Corollaries 6.2 and 6.5 (where the base schemes are Dedekind domains) are exactly the identity components of the Néron models of their respective generic fibers.

7. Nontrivial endomorphisms

In this section, we investigate the analogue of Question 1.2 which requires additionally that the semi-abelian schemes are equipped with endomorphism structures.

Let $\mathcal{O}$ be an order in a finite-dimensional semisimple algebra B over $\mathbb{Q}$, and let $\star : B \to B$ be a positive involution (i.e., $\mathrm{Tr}_{B/\mathbb{Q}}(xx^\star) > 0$ for all nonzero x in B), which we assume to stabilize $\mathcal{O}$. All endomorphism algebras of abelian varieties over algebraically closed fields are necessarily of this form, with the involution $\star$ given by the Rosati involution induced by some polarization. (See, for example, [17, Sec. 20–21], which contains a treatment of Albert's classification of finite-dimensional division algebras over $\mathbb{Q}$ with positive involutions. See also [14, Prop. 1.2.1.13 and 1.2.1.14] for a summary of possibilities when B is simple.)

Let us first expand the review of the theory of degeneration in §4. Let R, k, K, S, s, and η be as in the beginning of §4.

Definition 7.1. *With the setting as above, the category $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{PE},\mathcal{O}}(R)$ has objects consisting of triples (G, λ, i) over $S = \mathrm{Spec}(R)$, where (G, λ) is an object in $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{pol}}(R)$, and where $i : \mathcal{O} \to \mathrm{End}_S(G)$ is a homomorphism satisfying the* **Rosati condition** *that $\lambda \circ i(b^\star) = (i(b))^\vee \circ \lambda$ as homomorphisms from G to $G^\vee$, where $(i(b))^\vee : G^\vee \to G^\vee$ is the unique homomorphism* (*see* [19, IX, 1.4], [7, Ch. I, Prop. 2.7], *or* [14, Prop. 3.3.1.5]) *extending the dual homomorphism $(i(b))^\vee_\eta : G^\vee_\eta \to G^\vee_\eta$ of $(i(b))_\eta : G_\eta \to G_\eta$, for all $b \in \mathcal{O}$.*

Remark 7.2. As in the case of Definition 4.1 and Remark 4.2, the definition of pairs (G, λ, i) as in Definition 4.1 extends verbatim to the case where S is a noetherian normal local scheme.

By the theory of degeneration data (for polarized abelian schemes with endomorphism structures; see [14, Sec. 5.1.1; see, in particular, Thm. 5.1.1.4], with all abelian parts in the degenerations suppressed), there is an equivalence of categories

$$\mathrm{M}^{\mathrm{tor}}_{\mathrm{PE},\mathcal{O}}(R): \ \mathrm{DD}^{\mathrm{tor}}_{\mathrm{PE},\mathcal{O}}(R) \to \mathrm{DEG}^{\mathrm{tor}}_{\mathrm{PE},\mathcal{O}}(R) : (\underline{X}, \underline{Y}, \phi, \tau) \mapsto (G, \lambda, i)$$

realizing (G, λ, i) (up to isomorphism) as the image of an object in $\mathrm{DD}^{\mathrm{tor}}_{\mathrm{PE},\mathcal{O}}(R)$ given by the following data:

(1) The tuple $(\underline{X}, \underline{Y}, \phi, \tau)$ is an object in $\mathrm{DD}^{\mathrm{tor}}_{\mathrm{pol}}(R)$, as in §4.
(2) The étale shaves $\underline{X}$ and $\underline{Y}$ are equipped with the structures of étale sheaves of left $\mathcal{O}$-lattices (i.e., $\mathbb{Z}$-lattices with left $\mathcal{O}$-module structures), and the embedding $\phi : \underline{X} \to \underline{Y}$ is then $\mathcal{O}$-equivariant.
(3) The bimultiplicative homomorphism $\tau : \underline{Y}_\eta \times \underline{X}_\eta \to \mathbb{G}_{\mathrm{m},\eta}$ is compatible with the actions of $\mathcal{O}$ in the sense that $(b \times 1)^* \tau = (1 \times b^\star)^* \tau$, for all $b \in \mathcal{O}$.

We say that $(\underline{X}, \underline{Y}, \phi, \tau)$ is the *degeneration data* of (G, λ, i).

Suppose that we have tori T_s and $T_s^\vee$ over $s = \mathrm{Spec}(k)$ with character groups $\underline{X}_s$ and $\underline{Y}_s$ that are étale sheaves of left $\mathcal{O}$-lattices, together with an $\mathcal{O}$-equivariant embedding $\phi_s : \underline{Y}_s \hookrightarrow \underline{X}_s$ with finite cokernel, inducing an $\mathcal{O}$-equivariant isogeny $\lambda_{T_s} : T_s \twoheadrightarrow T_s^\vee$. By Lemma 3.1, the $\mathcal{O}$-equivariant embedding $\phi_s : \underline{Y}_s \hookrightarrow \underline{X}_s$ uniquely lifts to an $\mathcal{O}$-equivariant embedding $\phi : \underline{Y} \hookrightarrow \underline{X}$ over S of étale sheaves of left $\mathcal{O}$-lattices with finite cokernel, inducing an $\mathcal{O}$-equivariant isogeny $\lambda_T : T \twoheadrightarrow T^\vee$ (between isotrivial tori) lifting λ_{T_s}. Let $\widetilde{S}$, etc, be as in §3 and §5 such that there are left $\mathcal{O}$-lattices X and Y, isomorphisms $\xi : X \xrightarrow{\sim} \underline{X}_{\widetilde{\eta}}$ and $\xi^\vee : Y \xrightarrow{\sim} \underline{Y}_{\widetilde{\eta}}$, and representations $\rho : \mathrm{Gal}(\widetilde{K}/K) \to \mathrm{GL}_{\mathcal{O}}(X)$ and $\rho^\vee : \mathrm{Gal}(\widetilde{K}/K) \to \mathrm{GL}_{\mathcal{O}}(Y)$ defining the descent data for the étale sheaves $\underline{X}_\eta$ and $\underline{Y}_\eta$ (of left $\mathcal{O}$-lattices). Then we have an $\mathcal{O}$-equivariant embedding

