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Bifurcation Analysis for Minimal Complexity PaleoClimate Modeling

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Math and Climate Research Network

The Model

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A Low-Order Dynamical Model of Global Climatic Variability Over the Full Pleistocene

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The Model

$$\dot{I}' = a_0 I' - a_1 \mu' - a_2 M(t)$$

$$\dot{\mu}' = b_1 \mu' - (b_2 - b_3 N') N' - b_4 N'^2 \mu'$$

$$\dot{N}' = -c_0 I' - c_2 N'$$

- I = global ice mass
- N = North Atlantic Deep Water
- μ = Atmospheric CO_2
- $a_{0,1}$ $b_{1,2,3,4}$ and $c_{1,2}$ > O
- M(t) = Milankovitch Forcing (65˚ normalized to O mean and unit variance)
- Primes denote departures from an eq. state controled by possible ultraslow variation of solar constant or the tectonic state of the Earth.

The Model

Reduction:

$$\dot{X} = -X - Y - uM(t^*)$$
$$\dot{Y} = -pZ + rY + sZ^2 - Z^2Y$$
$$\dot{Z} = -q(X + Z)$$

Substitutions:

$$\mu' = \left[\frac{c_2}{a_1 c_0}\sqrt{\frac{a_0}{b_4}}\right] Y$$
$$N' = \left[\sqrt{\frac{a_0}{b_4}}\right] Z$$
$$I' = \left[\frac{c_2}{c_0}\sqrt{\frac{a_0}{b_4}}\right] X$$

Original Dynamical System:

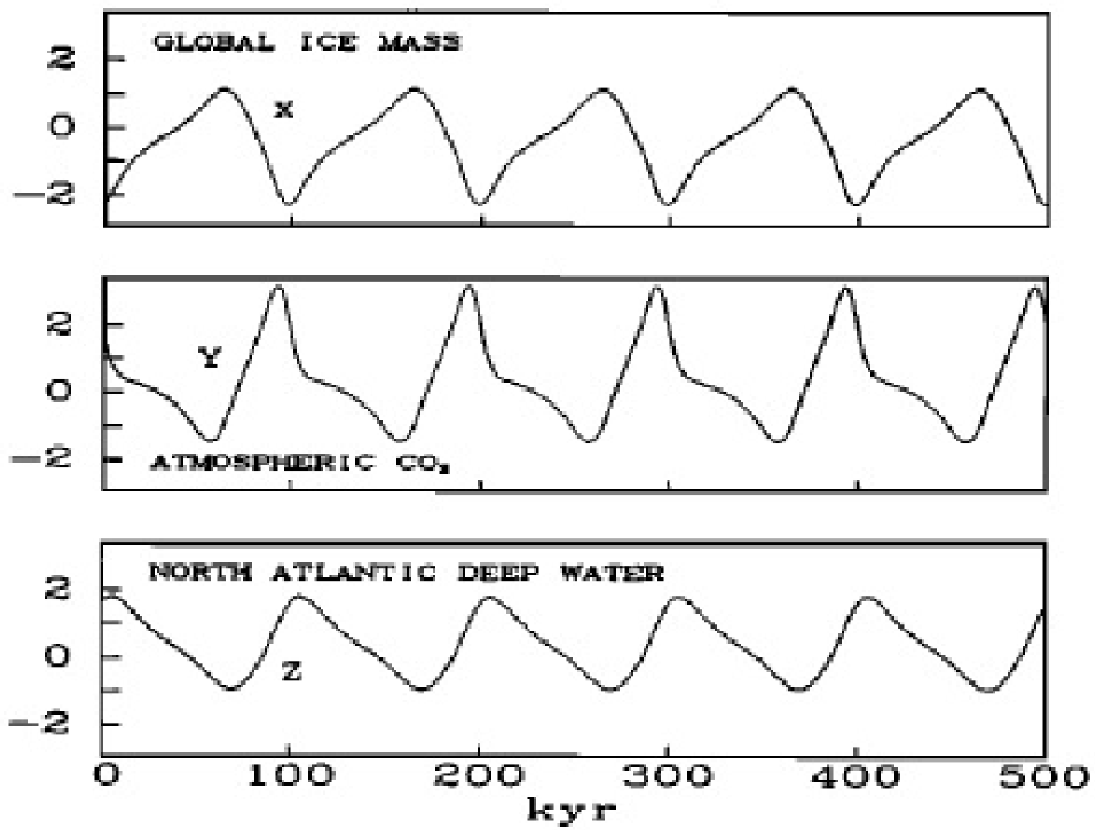
$$\dot{I}' = a_0 I' - a_1 \mu' - a_2 M(t)$$
$$\dot{\mu}' = b_1 \mu' - (b_2 - b_3 N')N' - b_4 N'^2 \mu'$$
$$\dot{N}' = -c_0 I' - c_2 N'$$

where $p = \frac{a_1 c_0 b_2}{a_0^2 c_2}$, $q = \frac{c_2}{a_0}$, $r = \frac{b_1}{a_0}$, $s = \frac{a_i b_3 c_0 \sqrt{a_0^3 b_4}}{c_2}$, and $u = \frac{a_2 c_0 \sqrt{\frac{b_4}{a_0^3}}}{c_2}$.

