A Finite Dimensional Model of Ice-Albedo Feedback

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Ice-Albedo Feedback

Acknowledgements

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Ice-Albedo Feedback

Budyko-Sellers Model

$$R\frac{\partial T}{\partial t} = \underbrace{Qs(y)}_{\text{insolation}}\underbrace{(1-\alpha(y,\eta))}_{\text{albedo}} - \underbrace{(A+BT)}_{\text{re-radiation}} + \underbrace{C(\bar{T}-T)}_{\text{transport}}$$

$T = T(y,t)$: annual mean surface temperature

$y = \sin(\text{latitude}) \quad y \in [0,1]$

Q: global annual mean insolation

$s(t)$: relative annual mean insolation $\quad \int_0^1 s(y)\,dy = 1$

$y = \eta$: ice boundary

$\alpha(y,\eta) = \begin{cases} \alpha_1, & y < \eta, \\ \alpha_2, & y > \eta. \end{cases}$ albedo

$\bar{T}(t) = \int_0^1 T(y,t)\,dy$: global annual mean temperature

Ice-Albedo Feedback

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$Q = 343 \ \text{W/m}^2 \qquad A = 202 \ \text{W/m}^2$

$B = 1.9 \ \text{W/m}^2/^\circ\text{C} \quad C = 3.04 \ \text{W/m}^2/^\circ\text{C}$

$\alpha_1 = 0.32$: (water and land)

$\alpha_2 = 0.62$: (ice)

y = proportion of Earth's surface between latitude $-\arcsin(y)$ and $+\arcsin(y)$,

η = proportion of Earth's surface that is ice-free.

If σ = surface area of Earth ($\approx 5.1\times10^{14}\ \text{m}^2$), then

$\sigma(1-\eta)$ = surface area of ice in square meters.

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Heat Capacity

R = heat capacity of Earth's surface ($\text{J/m}^2/^\circ\text{C}$).

Heat capacity of water ≈ 4 J/g/°C.

Mass of $1\ \text{cm}^3$ of $H_2O \approx 1$ g.

Assumption: Earth's surface consists of 100 m of H_2O.

Each square meter of Earth's surface consists of

$100\ \text{m}^3 = 10^8\ \text{cm}^3$ of H_2O, or about 10^8 g, so

$$\boxed{R \approx 4\times10^8\ \text{J/m}^2/^\circ\text{C}}$$

Ice-Albedo Feedback

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$$R\frac{\partial T}{\partial t} = \underbrace{Qs(y)}_{\text{insolation}}\underbrace{(1-\alpha(y,\eta))}_{\text{albedo}} - \underbrace{(A+BT)}_{\text{re-radiation}} + \underbrace{C(\bar{T}-T)}_{\text{transport}}$$

Time Scale

A Watt is a Joule per **second**, but all quantities are **annual** averages.
One year is approximately $\kappa = 3.16\times10^7$ seconds.
If we measure time in years instead of seconds, the equation becomes

$$\boxed{\frac{\partial T}{\partial t} = \frac{\kappa}{R}\left(Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)\right)}$$

Ice-Albedo Feedback

Budyko-Sellers Model

$$\frac{\partial T}{\partial t}=\frac{\kappa}{R}\Big(Qs(y)\big(1-\alpha(y,\eta)\big)-(A+BT)+C(\overline{T}-T)\Big)$$

Dynamics of the Ice Line

Widiasih equation:

$$\frac{d\eta}{dt}=\varepsilon(T_b-T_c)$$

where

$$T_b=\frac{T(\eta-)+T(\eta+)}{2},\qquad T_c=-10^\circ\mathrm{C}$$

$$\frac{d\eta}{dt}=\varepsilon\kappa(T_b-T_c)$$
$$\frac{\partial T}{\partial t}=\frac{\kappa}{R}\Big(Qs(y)\big(1-\alpha(y,\eta)\big)-(A+BT)+C(\overline{T}-T)\Big)$$

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Budyko-Sellers-Widiasih Model

$$\frac{d\eta}{dt}=\varepsilon\kappa(T_b-T_c)$$
$$\frac{\partial T}{\partial t}=\frac{\kappa}{R}\Big(Qs(y)\big(1-\alpha(y,\eta)\big)-(A+BT)+C(\overline{T}-T)\Big)$$
$$T_b=\frac{T(\eta-)+T(\eta+)}{2}$$
$$\overline{T}=\overline{T}(t)=\int_0^1 T(y,t)\,dy$$

What about ε ?

The parameter ε should be determined from climate data.
(good project!)