$$\phi_{\xi,\xi^\vee} := \xi^{-1} \, \phi_{\widetilde{\eta}} \, \xi^\vee : Y \hookrightarrow X \tag{7.3}$$

with finite cokernel (cf. (5.1)), and the following obvious analogue of Lemma 5.2:

Lemma 7.4. *We have*

$$\phi_{\xi,\xi^\vee}(\rho^\vee(\gamma) z) = \rho(\gamma)(\phi_{\xi,\xi^\vee}(z))$$

for all $\gamma \in \mathrm{Gal}(\widetilde{K}/K)$ *and* $z \in Y$.

Definition 7.5. *Given left* $\mathcal{O}$*-lattices* M *and* M'*, and an abelian group* M'' (*such as* $\mathbb{Z}$ *or* $\widetilde{K}^\times$), *we say that a* ($\mathbb{Z}$-)*bilinear* (*or bimultiplicative*) *pairing* $\langle \cdot , \cdot \rangle : M \times M' \to M''$ *is* $\mathcal{O}$**-compatible** *if* $\langle bz, w \rangle = \langle z, b^\star w \rangle$ *for all* $z \in M$, $w \in M'$, *and* $b \in \mathcal{O}$.

Remark 7.6. By [14, Lem. 1.1.4.5], the $\mathcal{O}$-compatible symmetric bilinear pairings $\langle \cdot , \cdot \rangle : M \times M \to \mathbb{Z}$ (as in Definition 7.5, with $M = M'$ and $M'' = \mathbb{Z}$) are exactly the traces of the Hermitian pairings $(\!| \cdot , \cdot |\!) : M \times M \to \mathrm{Diff}^{-1}_{\mathcal{O}/\mathbb{Z}}$ valued in the *inverse different* $\mathrm{Diff}^{-1}_{\mathcal{O}/\mathbb{Z}}$ (see [14, Def. 1.1.1.8]).

Then we have the following strengthening of Proposition 5.3:

Proposition 7.7. *With the setting as above, with the fixed choices of some ξ, $\xi^\vee$, ρ, and $\rho^\vee$, the datum of an object (G, λ, i) of $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{PE},\mathcal{O}}(R)$ (as in Definition 7.1) such that $\lambda_s : G_s \to G_s^\vee$ can be identified with $\lambda_{T_s} : T_s \to T_s^\vee$ (via $\mathcal{O}$-equivariant isomorphisms $G_s \cong T_s$ and $G_s^\vee \cong T_s^\vee$ over s) is equivalent to the datum of a bimultiplicative pairing*

$$\langle \cdot, \cdot \rangle_{\tau,\xi,\xi^\vee} : Y \times X \to \widetilde{K}^\times \tag{7.8}$$

satisfying the following conditions:

1. (**Galois equivariance**:) *$\langle \rho^\vee(\gamma)z, \rho(\gamma)w \rangle_{\tau,\xi,\xi^\vee} = \gamma\langle z, w\rangle_{\tau,\xi,\xi^\vee}$ for all $z \in Y$, $w \in X$, and $\gamma \in \mathrm{Gal}(\widetilde{K}/K)$.*
2. (**Symmetry**:) *$\langle z, \phi_{\xi,\xi^\vee}(w)\rangle_{\tau,\xi,\xi^\vee} = \langle w, \phi_{\xi,\xi^\vee}(z)\rangle_{\tau,\xi,\xi^\vee}$ for all $z, w \in Y$.*
3. (**Positivity**:) *$\widetilde{v}(\langle z, \phi_{\xi,\xi^\vee}(z)\rangle_{\tau,\xi,\xi^\vee}) > 0$, for every nonzero $z \in Y$, where $\widetilde{v} : \widetilde{K}^\times \to \mathbb{Z}$ is any nontrivial discrete valuation of $\widetilde{K}$.*
4. (**$\mathcal{O}$-compatibility**:) *$\langle bz, w\rangle_{\tau,\xi,\xi^\vee} = \langle z, b^\star w\rangle_{\tau,\xi,\xi^\vee}$ for all $z \in Y$, $w \in X$, and $b \in \mathcal{O}$.*

Let $\mathcal{O}_+$ (resp. $\mathcal{O}_-$) denote the $\mathbb{Z}$-submodule of $\mathcal{O}$ consisting of $b \in \mathcal{O}$ such that $b^\star = b$ (resp. $b^\star = -b$). Note that $\mathcal{O}_+ \cap \mathcal{O}_- = 0$ and $2\mathcal{O} \subset \mathcal{O}_+ + \mathcal{O}- \subset \mathcal{O}$, because $2b = (b + b^\star) + (b - b^\star)$ for all $b \in \mathcal{O}$. Let B_+ (resp. B_-) denote the $\mathbb{Q}$-subspace of $B = \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ spanned by $\mathcal{O}_+$ (resp. $\mathcal{O}_-$). Then $\mathcal{O}_+ + \mathcal{O}- \subset \mathcal{O}$ induces $B_+ \oplus B_- \cong B$.