The Model

Reference Parameters:

$(p, q, r, s) = (1.0, 1.2, 0.8, 0.8)$



Equilibrium Solutions

1. Let u = O.
2. Set $\dot{X} = \dot{Y} = \dot{Z} = 0$ and solve

$$0 = -X - Y - uM(t^*)$$
$$0 = -pZ + rY + sZ^2 - Z^2Y$$
$$0 = -q(X + Z)$$

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$$0 = -q(X + Z)$$

3. -X = Y = Z
4. $$\begin{aligned} 0 &= pX - rX + sX^2 + X^3 \\ &= X(X^2 + sX + (p - r)) \end{aligned}$$

Equilibrium Solutions

$$
\begin{aligned}
0 &= pX - rX + sX^2 + X^3 \\
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$$
X_0 = 0
$$

$$
X_{1,2} = \frac{-s \pm \sqrt{s^2 - 4(p-r)}}{2}.
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$$

- Thus for each point in the parameter space (p,q,r,s) there are 3 eq. solutions.
- For each of the 3 eq. pts are 3 eigenvalues, $\lambda_{1,2,3}$ for which Re(λ) will determine the stability of that eq. pt.

Linearised System

- To determine eigenvalues we must consider the linearised system at a given eq. point. By definition, the linearised system is:

$$\hat{f}(X,Y,Z) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = Df|_{\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$$

Linearised System

Reduced Dynamical System:

$$\dot{X} = -X - Y - uM(t^*)$$
$$\dot{Y} = -pZ + rY + sZ^2 - Z^2Y$$
$$\dot{Z} = -q(X + Z)$$

$$Df = \begin{bmatrix} -1 & -1 & 0 \\ 0 & r - Z^2 & (-p + 2sZ - 2YZ) \\ -q & 0 & -q \end{bmatrix}$$

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Recall
-X=Y=Z

$$\Rightarrow \quad Df = \begin{bmatrix} -1 & -1 & 0 \\ 0 & r - X^2 & (-p - 2sX - 2X^2) \\ -q & 0 & -q \end{bmatrix}$$

Linearised System

- Thus to linearise about $(-\alpha, \alpha, \alpha)$:

$$\hat{f}_{(\alpha,-\alpha,-\alpha)}(X,Y,Z) = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & r-\alpha^2 & (-p+2s\alpha-2\alpha^2) \\ -q & 0 & -q \end{bmatrix} \begin{bmatrix} X-\alpha \\ Y-\alpha \\ Z-\alpha \end{bmatrix}$$

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- Recall if Re(λ) < O for all λ then the eq. pt is stable.
- If Re(λ) >O for any λ then the eq. pt is unstable.
- We must now solve: $|Df - \lambda I| = 0$

Eigenvalues

- Characteristic Polynomial:

$$\lambda^3 + (1 + q + X^2 - r)\lambda^2 + (q(1 + X^2 - r) - r + X^2)\lambda + q(p + 2sX + X^2 - r)$$

- For reference parameters (p,q,r,s) = (1,1.2,0.8,0.8):

$$\lambda^3 + (1.4 + X^2)\lambda^2 + (2.2X^2 - 0.56)\lambda + 3.6X^2 + 1.92X + 0.24 = 0$$

- Solving λ for each of the three eq. pts:

eq pt	λ_1	λ_2	λ_3
$(0,0,0)$	-1.7882	$0.0194 - 0.3107i$	$0.0194 + 0.3107i$
$(-0.4 + 0.2i, 0.4 - 0.2i, 0.4 - 0.2i)$	$-1.6625 + 0.1665i$	$-0.2408 - 0.1305i$	$0.3834 + .02739i$
$((-0.4 - 0.2i, 0.4 + 0.2i, 0.4 + 0.2i)$	$-1.6625 - 0.1665i$	$-0.2408 + 0.1305i$	$0.3834 - .02739i$

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- The system is hyperbolic for this parameter, thus the linearised system is an accurate representation for the non-linear system locally.

Eigenvalues

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- The origin is spirally unstable with a 2D unstable space and a 1D stable space.
- The other two points don't have any physical meaning because the eq. pts are complex valued.

Eigenvalues

- Considering the system as a function of p. Now try to understand how the stability of the system changes as p changes.