Best current guess:

$$\varepsilon\approx 10^{-12}$$

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Budyko-Sellers-Widiasih Model

Heat of Fusion

$$\frac{d\eta}{dt}=\varepsilon\kappa(T_b-T_c)$$

To move the ice line, we must melt or freeze water.
Heat of fusion of water: 334 J/g , or $3.34\text{x}10^8$ J/m^3

Assumption: Average thickness of ice is 450 m.
Energy to melt 1 m^2 of ice: Ω = $1.5\text{x}10^{11}$ Joules
Energy to move ice line from η to $\eta+\Delta\eta$:

$$\Omega\sigma\Delta\eta\ \text{ Joules, or }\ \Omega\Delta\eta\ \text{ J/m}^2$$

(σ = surface area of Earth.)

Energy needed to move the ice line:

$$\Omega\frac{d\eta}{dt}\ \text{ J/yr/m}^2$$

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Budyko-Sellers-Widiasih Model

$$\frac{d\eta}{dt}=\varepsilon\kappa(T_b-T_c)$$
$$\frac{\partial T}{\partial t}=\frac{\kappa}{R}\Big(Qs(y)\big(1-\alpha(y,\eta)\big)-(A+BT)+C(\overline{T}-T)\Big)$$

Energy to raise surface temperature: $R\frac{\partial T}{\partial t}$ J/yr/m^2

Energy to move ice line: $\Omega\frac{d\eta}{dt}$ J/yr/m^2

$$R\frac{\partial T}{\partial t}+\Omega\frac{d\eta}{dt}=\kappa\Big(Qs(y)\big(1-\alpha(y,\eta)\big)-(A+BT)+C(\overline{T}-T)\Big)$$

Altogether:

$$\frac{d\eta}{dt}=\varepsilon\kappa(T_b-T_c)$$
$$\frac{\partial T}{\partial t}=\frac{\kappa}{R}\Big(Qs(y)\big(1-\alpha(y,\eta)\big)-(A+BT)+C(\overline{T}-T)-\varepsilon\Omega(T_b-T_c)\Big)$$

Ice-Albedo Feedback

Budyko-Sellers-Widiasih Model

$$\frac{d\eta}{dt}=\varepsilon\kappa(T_b-T_c)$$
$$\frac{\partial T}{\partial t}=\frac{\kappa}{R}\Big(Qs(y)\big(1-\alpha(y,\eta)\big)-(A+BT)+C(\overline{T}-T)-\varepsilon\Omega(T_b-T_c)\Big)$$

State Space

η lives in [0,1].
T lives in a space of functions on [0,1].

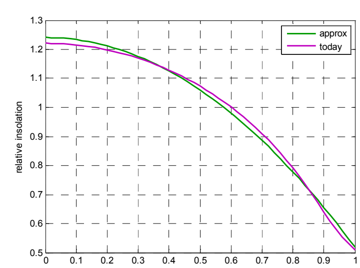
What space of functions?

Ice-Albedo Feedback

Relative Insolation Function

green = quadratic approximation (Tung and North)

mauve = formula using obliquity of 23.5°



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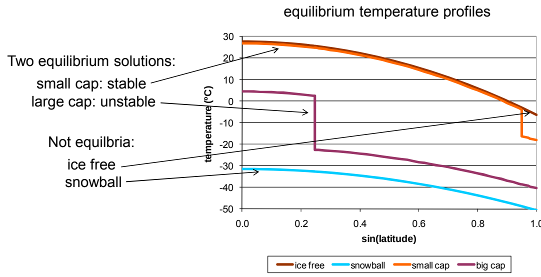
If s is assumed to be quadratic, then the equilibrium solutions are piecewise quadratic with a discontinuity at the ice boundary.



Ice-Albedo Feedback

Budyko-Sellers-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon\kappa\left(T_b - T_c\right)$$

$$\frac{\partial T}{\partial t} = \frac{\kappa}{R}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A + BT\right) + C\left(\overline{T} - T\right) - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

State Space

What space of functions?

Assume that T lives in the space of functions on [0,1] which are piecewise quadratic with a discontinuity at the ice boundary.

$$T(y) = \begin{cases} U(y), & y < \eta, \\ V(y), & y > \eta, \\ \left(U(\eta) + V(\eta)\right)/2, & y = \eta, \end{cases}$$

where U and V are quadratic on [0,1] .