We have the following rather elaborate analogue of Proposition 5.5:

Proposition 7.9. *In the setting of Proposition 7.7, suppose that $B = \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a division algebra. Suppose that the pairing $\langle \cdot, \cdot \rangle_{\tau,\xi,\xi^\vee}$ corresponding to (G, λ) satisfies the additional condition that*

$$\langle z, w\rangle_{\tau,\xi,\xi^\vee} \neq 1 \tag{7.10}$$

for all nonzero $z \in Y$ and $w \in X$, except when $\phi_{\xi,\xi^\vee}(z) = cw$ for some nonzero $c \in B_-$. Then G_η and hence $G_\eta^\vee$ must be absolutely simple, except when B belongs to one of the following two special cases (see Remark 7.12 below):

- (A) *The center F of the algebra B is a totally real field, and $B \underset{F,\sigma}{\otimes} \mathbb{R}$ is isomorphic to the Hamilton quaternion algebra $\mathbb{H}$ over $\mathbb{R}$ for every field homomorphism $\sigma : F \to \mathbb{R}$, with the positive involution $\star$ of B induced by the canonical positive involution $x \mapsto \mathrm{Tr}_{\mathbb{H}/\mathbb{R}}(x) - x$ on $\mathbb{H}$. (This is exactly the third case in [14, Prop. 1.2.1.13].) Also, $Y \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong B$ as B-modules.*
- (B) *The center F of the algebra B is a CM field (i.e., is a totally imaginary extension of a totally real field F_+), and $B = F$. (This is a very special case of the fourth case in [14, Prop. 1.2.1.13].) Also, $Y \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong B$ as B-modules.*

Remark 7.11. The assumption that $B = \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a division algebra is natural for our purpose, because the existence of any nontrivial idempotent in B will force G_η to be isogenous to a nontrivial product of abelian varieties of smaller dimensions.

Remark 7.12. The two exceptional cases (A) and (B) in Proposition 7.9 (and therefore also in Theorems 7.20, 7.22, and 7.24 below) are unavoidable for the following reasons: Suppose $\langle \cdot, \cdot \rangle_{\tau,\xi,\xi^\vee}$ is any pairing, with associated G over S, as in Proposition 7.7. For simplicity, let us identify Y with its image $\phi_{\xi,\xi^\vee}(Y)$ in X, and suppress $\phi_{\xi,\xi^\vee}$ from the notation. In the case (A), there exist $e, f \in \mathcal{O}_-$ such that

$B = B' \oplus fB'$ as modules of the (commutative) CM field $B' := F + Fe$, by letting B' act on B by right multiplication. Since $f^\star = -f$, for any $z \in Y \subset X$, we have

$$\begin{aligned}\langle az, 4fbz\rangle_{\tau,\xi,\xi^\vee} &= \langle az, 2fbz\rangle_{\tau,\xi,\xi^\vee} \cdot \langle -2faz, bz\rangle_{\tau,\xi,\xi^\vee} = \langle z, 2(a^\star fb - b^\star fa)z\rangle_{\tau,\xi,\xi^\vee} \\ &= \langle z, (a^\star fb - b^\star fa)z\rangle_{\tau,\xi,\xi^\vee} \cdot \langle (b^\star fa - a^\star fb)z, z\rangle_{\tau,\xi,\xi^\vee} = 1,\end{aligned}$$

for all $a, b \in \mathcal{O}' := \mathcal{O} \cap B'$. Similarly, in the case (B), there exists $e \in \mathcal{O}_-$ such that $B = F = B' \oplus B'e$ as modules of the (commutative) totally real field $B' := F_+$. Since $e^\star = -e$, and since $B = F$ is commutative, for any $z \in Y \subset X$, we have

$$\langle az, 2bez\rangle_{\tau,\xi,\xi^\vee} = \langle abz, ez\rangle_{\tau,\xi,\xi^\vee} \cdot \langle -ez, abz\rangle_{\tau,\xi,\xi^\vee} = 1,$$

for all $a, b \in \mathcal{O}' := \mathcal{O}_+ = \mathcal{O} \cap B'$. Hence, in both cases, up to replacing Y and X with some sublattices of finite indices, the pairing $\langle \cdot\,, \cdot \rangle_{\tau,\xi,\xi^\vee}$ decomposes into an orthogonal direct sum of two pairings that are symmetric, positive, and compatible with the order $\mathcal{O}'$ in B' (but not necessarily Galois equivariant) as in Proposition 7.7. Therefore, the base change $G_{\bar\eta}$ of G_η to any geometric point $\bar\eta \to S$ above $\eta \to S$ is isogenous to a product of two abelian varieties over $\bar\eta$ with endomorphisms by $\mathcal{O}'$, both of which are simple by applying Proposition 7.9 with $\mathcal{O}$ replaced with $\mathcal{O}'$. (Moreover, these two simple abelian varieties are isogenous to each other by the same argument as in the second paragraph of the proof of Proposition 7.9 below.) In particular, G_η can never be absolutely simple in these two exceptional cases.

Proof of Proposition 7.9. Without loss of generality, we may and we shall assume that $Y \neq 0$, so that G_η is a nontrivial abelian variety over η.