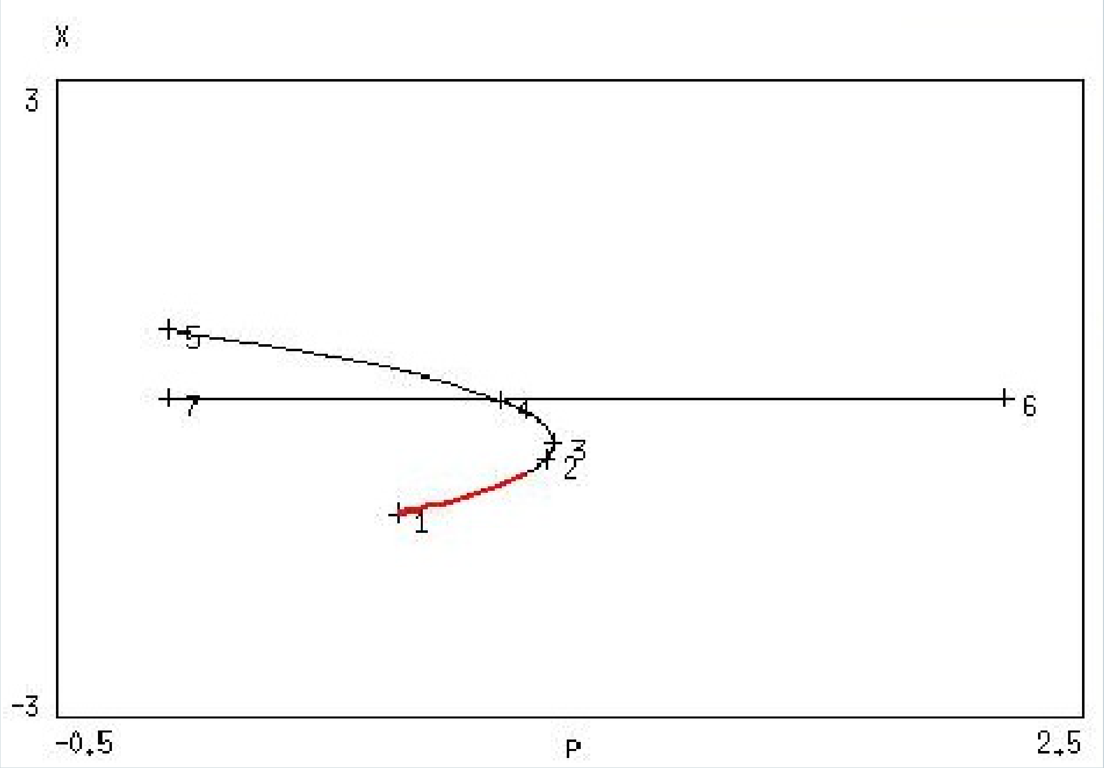
Eigenvalues

- Considering the system as a function of p. Now try to understand how the stability of the system changes as p changes.

- Fix q ,r and s at reference values.
- Initially we can see eq pts $X_{2,3}$ are only real for $p < 0.96$

$$X_{1,2} = \frac{-s \pm \sqrt{s^2 - 4(p-r)}}{2}.$$

Bifurcations



Bifurcations

Claim: For eq. pt. X2= $\frac{(-s-\sqrt{s^2-4*(p-r)})}{2}$ there exists a Hopf's Bifurcation between p = 0.9353 & p = 0.9354.

Proof: We will use the following theorem

A Hopf's bifurcation occurs when all eigenvalues of Df have $Re(\lambda_0) < O$ except one conjugate pair $\lambda_{1,2} = \iota\omega$.

Bifurcation

For (p,q,r,s)= (0.9353, 1.2, 0.8, 0.8) the eigenvalues are:

$\lambda == -1.71039$ $\lambda == -0.0000220078 \pm 0.350529i$

For (p,q,r,s)= (0.9354, 1.2, 0.8, 0.8) the eigenvalues are:

$\lambda == -1.7103$ $\lambda == 0.000114597 \pm 0.350082i$

1. $\mathrm{Re}(\lambda_0) < 0$ as required.
2. The next claim is that the $\mathrm{Re}(\lambda_{1,2}) = 0$ at some point $0.9353 < \boldsymbol{p} < 0.9354$.

