A Review of Energy Balance Models

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Energy Balance Models

References

Classic Papers:

M. I. Budyko, The effect of solar radiation variation on the climate of the Earth, *Tellus* **21** (1969), 611-619.

W. D. Sellers, A Global Climatic Model Based on the Energy Balance of the Earth-Atmosphere System, *Journal of Applied Meteorology* **8** (1969), 392-400.

Recent Interpretation:

K.K. Tung, Topics in Mathematical Modeling, Princeton University Press, 2007. (Chapter 8)

Energy Balance Models

Insolation

Insolation = **In**coming **sol**ar radi**ation**

solar intensity at average distance from the sun: 1368 W/m^2

radius of the Earth: ρ meters
cross sectional area: $\pi\rho^2$ m^2
intercepted power: 1368 $\pi\rho^2$ Watts
surface area: $4\pi\rho^2$ m^2

average insolation: 1368/4 W/m^2 = 342 W/m^2

Energy Balance Models

Reradiation

$$T^4 = kS$$

where T = surface temperature (ºK)

S = solar influx (W/m^2)

k = constant depending on emissivity of the surface and the Stefan-Boltzmann constant.

We know S and k, so we can solve for T

$$T = 255°K = -18°C = \boxed{0°F}$$

Why isn't the Earth a Snowball?

C.Lorius, The ice-core record: climate sensitivity and future greenhouse warming, *Nature* 347 (1990), pp.139-145

Energy Balance Models

Why isn't the Earth a Snowball?

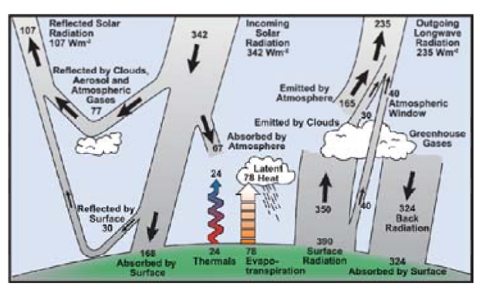
The Greenhouse Effect!

Joseph Fourier, *Mémoires de l'Académie des Sciences de l'Institut de France*, t. vii. 1827.

Svante Arrhenius, "On the Influence of Carbonic Acid in the Air upon the Temperature of the Ground," *Philosophical Magazine and Journal of Science (Fifth Series)* 41, pp. 237-276, 1896.

Energy Balance Models

Earth's Energy Balance



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Energy Balance Models

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

T = global mean temperature (°C)
Q = mean solar input (W/m²)
α = mean albedo
$A+BT$ = outward radiation (linear approximation)
R = heat capacity of Earth's surface

Tung's values:

T = global mean temperature (°C)
Q = 343 W/m²
A = 202 W/m²
B = 1.9 W/(m² °C)
$\alpha = \alpha_1 = 0.32$ (water and land)
$\alpha = \alpha_2 = 0.62$ (ice)

Energy Balance Models

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

Equilibrium temperature

$$T_{eq} = \frac{Q(1-\alpha) - A}{B}$$

ice free Earth: $\alpha = \alpha_1$, T_{eq} = 16.4 °C
snowball Earth: $\alpha = \alpha_2$, T_{eq} = -37.7 °C

Glaciers form if $T < T_c$ = -10 °C and melt if $T > T_c$.

Since 16.4 > -10, no glacier would form on an ice free Earth.
Since -37.7 < -10, no glacier would melt on a snowball Earth.

Why isn't the Earth a snowball?
Why isn't the Earth ice free?

Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\big(1-\alpha(y,\eta)\big) - (A+BT) + C(\bar{T} - T)$$

Now the annual average surface temperature T is a function of y = sine(latitude).

The albedo α is a function of y and the location η of the ice boundary.

The outward radiation $A+BT$ is as before.

Heat transport across latitudes is assumed to be linear and is given by $C(\bar{T} - T)$ where C = 3.04 W/m²/°C.

The global annual average insolation is Q, with the same value as above, while $s(y)$ is the relative insolation, normalized to satisfy

$$\int_0^1 s(y)\,dy = 1$$

Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\big(1-\alpha(y,\eta)\big) - (A+BT) + C(\bar{T} - T)$$

The variable y is chosen instead of the latitude, because the global annual mean temperature is given by

$$\bar{T}(t) = \int_0^1 T(y,t)\,dy$$

We assume symmetry with respect to the equator, so the variable y takes on values between 0 and 1.