The condition (7.10) implies the weaker condition that, for any given nonzero $z \in Y$ and $w \in X$, there exists some $b \in \mathcal{O}$ such that $\langle v, bw\rangle_{\tau,\xi,\xi^\vee} \neq 1$. By the same argument as in the proof of Proposition 5.5, this implies that G_η and hence $G_\eta^\vee$ are *absolutely simple in the category of abelian varieties with endomorphisms by* $\mathcal{O}$, in the sense that their base changes $G_{\bar\eta}$ and $G_{\bar\eta}^\vee$ to any geometric point $\bar\eta \to S$ above $\eta \to S$ are not isogenous to nontrivial products of abelian varieties of smaller dimensions with endomorphisms by $\mathcal{O}$. Moreover, if $G_{\bar\eta}$ is isogenous to a product $A_1 \underset{\bar\eta}{\times} A_2 \underset{\bar\eta}{\times} \cdots \underset{\bar\eta}{\times} A_r$ of simple abelian subvarieties over $\bar\eta$, then the actions of elements of $\mathcal{O}$ on $G_{\bar\eta}$ induce quasi-isogenies between all possible pairs A_i and A_j, with $1 \leq i, j \leq r$, because otherwise $G_{\bar\eta}$ cannot be absolutely simple in the category of abelian varieties with endomorphisms by $\mathcal{O}$. Hence, we may assume that $G_{\bar\eta}$ is isogenous to A^r for some (nontrivial) simple abelian variety A over $\bar\eta$, for some integer $r \geq 1$. Note that $\dim(A) \leq \frac{1}{2}\dim(G_{\bar\eta})$ when $r \geq 2$.

For simplicity, let us identify Y with its image $\phi_{\xi,\xi^\vee}(Y)$ in X, and suppress $\phi_{\xi,\xi^\vee}$ from the notation. Suppose $e_1, e_2, \ldots, e_n$ are elements of Y such that the assignment $B^n \to Y \underset{\mathbb{Z}}{\otimes} \mathbb{Q} : (a_i)_{1\leq i\leq n} \mapsto \sum\limits_{1\leq i\leq n} a_i e_i$ induces an isomorphism of left B-modules (which is possible because B is a division algebra, whose simple modules are all isomorphic to B itself; see [14, Lem. 1.1.2.4]).

Suppose $n \geq 2$. Then we have $\langle e_1, be_j\rangle_{\tau,\xi,\xi^\vee} \neq 1$ for all $j > 1$ and all $b \in \mathcal{O}$. Note that the dimension of the $\mathbb{Q}$-span of e_1 and all the be_j for all $j > 1$ and all $b \in \mathcal{O}$ is greater than $(n-1)\dim_{\mathbb{Q}}(B) = \frac{n-1}{n}\dim(G_{\bar\eta}) \geq \frac{1}{2}\dim(G_{\bar\eta})$. This shows that $\dim(A) > \frac{1}{2}\dim(G_{\bar\eta})$, so that $r = 1$. That is, $G_{\bar\eta}$ is simple whenever $n \geq 2$.

Suppose $n = 1$, so that $\dim(G_{\bar\eta}) = \dim_{\mathbb{Q}}(B)$. Let F denote the center of B. We have four cases of the division algebra B with the positive involution $\star$, as in [17, Sec. 21] or [14, Prop. 1.2.1.13]:

(I) F is a totally real field, and $B = F$, with the trivial involution $\star$. In this case, we have $B_- = 0$, and $G_{\bar{\eta}}$ is simple by Proposition 5.5, because the condition (7.10) here is identical to the corresponding condition (5.6) there.

(II) F is a totally real field, and $B \underset{F,\sigma}{\otimes} \mathbb{R}$ is isomorphic to $\mathrm{M}_2(\mathbb{R})$ over $\mathbb{R}$ for every field homomorphism $\sigma : F \to \mathbb{R}$, with the positive involution $\star$ of B induced by some conjugation of the canonical positive involution $x \mapsto {}^t x$ of $\mathrm{M}_2(\mathbb{R})$. In this case, since $\langle e_1, be_1 \rangle_{\tau,\xi,\xi^\vee} \neq 1$ for all $b \in \mathcal{O}_+$, and since $\dim_{\mathbb{Q}}(B_+) = \frac{3}{4}\dim_{\mathbb{Q}}(B) > \frac{1}{2}\dim_{\mathbb{Q}}(B) = \frac{1}{2}\dim(G_{\bar{\eta}})$, we can conclude as before that $r = 1$, so that $G_{\bar{\eta}}$ is simple.

(III) F is a totally real field, and $B \underset{F,\sigma}{\otimes} \mathbb{R}$ is isomorphic to the Hamilton quaternion algebra $\mathbb{H}$ over $\mathbb{R}$ for every field homomorphism $\sigma : F \to \mathbb{R}$. This is exactly the exceptional case (A) in the proposition.

(IV) F is a totally imaginary extension of a totally real field F_+, with the restriction of the involution $\star$ to F given by the complex conjugation over F_+. (We shall ignore the additional conditions satisfied by B.) In this case, since $\langle e_1, be_1 \rangle_{\tau,\xi,\xi^\vee} \neq 1$ for all $b \in \mathcal{O}_+$, and since $\dim_{\mathbb{Q}}(B_+) = \frac{1}{2}\dim_{\mathbb{Q}}(B)$, we have $r \leq 2$. If $r = 1$, then $G_{\bar{\eta}}$ is simple, as desired.