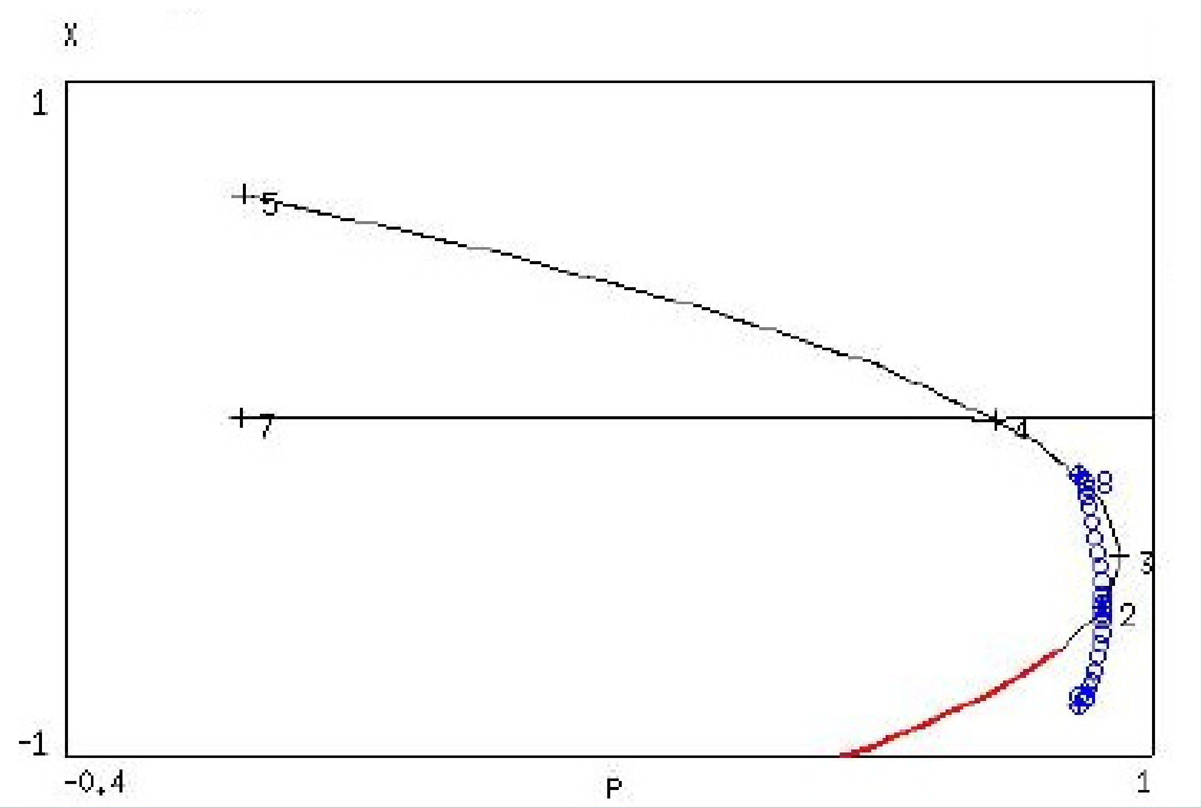
Bifurcations

- We can view the system completely as a function of p.

$$Df = \begin{bmatrix} -1 & -1 & 0 \\ 0 & r-(X(p))^2 & (-p-2s(X(p))-2(X(p))^2) \\ -q & 0 & -q \end{bmatrix}$$

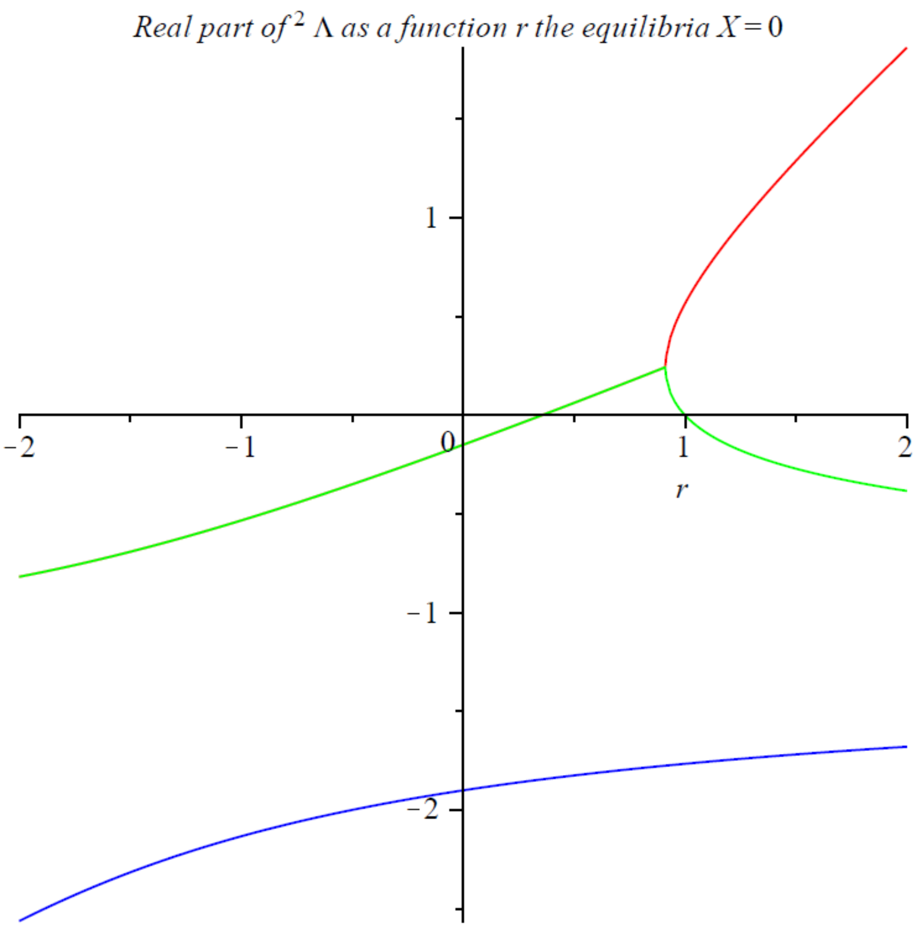
- X(p) is a continuous function of p.
- Det[Df] is thus a continuous function of p.
- The Re(λ_i) are continuous with respect to p.
- Thus by Intermediate Value Theorem there exists a ***p***, $0.9353 < \boldsymbol{p} < 0.9354$ such that Re($\lambda_{1,2}$) = 0. $\square$