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Budyko-Sellers-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon\kappa\left(T_b - T_c\right)$$

$$\frac{\partial U}{\partial t} = \frac{\kappa}{R}\left(Qs(y)\left(1-\alpha_1\right) - \left(A + BU\right) + C\left(\overline{T} - U\right) - \varepsilon\Omega\left(T_b - T_c\right)\right), \quad y < \eta$$

$$\frac{\partial V}{\partial t} = \frac{\kappa}{R}\left(Qs(y)\left(1-\alpha_2\right) - \left(A + BV\right) + C\left(\overline{T} - V\right) - \varepsilon\Omega\left(T_b - T_c\right)\right), \quad y > \eta$$

Ice boundary temperature:

$$T_b = \frac{U(\eta) + V(\eta)}{2}$$

Global mean temperature:

$$\overline{T} = \int_0^\eta U(y,t)\,dy + \int_\eta^1 V(y,t)\,dy$$

Ice-Albedo Feedback

Budyko-Sellers-Widiasih Model

Legendre Polynomials

Since s is even, we assume that U and V are even, and expand using the first two even Legendre polynomials:

$$p_0(y) = 1$$

$$p_2(y) = \frac{1}{2}\left(3y^2 - 1\right)$$

$$U(y,t) = u_0(t)p_0(y) + u_2(t)p_2(y) = u_0(t) + u_2(t)p_2(y),$$

$$V(y,t) = v_0(t)p_0(y) + v_2(t)p_2(y) = v_0(t) + v_2(t)p_2(y),$$

$$s(y) = s_0 p_0(y) + s_2 p_2(y) = 1 + s_2 p_2(y).$$

$$\frac{\partial U}{\partial t} = \dot{u}_0 + \dot{u}_2 p_2(y), \qquad \left(\dot{u} = \frac{du}{dt}\right)$$

$$\frac{\partial V}{\partial t} = \dot{v}_0 + \dot{v}_2 p_2(y).$$

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$$\frac{\partial U}{\partial t} = \frac{\kappa}{R}\left(Qs(y)\left(1-\alpha_1\right) - \left(A + BU\right) + C\left(\overline{T} - U\right) - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\frac{\partial V}{\partial t} = \frac{\kappa}{R}\left(Qs(y)\left(1-\alpha_2\right) - \left(A + BV\right) + C\left(\overline{T} - V\right) - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

becomes

$$\dot{u}_0 + \dot{u}_2 p_2(y) = \frac{\kappa}{R}\left(Q\left(1 + s_2 p_2(y)\right)\left(1-\alpha_1\right) - A + C\overline{T} - \left(B + C\right)\left(u_0 + u_2 p_2(y)\right) - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\dot{v}_0 + \dot{v}_2 p_2(y) = \frac{\kappa}{R}\left(Q\left(1 + s_2 p_2(y)\right)\left(1-\alpha_2\right) - A + C\overline{T} - \left(B + C\right)\left(v_0 + v_2 p_2(y)\right) - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

Equate coefficients:

$$\boxed{\begin{aligned} \dot{u}_0 &= \frac{\kappa}{R}\left(Q\left(1-\alpha_1\right) - A + C\overline{T} - \left(B + C\right)u_0 - \varepsilon\Omega\left(T_b - T_c\right)\right) \\ \dot{u}_2 &= \frac{\kappa}{R}\left(Qs_2\left(1-\alpha_1\right) - \left(B + C\right)u_2\right) \\ \dot{v}_0 &= \frac{\kappa}{R}\left(Q\left(1-\alpha_2\right) - A + C\overline{T} - \left(B + C\right)v_0 - \varepsilon\Omega\left(T_b - T_c\right)\right) \\ \dot{v}_2 &= \frac{\kappa}{R}\left(Qs_2\left(1-\alpha_2\right) - \left(B + C\right)v_2\right) \end{aligned}}$$

Ice-Albedo Feedback

Budyko-Sellers-Widiasih Model

Global Mean Temperature

$$\overline{T} = \int_0^1 T(y)\,dy = \int_0^\eta U(y)\,dy + \int_\eta^1 V(y)\,dy$$

$$= \int_0^\eta \left(u_0 + u_2 p_2(y)\right)dy + \int_\eta^1 \left(v_0 + v_2 p_2(y)\right)dy$$

$$= \eta u_0 + \left(1-\eta\right)v_0 + u_2\int_0^\eta p_2(y)\,dy + v_2\int_\eta^1 p_2(y)\,dy$$

Recall: $p_2(y) = \frac{1}{2}\left(3y^2 - 1\right)$

Hence:

$$\int_0^\eta p_2(y)\,dy = \frac{1}{2}\left(\eta^3 - \eta\right) = P_2(\eta), \qquad \int_\eta^1 p_2(y)\,dy = -P_2(\eta)$$

$$\boxed{\overline{T} = \eta u_0 + \left(1-\eta\right)v_0 + P_2(\eta)\left(u_2 - v_2\right)}$$