Otherwise, we necessarily have $r = 2$. Let $D := \mathrm{End}_{\bar{\eta}}(A) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, which is a division algebra with a positive involution, because A is a simple abelian variety. (See, for example, [17, Sec. 20–21].) Then we have an injective homomorphism $B \hookrightarrow \mathrm{M}_2(D)$ of algebras over $\mathbb{Q}$, which cannot be an isomorphism because B is a division algebra but $\mathrm{M}_2(D)$ is not. Since the action of the order $\mathrm{End}_{\bar{\eta}}(A)$ in D induces an action of the same order on the character group of the torus degeneration of A, we have an action of D on B stabilizing the $\mathbb{Q}$-subspaces B_+ and B_- of the same dimension, so that $B \cong B_+ \oplus B_-$ and $B_+ \cong D^{r'} \cong B_-$ as D-modules, for some integer $r' \geq 1$. Then we have $2r' \dim_{\mathbb{Q}}(D) \leq \dim_{\mathbb{Q}}(B) < 4\dim_{\mathbb{Q}}(D)$, forcing $r' = 1$.

Let E denote the center of D, which is a field and is also the center of $\mathrm{M}_2(D)$. Then $E_0 := B \cap E$ is a subfield of the center F of B, because E commutes with all elements of B. If $E_0 = E$, then E is a subfield of F. Otherwise, the E-span B' of the image of B in $\mathrm{M}_2(D)$ form an E-subalgebra of $\mathrm{M}_2(D)$ such that $\dim_{\mathbb{Q}}(B') \geq 2\dim_{\mathbb{Q}}(B) = 4\dim_{\mathbb{Q}}(D)$, which must be the whole of $\mathrm{M}_2(D)$. Therefore, $B \underset{E_0}{\otimes} E \cong \mathrm{M}_2(D)$ is simple, which implies that $F = E_0$. In particular, we have either $E \subset F$ or $F \subset E$.

Since E is the center of an endomorphism algebra of a simple abelian variety, it is either a totally real field or a CM field. If E is a CM field, then there exists some nonzero $c \in E_0 = E \cap F$ such that $c^\star = -c$, so that $cB_+ = B_-$, contradicting the assumption that D stabilizes B_+. Hence, E is necessarily a totally real subfield, in which case $[D : E]$ is either 4 or 1, and $E \subset F$ (because $F \subset E$ cannot happen, since F is a CM field).

Let C denote the centralizer of B in $\mathrm{M}_2(D)$. By [10, Ch. 5, Sec. 11, Thm. 19], we have $[B : E][C : E] = [\mathrm{M}_2(D) : E] = 4[D : E]$.

If $[D : E] = 4$, then $[F : E]^2 | [B : E][C : E] = 16$, and so $[F : E]$ is either 4 or 2, and $[B : F]$ is either 1 or 4. If $[B : F] = 1$, then the torus degeneration of A is of dimension $\frac{1}{2}\dim_{\mathbb{Q}}(B) = \frac{1}{2}\dim_{\mathbb{Q}}(F) \leq 2\dim_{\mathbb{Q}}(E)$, which cannot admit an action of the quaternion algebra D over E. If $[B : F] = 4$, then $[F : E] = 2$ and $C = F$, forcing $F_+ = E$. Consider any $f \in F$ such that

$f^\star = -f$, so that $F = E[f] = E + Ef$. Then $B_- = B_+ \cdot f$ and $B_+ = B_- \cdot f$, and the left actions of D and F on B commute with each other because the left action of f on B can be equivalently given by the right multiplication by f, and because $f^2 \in F_+ = E$ commutes with D. Then $D \underset{E}{\otimes} F$ can be canonically identified with an E-subalgebra of $\mathrm{M}_2(D)$, and the centralizers of $E \subset F \subset \mathrm{M}_2(E)$ are $\mathrm{M}_2(D) \supset D \underset{E}{\otimes} F \supset D$, respectively. Consequently, $B = D \underset{E}{\otimes} F$ as E-subalgebras of $\mathrm{M}_2(D)$, because the centralizer of $C = F$ is B, by [10, Ch. 5, Sec. 11, Thm. 19] again. Since the positive involution $\star$ of B restricts to a positive involution on D, and since this restriction cannot be trivial because D is a quaternion algebra over E (and because of Albert's classification), there exists some nonzero $d \in D \cap B_-$. But then d^{-1} exists in D, and the left multiplication by $d^{-1} \in D$ maps $d \in B_-$ to the nonzero $1 \in B_+$, contradicting the assumption that D stabilizes B_-.

Finally, if $[D : E] = 1$, then $[F : E]^2 | [B : E][C : E] = 4$, and so $[F : E] = 2$ and $B = F = C$. This is exactly the exceptional case (B) in the proposition.

Thus, we have shown that G_η and hence $G_\eta^\vee$ are absolutely simple except in the two cases (A) and (B) in the proposition, as desired. □

Hence, the key point is to establish the following analogue of Proposition 5.14:

Proposition 7.13. *Suppose that $B = \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a division algebra. Then there exists a bimultiplicative pairing*

$$\langle \cdot , \cdot \rangle_{\tau,\xi,\xi^\vee} : Y \times X \to \widetilde{K}^\times$$

as in (7.8) *satisfying all the conditions in Propositions* 7.7 *and* 7.9.

Proof. The idea is similar to that in the proof of Proposition 5.14, but the implementation here is, perhaps unavoidably, more tedious. Let

$$\langle \cdot , \cdot \rangle_0 : X \times X \to \mathbb{Z}$$

be any positive definite $\mathcal{O}$-compatible (symmetric) bilinear pairing (which exists by the same argument as in the proof of, for example, [13, Lem. 2.5]).