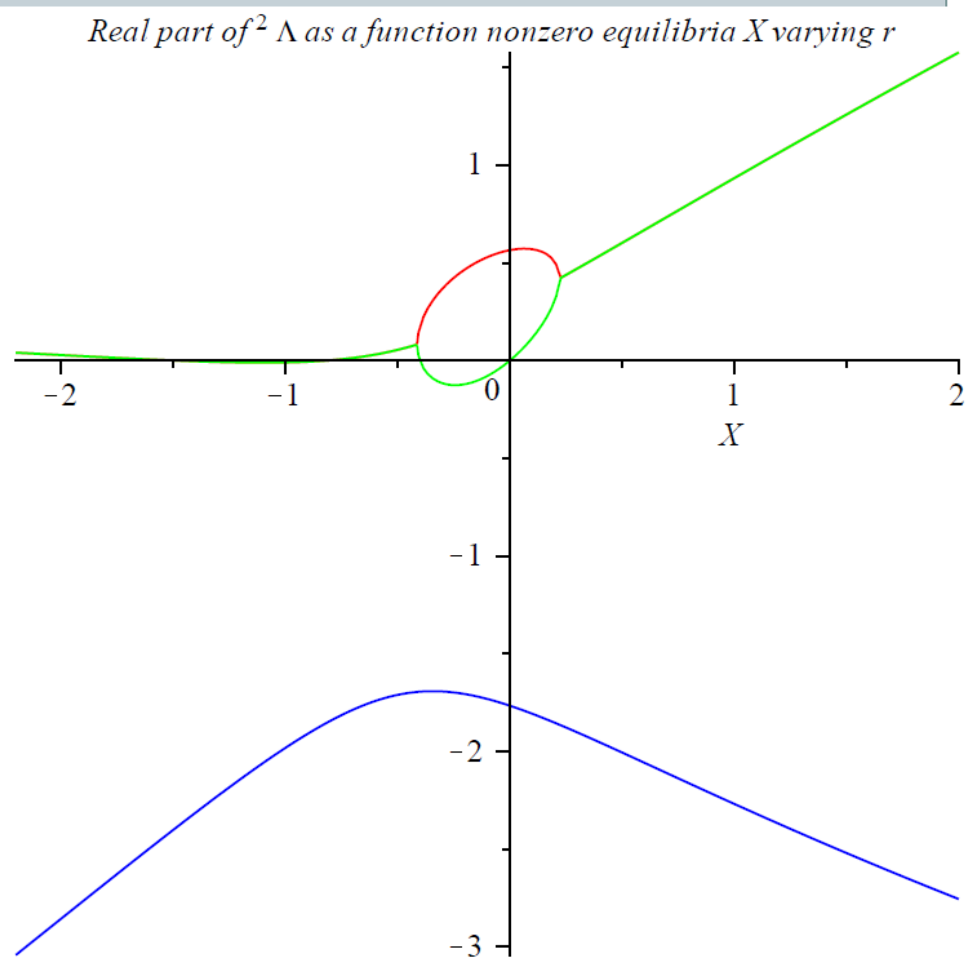
Bifurcations



- Next we'll take a quick review of the bifurcation diagrams for r and s.

Bifurcations





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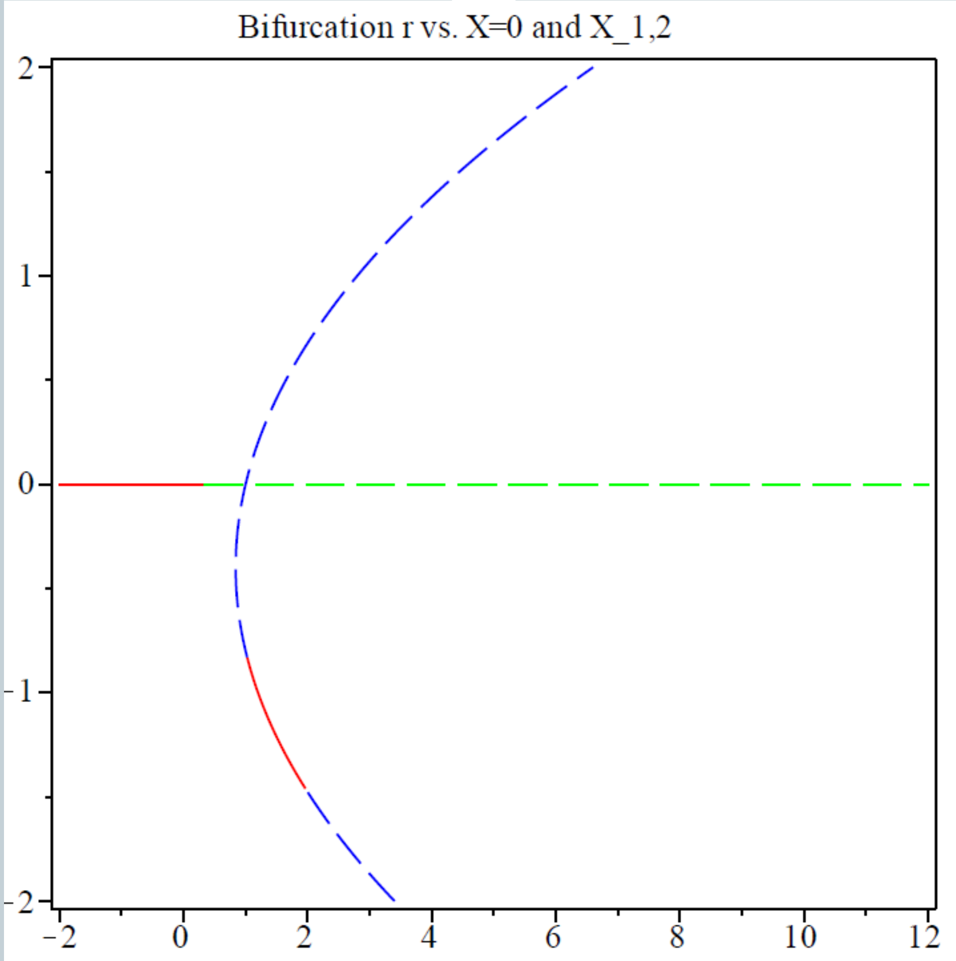