Ice-Albedo Feedback

Budyko-Sellers-Widiasih Model

Ice Line Temperature

$$T_b = \frac{U(\eta)+V(\eta)}{2} = \frac{1}{2}\left(u_0 + u_2 p_2(\eta) + v_0 + v_2 p_2(\eta)\right)$$

$$\boxed{T_b = \frac{u_0+v_0}{2} + \frac{u_2+v_2}{2} p_2(\eta)}$$

Ice-Albedo Feedback

Budyko-Sellers-Widiasih Model

Summary

$$\dot{\eta} = \varepsilon\kappa\left(T_b - T_c\right)$$

$$\dot{u}_0 = \frac{\kappa}{R}\left(Q(1-\alpha_1) - A + C\overline{T} - (B+C)u_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\dot{v}_0 = \frac{\kappa}{R}\left(Q(1-\alpha_2) - A + C\overline{T} - (B+C)v_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\dot{u}_2 = \frac{\kappa}{R}\left(Qs_2(1-\alpha_1) - (B+C)u_2\right)$$

$$\dot{v}_2 = \frac{\kappa}{R}\left(Qs_2(1-\alpha_2) - (B+C)v_2\right)$$

$$T_b = \frac{u_0+v_0}{2} + \frac{u_2+v_2}{2} p_2(\eta)$$

$$\overline{T} = \eta u_0 + (1-\eta)v_0 + P_2(\eta)(u_2 - v_2)$$

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Budyko-Sellers-Widiasih Model

Quadratic Modes

$$\dot{u}_2 = \frac{\kappa}{R}\left(Qs_2(1-\alpha_1) - (B+C)u_2\right)$$

$$\dot{v}_2 = \frac{\kappa}{R}\left(Qs_2(1-\alpha_2) - (B+C)v_2\right)$$

The dynamics of u_2 and v_2 are independent of each other and of the other variables (including η).
These variables decay exponentially to

$$u_2^* = \frac{Qs_2(1-\alpha_1)}{(B+C)} \qquad v_2^* = \frac{Qs_2(1-\alpha_2)}{(B+C)}$$

Exercise: All higher order modes behave the same way.

Ice-Albedo Feedback

Budyko-Sellers-Widiasih Model

Globally Attracting 3D Subspace

$$\boxed{\begin{aligned}
\dot{\eta} &= \varepsilon\kappa\left(T_b - T_c\right)\\
\dot{u}_0 &= \frac{\kappa}{R}\left(Q(1-\alpha_1) - A + C\overline{T} - (B+C)u_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)\\
\dot{v}_0 &= \frac{\kappa}{R}\left(Q(1-\alpha_2) - A + C\overline{T} - (B+C)v_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)
\end{aligned}}$$

where

$$T_b = \frac{u_0+v_0}{2} + \frac{u_2^*+v_2^*}{2} p_2(\eta) = \frac{u_0+v_0}{2} + p_2(\eta)\frac{Qs_2(1-\alpha_0)}{B+C}, \quad \alpha_0 = \frac{\alpha_1+\alpha_2}{2}$$

$$\overline{T} = \eta u_0 + (1-\eta)v_0 + P_2(\eta)\left(u_2^* - v_2^*\right) = \eta u_0 + (1-\eta)v_0 + P_2(\eta)\frac{Qs_2(\alpha_2-\alpha_1)}{B+C}$$

Ice-Albedo Feedback

Budyko-Sellers-Widiasih Model

Globally Attracting 3D Subspace
Summary

$$\dot{\eta} = \varepsilon\kappa\left(T_b - T_c\right)$$

$$\dot{u}_0 = \frac{\kappa}{R}\left(Q(1-\alpha_1) - A + C\overline{T} - (B+C)u_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

$$\dot{v}_0 = \frac{\kappa}{R}\left(Q(1-\alpha_2) - A + C\overline{T} - (B+C)v_0 - \varepsilon\Omega\left(T_b - T_c\right)\right)$$

Ice Boundary Temperature

$$T_b = \frac{u_0+v_0}{2} + p_2(\eta)\frac{Qs_2(1-\alpha_0)}{B+C}$$

Global Mean Temperature

$$\overline{T} = \eta u_0 + (1-\eta)v_0 + P_2(\eta)\frac{Qs_2(\alpha_2-\alpha_1)}{B+C}$$