Suppose $e_1, e_2, \ldots, e_n$ are elements of $X \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that the assignment $B^n \to X \underset{\mathbb{Z}}{\otimes} \mathbb{Q} : (a_i)_{1 \leq i \leq n} \mapsto \sum_{1 \leq i \leq n} a_i e_i$ induces an isomorphism of left B-modules (which, again, is possible because B is a division algebra, whose simple modules are all isomorphic to B itself; see [14, Lem. 1.1.2.4]), and such that X is contained in the left $\mathcal{O}$-lattice $Z := \sum_{1 \leq i \leq n} (\mathcal{O} e_i)$, which is the image of $\mathcal{O}^n$ under the last isomorphism. Then there exists a sufficiently divisible integer $r \in \mathbb{Z}_{\geq 1}$ such that $r \langle e_i, b e_j \rangle_0 \in \mathbb{Z}$ for all $b \in \mathcal{O}$ and $1 \leq i, j \leq n$; namely, $r \langle \cdot , \cdot \rangle_0$ induces a positive definite $\mathcal{O}$-compatible (symmetric) bilinear pairing $Z \times Z \to \mathbb{Z}$.

Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_h$ be a $\mathbb{Z}$-basis of $\mathcal{O}_+$, and let $\varepsilon_{h+1}, \varepsilon_{h+2}, \ldots, \varepsilon_m$ be a $\mathbb{Z}$-basis of $\mathcal{O}_-$. Let ϖ be any element of $K^\times$ of positive valuation. Let $\{u_{ijl}\}_{1 \leq i \leq j \leq m; 1 \leq l \leq m'_{ij}}$, where $m'_{ij} = h$ when $i = j$ and $m'_{ij} = n$ when $i < j$, be elements in $\widetilde{R}^\times$ that are multiplicatively independent as in Lemma 5.10. Then we define a symmetric bimultiplicative pairing

$$\langle \cdot , \cdot \rangle_1 : Z \times Z \to \widetilde{K}^\times$$

satisfying the compatibility

$$\langle bz, w\rangle_1 = \langle z, b^\star w\rangle_1 \tag{7.14}$$

for all $b \in \mathcal{O}_+ \oplus \mathcal{O}_-$ and $z, w \in Z$ by setting

(1) $\langle e_i, \varepsilon_l e_i\rangle_1 := u_{ijl} \cdot \varpi^{r\langle e_i, \varepsilon_l e_i\rangle_0}$ for all $1 \le i \le n$ and $1 \le l \le h$ (in which case $\varepsilon_l^\star = \varepsilon_l$ is compatible with $\langle e_i, \varepsilon_l^\star e_i\rangle_0 = \langle \varepsilon_l^\star e_i, e_i\rangle_0 = \langle e_i, \varepsilon_l e_i\rangle_0$);
(2) $\langle e_i, \varepsilon_l e_i\rangle_1 := 1$ for all $1 \le i \le n$ and $h < l \le m$ (in which case $\varepsilon_l^\star = -\varepsilon_l$ forces $\langle e_i, \varepsilon_l^\star e_i\rangle_0 = \langle \varepsilon_l^\star e_i, e_i\rangle_0 = \langle e_i, \varepsilon_l e_i\rangle_0 = 0$); and
(3) $\langle e_i, \varepsilon_l e_j\rangle_1 := u_{ijl} \cdot \varpi^{r\langle e_i, \varepsilon_l e_j\rangle_0}$ for all $1 \le i < j \le n$ and $1 \le l \le m$;

and by extending the values of the pairing to the whole domain $Z \times Z$ by symmetry, bimultiplicativity, and the rule that $\langle ae_i, be_j\rangle_1 := \langle e_i, a^\star be_j\rangle_1$ for all $a, b \in \mathcal{O}_+ \oplus \mathcal{O}_-$ and $1 \le i \le j \le n$. Note that this is well defined because $Be_i \cong B$ as left B-modules, for all $1 \le i \le n$. Also, note that $\widetilde{v} \circ \langle\,\cdot\,,\,\cdot\,\rangle_1$ is a positive multiple of $\langle\,\cdot\,,\,\cdot\,\rangle_0$ for every nontrivial discrete valuation $\widetilde{v} : \widetilde{K}^\times \to \mathbb{Z}$.

Next, we define an $\mathcal{O}$-compatible symmetric bilinear pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_2 : X \times X \to \widetilde{K}^\times$$

satisfying the Galois equivariance

$$\langle \rho(\gamma)z, \rho(\gamma)w\rangle_2 = \gamma\langle z, w\rangle_2 \tag{7.15}$$

for all $\gamma \in \mathrm{Gal}(\widetilde{K}/K)$ and $z, w \in X$ (cf. (5.15)) by setting

$$\langle z, w\rangle_2 := \prod_{\gamma \in \mathrm{Gal}(\widetilde{K}/K)} \gamma^{-1}\langle \rho(\gamma)z, \rho(\gamma)w\rangle_1 \tag{7.16}$$

for all $z, w \in X$. Note that $\langle \rho(\gamma)z, \rho(\gamma)w\rangle_1$ are defined for all $z, w \in X$ because X is contained in Z, and that $\widetilde{v} \circ \langle\,\cdot\,,\,\cdot\,\rangle_2$ is still positive definite for every nontrivial discrete valuation $\widetilde{v} : \widetilde{K}^\times \to \mathbb{Z}$.