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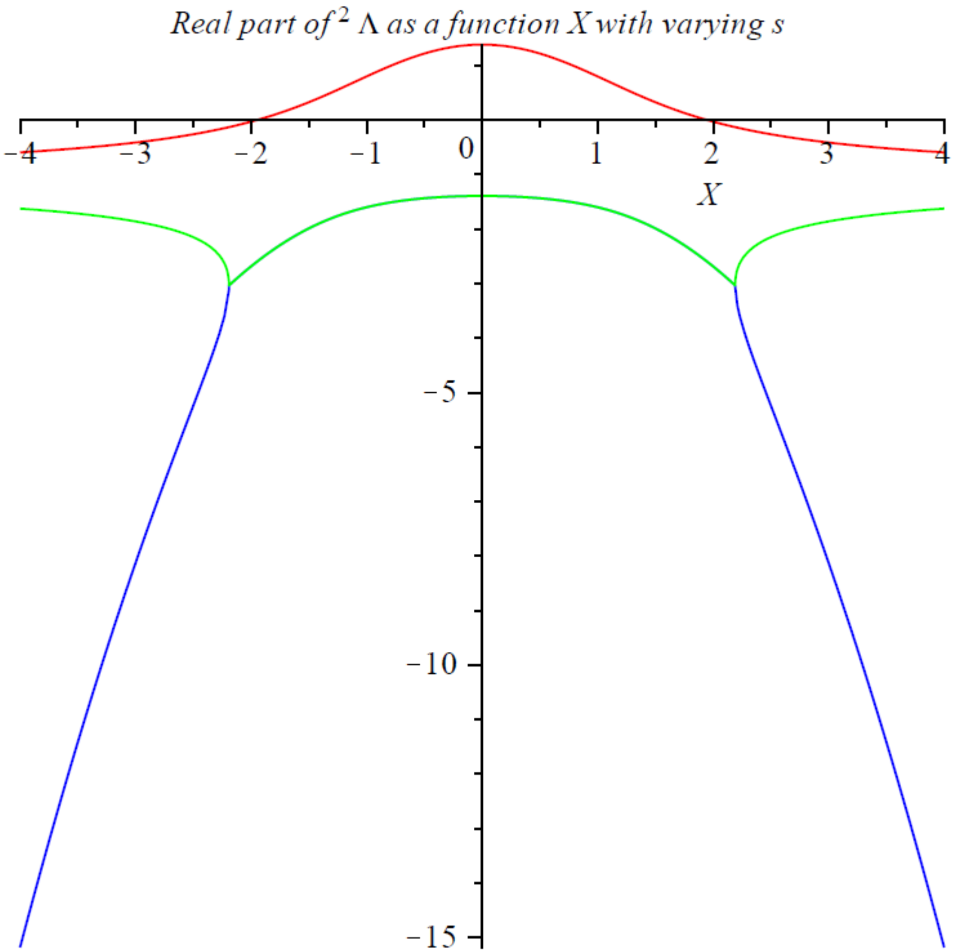