We assume an ice boundary at $y = \eta$, with ice toward the pole and no ice toward the equator. The albedo is therefore

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y < \eta, \\ \alpha_2, & y > \eta. \end{cases}$$

Rate of solar energy absorption at y = sine(latitude):

$$Qs(y)\big(1-\alpha(y,\eta)\big)$$

Energy Balance Models

Inhomogeneous Earth

$$R\frac{\partial T}{\partial t} = Qs(y)\big(1-\alpha(y,\eta)\big) - (A+BT) + C(\bar{T} - T)$$

Look for an equilibrium solution having an ice line at $y = \eta$

$$T = T_\eta^*(y)$$

This equilibrium satisfies

$$\boxed{Qs(y)\big(1-\alpha(y,\eta)\big) - \big(A + BT_\eta^*(y)\big) + C\big(\bar{T}_\eta^* - T_\eta^*(y)\big) = 0}$$

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Mathematics

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Energy Balance Models

Inhomogeneous Earth

Equilibrium temperature (given ice line):

$$\boxed{T_\eta^*(y) = \frac{1}{B+C}\Big(Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}_\eta^*\Big)}$$

where

$$\bar{T}_\eta^* = \frac{1}{B}\Big(Q\big(1-\bar{\alpha}(\eta)\big) - A\Big)$$

is the global mean temperature
and where

$$\bar{\alpha}(\eta) = \alpha_2 - (\alpha_2 - \alpha_1)\int_0^\eta s(y)\,dy$$

is the global mean albedo.

Energy Balance Models

Inhomogeneous Earth

$$\boxed{T_\eta^*(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right)-A+C\overline{T}_\eta^*\right)}$$

Additional assumption: At equilibrium, the average temperature across the ice boundary is T_c = -10 °C. *I.e.*,

$$\frac{T_\eta^*(\eta-)+T_\eta^*(\eta+)}{2}=\frac{1}{B+C}\left(Qs(\eta)\left(1-\frac{\alpha_1+\alpha_2}{2}\right)-A+C\overline{T}_\eta^*\right)=T_c=-10$$

Solve for η .

Mathematics

Energy Balance Models

Inhomogeneous Earth

The ice boundary η satisfies.

$$h(\eta)=\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0)+\frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy\right)\right)-\frac{A}{B}-T_c=0$$

which can be solved numerically.

What about $s(y)$, the relative insolation function?

$$s(y)=\frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\gamma-y\cos\beta\right)^2}\,d\gamma$$

where β = obliquity. (Current value is about 23.5°.)

Tung and North's quadratic approximation:

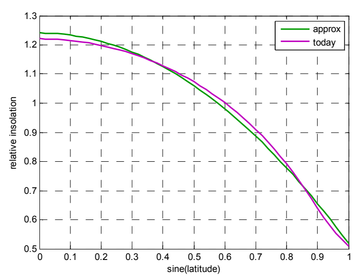
$$s(y)\approx 1-0.241\left(3y^2-1\right)$$

Energy Balance Models

Inhomogeneous Earth

green = quadratic approximation (Tung and North)

mauve = formula using obliquity of 23.5°



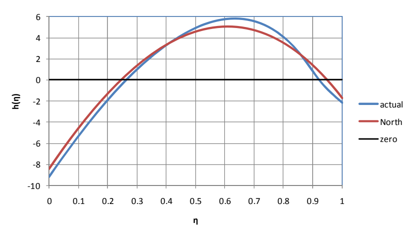
Energy Balance Models

Inhomogeneous Earth

Ice boundary condition

$$h(\eta)=\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0)+\frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy\right)\right)-\frac{A}{B}-T_c=0$$

If s is quadratic, then the integral is cubic, which is easier to work with.



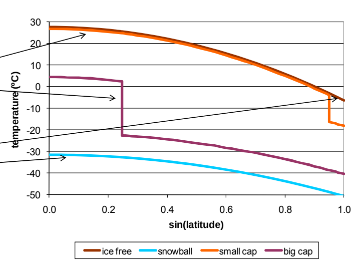
Energy Balance Models

Inhomogeneous Earth

equilibrium temperature profiles

Two equilibrium solutions:
small cap: stable
large cap: unstable

Not equilbria:
ice free
snowball



Energy Balance Models

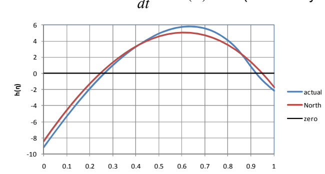
Widiasih Dynamic Iceline

$$R\frac{\partial T}{\partial t}=Qs(y)\left(1-\alpha(y,\eta)\right)-(A+BT)+C\left(\overline{T}-T\right)$$

$$\boxed{\frac{d\eta}{dt}=\varepsilon\left(T(\eta)-T_c\right)}$$

This infinite dimension system has a one dimensional attracting invariant manifold. On the manifold, the system reduces to

$$\frac{d\eta}{dt}\approx\varepsilon h(\eta) \qquad \text{(sufficiently small } \varepsilon\text{)}$$



Energy Balance Models

Widiasih Dynamic Iceline

$$\frac{d\eta}{dt} \approx \varepsilon h(\eta)$$

Why isn't the Earth ice free?

Because h(1)<0. (current conditions)

Why isn't the Earth a snowball?

Could be, since h(0)<1. (current conditions)

But we would have to get there from here.