Finally, we define the desired bimultiplicative pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\xi,\xi^\vee} : Y \times X \to \widetilde{K}^\times$$

by setting

$$\langle z, w\rangle_{\tau,\xi,\xi^\vee} := \langle \phi_{\xi,\xi^\vee}(z), 2w\rangle_2 \tag{7.17}$$

for all $z \in Y$ and $w \in X$, where $\phi_{\xi,\xi^\vee}$ is as in (7.3) (cf. (5.17); but note the coefficient 2 of w). Then $\langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\xi,\xi^\vee}$ satisfies the first three conditions (1), (2), and (3) in Proposition 7.7 by the symmetry and positive definiteness of $\langle\,\cdot\,,\,\cdot\,\rangle_0$; by the definitions of the pairings $\langle\,\cdot\,,\,\cdot\,\rangle_1$ and $\langle\,\cdot\,,\,\cdot\,\rangle_2$; by the choices of ϖ (of positive valuation in $\widetilde{K}^\times$) and $\{u_{ijl}\}_{1\le i\le j\le n; 1\le l\le m'_{ij}}$ (of zero valuation in $\widetilde{K}^\times$); and by Lemma 7.4 and the relations (7.15) and (7.17). As for the remaining condition (4) in Proposition 7.7, it suffices to note that, since $2\mathcal{O} \subset \mathcal{O}_+ \oplus \mathcal{O}_-$, the compatibility (7.14) implies that $\langle bz, (2w)\rangle_2 = \langle (2b)z, w\rangle_2 = \langle z, (2b^\star)w\rangle_2 = \langle z, b^\star(2w)\rangle_2$ for all $b \in \mathcal{O}$ and $z, w \in Z$.

It remains to show that $\langle z, w\rangle_{\tau,\xi,\xi^\vee} \neq 1$ (as in (7.10)) for all nonzero $z \in Y$ and $w \in X$, except when $\phi_{\xi,\xi^\vee}(z) = cw$ for some nonzero $c \in B_-$. Since $\phi_{\xi,\xi^\vee}$ is an embedding, by the defining relation (7.17), it suffices to show that $\langle z, 2w\rangle_2 \neq 1$ for all nonzero $z, w \in Z$, except when $z = cw$ for some nonzero $c \in B_-$. By the choice of $\{u_{ijl}\}_{1\le i\le j\le n; 1\le l\le m'_{ij}}$, the terms in the product (7.16) indexed by different elements γ have multiplicatively independent values (up to powers of ϖ), and so it suffices to show that $\langle z, 2w\rangle_1 \neq 1$ for all nonzero $z, w \in Z$, except when

$z = cw$ for some nonzero $c \in B_-$. Suppose that there are some $z = \sum_{1\leq i\leq n} (a_i e_i)$ and $w = \sum_{1\leq i\leq n} (b_i e_i)$ in Z, where $(a_i)_{1\leq i\leq n}$ and $(b_i)_{1\leq i\leq n}$ are nonzero n-tuples of elements in $\mathcal{O}$, such that $\langle z, 2w\rangle_1 = 1$. Then we have

$$\left(\prod_{1\leq i\leq n} \langle e_i, 2a_i^\star b_i e_i\rangle_1\right) \cdot \left(\prod_{1\leq i<j\leq n} \langle e_i, 2(a_i^\star b_j + b_i^\star a_j) e_j\rangle_1\right) = 1 \tag{7.18}$$

in $\widetilde{K}^\times$ (using again the relation $2\mathcal{O} \subset \mathcal{O}_+ \oplus \mathcal{O}_-$ and the compatibility (7.14)). By pulling out all powers of ϖ, and by the choice of $\{u_{ijl}\}_{1\leq i\leq j\leq n; 1\leq l\leq m'_{ij}}$ again, the identity (7.18) is possible only when

$$a_i^\star b_j + b_i^\star a_j = 0 \tag{7.19}$$

in B, for all $1 \leq i \leq j \leq n$. Let i_0 (resp. j_0) be the smallest index i (resp. j) such that $a_i \neq 0$ (resp. $b_j \neq 0$), which exists by assumption. If $i_0 \neq j_0$, then $a_{i_0}^\star b_{j_0} + b_{i_0}^\star a_{j_0} \neq 0$, which contradicts the condition (7.19). If $i_0 = j_0$, then the condition (7.19) implies that

$$c := a_{i_0} b_{i_0}^{-1}$$

is a nonzero element of B_-, because

$$c^\star = (b_{i_0}^\star)^{-1} a_{i_0}^\star = (b_{i_0}^\star)^{-1}(a_{i_0}^\star b_{i_0}) b_{i_0}^{-1} = (b_{i_0}^\star)^{-1}(-b_{i_0}^\star a_{i_0}) b_{i_0}^{-1} = -a_{i_0} b_{i_0}^{-1} = -c.$$

Moreover, for every $j \geq i_0$, the condition (7.19) implies that

$$a_j = (b_{i_0}^\star)^{-1}(b_{i_0}^\star a_j) = (b_{i_0}^\star)^{-1}(-a_{i_0}^\star b_j) = -c^\star b_j = c b_j.$$

Thus, $z = cw$ for some nonzero $c \in B_-$, as desired. □

Based on the combination of Propositions 7.7, 7.9, and 7.13, the same arguments as before give the following analogues of Theorem 6.1, Corollary 6.2, Theorem 6.4, and Corollary 6.5:

Theorem 7.20. *Let R, k, K, S, s, and η be as in the beginning of §4. Let $\mathcal{O}$ be as in the beginning of this §7, but assume moreover that $B = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a division algebra. Given any tori T_s and $T_s^\vee$ over s with character groups $\underline{X}_s$ and $\underline{Y}_s$, respectively, that are étale sheaves of left $\mathcal{O}$-lattices, and an $\mathcal{O}$-equivariant embedding $\phi_s : \underline{Y}_s \hookrightarrow \underline{X}_s$ with finite cokernel, inducing an $\mathcal{O}$-equivariant isogeny $\lambda_{T_s} = \phi_s^* : T_s \to T_s^\vee$, there exists an object (G, λ, i) in $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{PE},\mathcal{O}}(R)$ (see Definition 7.1) such that $\lambda_s : G_s \to G_s^\vee$ can be identified with $\lambda_{T_s} : T_s \to T_s^\vee$ (via some $\mathcal{O}$-equivariant isomorphisms $G_s \cong T_s$ and $G_s^\vee \cong T_s^\vee$ over s), and such that G_η is absolutely simple except in the two cases* (A) *and* (B) *in Proposition* 7.9.