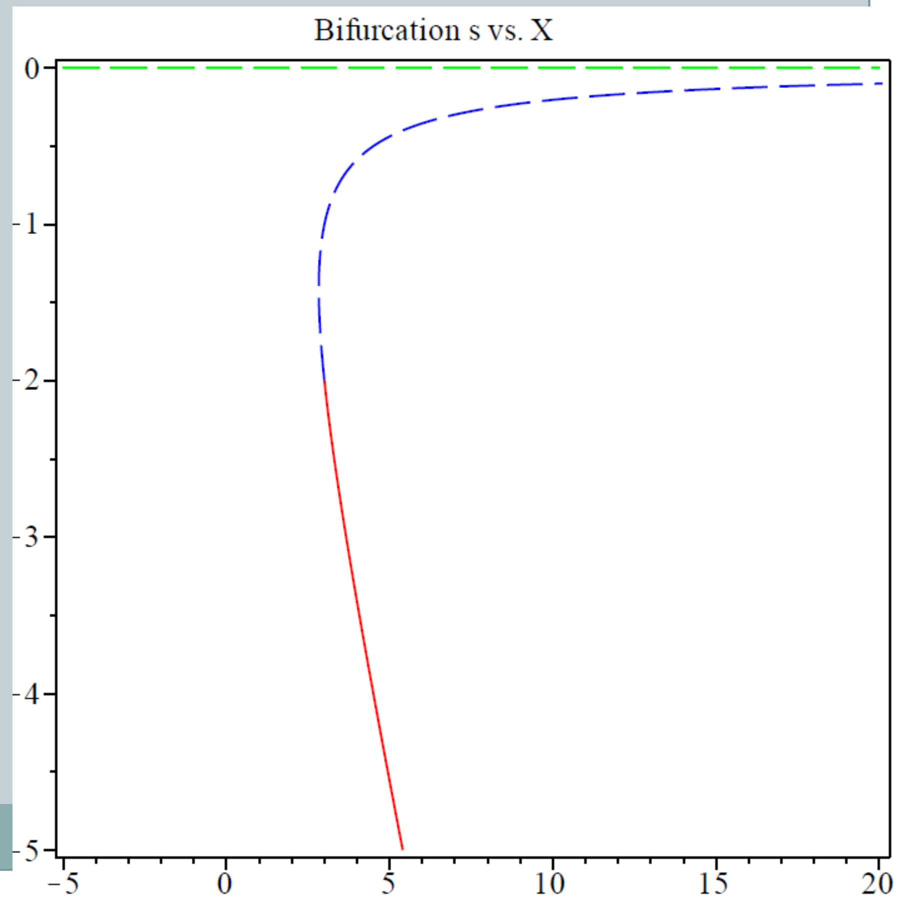

Varying Parameters

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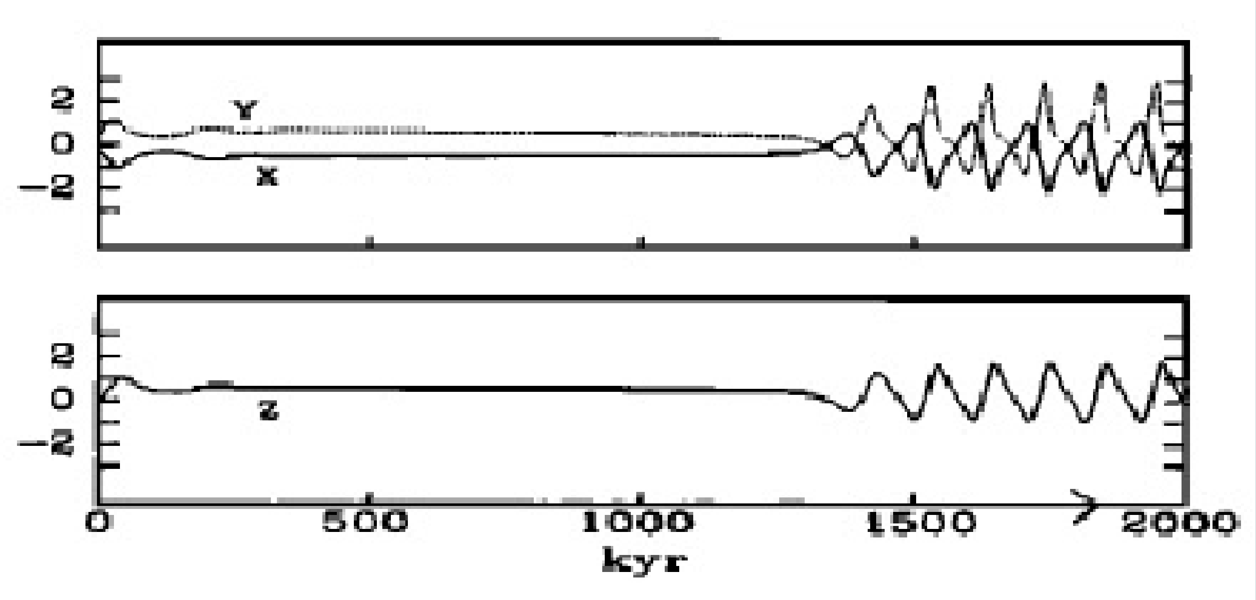
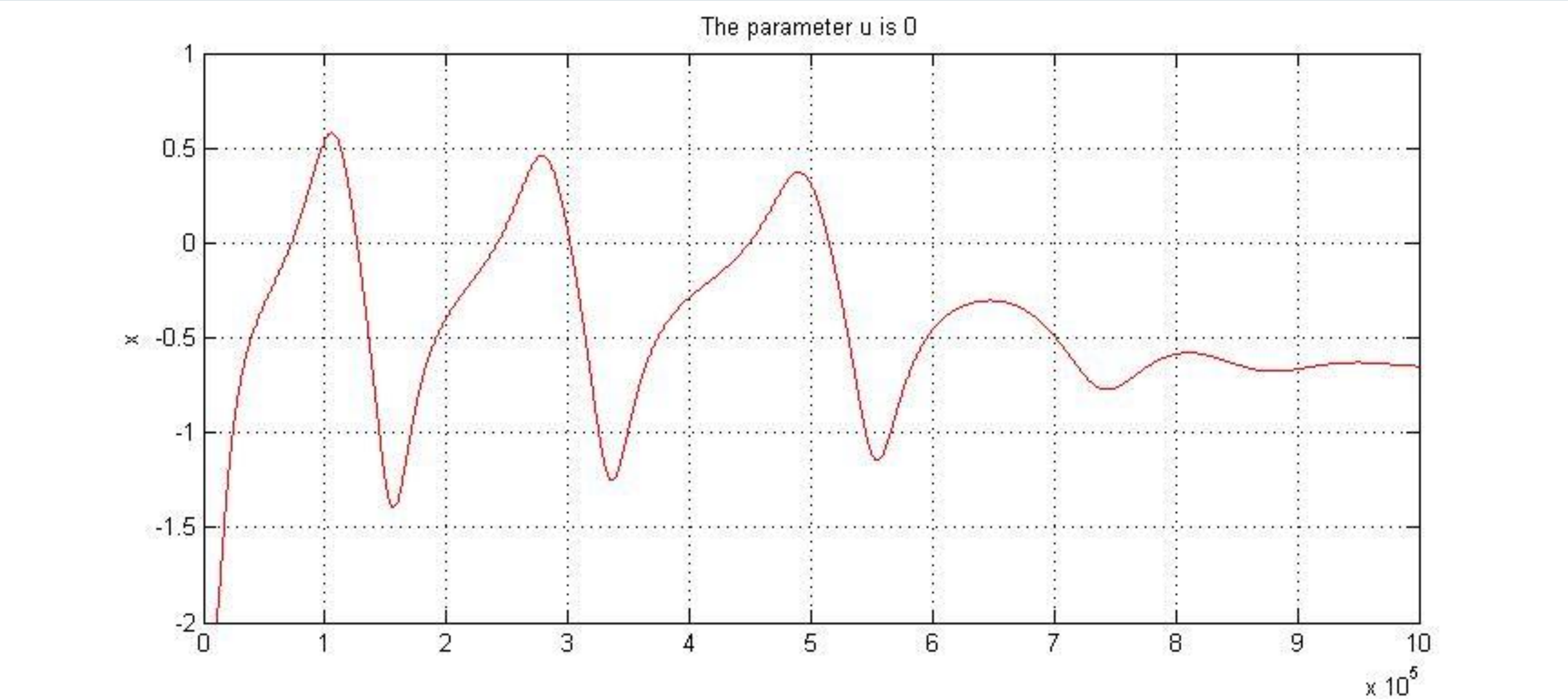
- Below is the **published** solution curve for q = 1.2, s = 0.8 and p and r linearly varying between 0.8 -> 1 and 0.7 ->0.8 respectively.



