MATH 8402: Assignment 3

Problem 1.

Let $\mathbf{T}(\mathbf{x}, t)$ denote the Cauchy stress tensor at the point $\mathbf{x} \in \Omega_t$ and at time $t > 0$. Suppose that the corresponding surface force on a given plane $\mathcal{S}$ is perpendicular to $\mathcal{S}$, while the surface force on any plane perpendicular to $\mathcal{S}$ is zero. Let $\mathbf{n}$ be the unit normal vector to $\mathcal{S}$. Show that $\mathbf{T}$ is a pure tension along the direction of $\mathbf{n}$. [Definition: $\mathbf{T}$ is a pure tension along the direction of $\mathbf{n}$ if there is a scalar λ such that $\mathbf{t} = \mathbf{T}\mathbf{n} = \lambda\mathbf{n}$.

Problem 2.

Let Ω be a parallelepiped $-h \leq z \leq h$, $-c \leq y \leq c$ and $0 \leq x \leq L$. Suppose that it is subject to a system of stress expressed by the (Cauchy) stress tensor

$$\mathbf{T} = \begin{bmatrix} \frac{3P}{2c^3}(L-x)y & -\frac{3P}{4c}(1-\frac{y^2}{c^2}) & 0 \\ -\frac{3P}{4c}(1-\frac{y^2}{c^2}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

P, h, c and L denote positive constants. Calculate the tractions (surface forces) on the boundaries of Ω.

Problem 3.

Let the map $\mathbf{\Phi}(\cdot)$ denote a deformation. The *displacement* vector $\mathbf{u}$ is defined as $\mathbf{u} = \mathbf{\Phi}(\mathbf{X}) - \mathbf{X}$. We define the *infinitesimal strain* tensor $\mathbf{E}$ as $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$. Show the following:

- $\mathbf{C} = \mathbf{I} + 2\mathbf{E} + \nabla\mathbf{u}^T\nabla\mathbf{u}$,
- $\mathbf{B} = \mathbf{I} + 2\mathbf{E} + \nabla\mathbf{u}\nabla\mathbf{u}^T$.

Let $\mathbf{u}$ be of class C^2 and suppose that Ω_0 is bounded and $\mathbf{u} = \mathbf{0}$ on $\partial\Omega_0$. Show (*Korn's inequality*)

$$\int_{\Omega_0} |\nabla\mathbf{u}|^2 d\mathbf{X} \leq k \int_{\Omega_0} |\mathbf{E}|^2 d\mathbf{X},$$

where $k > 0$ is constant. Also, idenfify the value of k in the inequality.

Problem 4.

Let the Cauchy-Green deformation tensor $C = F^T F$ of a given deformation be given by

$$C = \begin{bmatrix} 3 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & 14 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- Calculate the principal stretches of the deformation,
- Calculate the principal directions of stretch.

Problem 5.

Consider a deformation with deformation gradient matrix

$$F = \begin{bmatrix} \frac{2}{5} & -1 \\ \frac{11}{5} & 2 \end{bmatrix}.$$

- Calculate the matrices R, U and V of the polar decomposition $F = RU = VR$,
- Consider the (reference configuration) fiber $\mathbf{N} = \frac{1}{\sqrt{2}}(1,1)$. How does it transform under the given deformation? (Find the stretch and rotation angle of the fiber).

Problem 6.

Let $\mathbf{e}_i$, $i = 1,2,3$ denote the orthonormal system associated with an observer $\mathcal{O}$. Suppose that the Cauchy stress tensor measured by the observer, is given by

$$\mathbf{T} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & -\sqrt{3} \\ 0 & -\sqrt{3} & 9 \end{bmatrix}.$$

Consider the observer $\mathcal{O}^*$ with orthonormal system $\mathbf{e}_i^*$ obtained by rotating $\mathbf{e}_i$ by 60 degrees about the $\mathbf{e}_1$ direction.

- Determine $\mathbf{T}^*$,
- Determine and compare the principal stresses and principal directions of $\mathbf{T}$ and $\mathbf{T}^*$.

Problem 7.

Let $\mathbf{T}(\mathbf{x})$ denote the (symmetric) Cauchy stress tensor of a deformed body Ω. If there are no body forces applied, Ω is at equilibrium provided

$$\nabla \cdot \mathbf{T}(\mathbf{x}) = 0$$

holds for all $\mathbf{x} \in \Omega$. Consider the components of the stress tensor given by

$$\begin{aligned} T_{11} &= 3x^2 + 4xy - 8y^2 \\ T_{22} &= 2x^2 + xy + 3y^2 \\ T_{12} &= \frac{1}{2}x^2 - 6xy - 2y^2 \\ T_{33} &= T_{13} = T_{23} = 0. \end{aligned}$$

Determine whether Ω is at equilibrium.

The assignment is due on Wednesday, March 26.