Corollary 7.21. *Let $\mathcal{O}$ be as in Theorem* 7.20. *Given any field k and any complete discrete valuation ring R with residue field k, any torus with endomorphisms by $\mathcal{O}$ over k can be realized as the special fiber of some semi-abelian scheme over R whose generic fiber is an abelian variety with endomorphisms by $\mathcal{O}$ over K (which can be principally polarized) that is absolutely simple except in the two cases* (A) *and* (B) *in Proposition* 7.9.

Theorem 7.22. *Let $\mathcal{O}$ be as in Theorem* 7.20. *Given any (G, λ, i) in $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{PE},\mathcal{O}}(R)$ as in Theorem* 7.20, *there exists some (G_1, λ_1, i_1) in $\mathrm{DEG}^{\mathrm{tor}}_{\mathrm{PE},\mathcal{O}}(R_1)$ (see Remark* 7.2) *such that $(G_1, \lambda_1, i_1) \otimes_{R_1} k \cong (G, \lambda, i) \otimes_{R_1} k$, and such that the generic fiber G_{1,η_1}*

is absolutely simple except in the two cases (A) *and* (B) *in Proposition* 7.9*. Consequently, the special fiber* $\lambda_{1,s} : G_{1,s} \to G_{1,s}^\vee$ *of* λ_1 *can also be identified with* $\lambda_{T_s} : T_s \to T_s^\vee$ *(via some* $\mathcal{O}$*-equivariant isomorphisms* $G_{1,s} \cong T_s$ *and* $G_{1,s}^\vee \cong T_s^\vee$ *over* s*).*

These assertions remain true with $S_1 = \operatorname{Spec}(R_1)$ *replaced with some connected affine étale neighborhood* $s \to U \to S_0 = \operatorname{Spec}(R_0)$ *of* s*.*

Corollary 7.23. *Let* $\mathcal{O}$ *be as in Theorem* 7.20*. Given any field* k *and any complete discrete valuation ring* R *with residue field* k*, any torus with endomorphisms by* $\mathcal{O}$ *over* k *can be realized as the special fiber of some semi-abelian scheme over some connected affine étale neighborhood* $s \to U \to S_0$ *of* s *that is generically an abelian variety with endomorphisms by* $\mathcal{O}$ *(which can be principally polarized) that is absolutely simple except in the two cases* (A) *and* (B) *in Proposition* 7.9*.*

By combining Corollaries 7.21 and 7.23, we obtain the following strengthening of our main theorem:

Theorem 7.24. *In Theorem* 1.3*, suppose* T *has endomorphisms by* $\mathcal{O}$*, where* $\mathcal{O}$ *is as in the beginning of this* §7*, but with the additional assumption that* $B = \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *is a division algebra. Then we may assume that the semi-abelian scheme* G *also has endomorphisms by* $\mathcal{O}$*, and—instead of* (2) *in Theorem* 1.3*—that* G_η *is absolutely simple except in the two cases* (A) *and* (B) *in Proposition* 7.9*.*

Remark 7.25*.* Conversely, as mentioned in the introduction, by [19, IX, 1.4], [7, Ch. I, Prop. 2.7], or [14, Prop. 3.3.1.5], any endomorphism structure on the generic fiber G_η necessarily and uniquely extends to the whole semi-abelian scheme G, and also (by reduction) to the torus T we stared with. Hence, there has been no loss of generality in considering only tori with endomorphisms by the same orders.

Remark 7.26*.* Certainly, there are many examples of orders $\mathcal{O}$ (with positive involutions) defining endomorphism structures of abelian varieties of certain dimension n, but cannot possibly have actions on any n-dimensional tori. For example, already when $n = 1$, no CM elliptic curve can degenerate to any one-dimensional torus. In fact, one can generalize this to show that certain abelian varieties must have potential good reductions everywhere—see [11, Sec. 4.2] (and the review of the literature there) for a systematic discussion.

Remark 7.27*.* The two exceptional cases (A) and (B) in Proposition 7.9, and therefore also in Theorems 7.20, 7.22, and 7.24, correspond to the two exceptional cases (b) and (d) in [20, Sec. 4, Thm. 5], which are treated in more detail in [20, Sec. 4, Prop. 17 and 18]. These results in [20, Sec. 4] are relevant for our investigation of degenerations of abelian varieties into tori, because the corresponding complex moduli can be noncompact, and because their toroidal compactifications can be compared with the corresponding ones in [14, Thm. 6.4.1.1] in a way that respects the degenerations of abelian varieties into semi-abelian varieties along the boundary, by [12, Thm. 4.1.1 and Prop. 4.2.2]. (More precisely, some subcases of the two exceptional cases (b) and (d) in [20, Sec. 4, Thm. 5] are still irrelevant, because the corresponding complex moduli are compact, and therefore cannot parameterize nontrivial degenerations of abelian varieties, by [11, Sec. 4.2]. Similarly, the remaining exceptional cases (a), (c), and (e) in [20, Sec. 4, Thm. 5], which are treated in more detail in [20, Sec. 4, Prop. 14, 15, and 19], are also irrelevant, because the corresponding complex moduli are compact.)

Acknowledgements

We would like to thank the anonymous referees for their helpful comments.

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