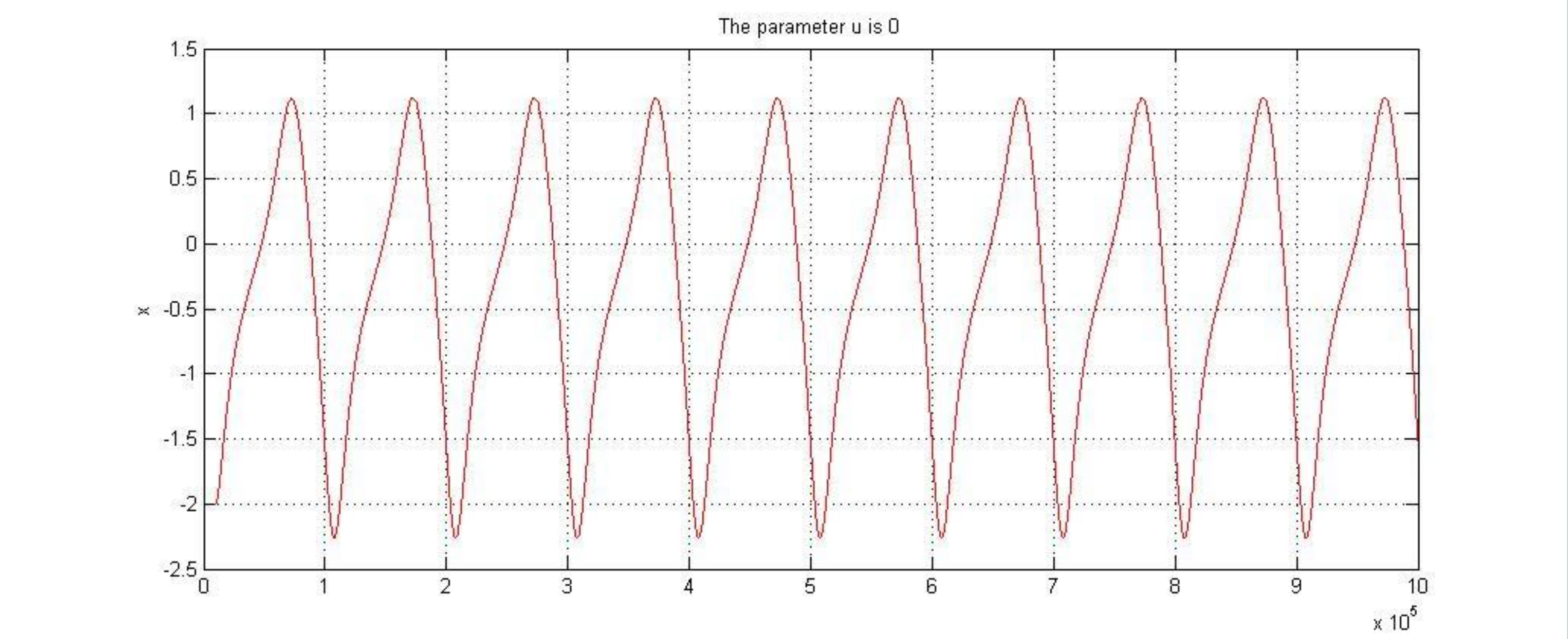
Varying Parameters

- Below is **my** solution curve for q = 1.2, s = 0.8 and p and r linearly varying between 0.8 -> 1 and 0.7 ->0.8 respectively.



Varying Parameters

- Below is **my** solution curve for q = 1.2, s = 0.8 and p and r linearly varying between **1 -> .08** and **0.8 ->0.7** respectively.



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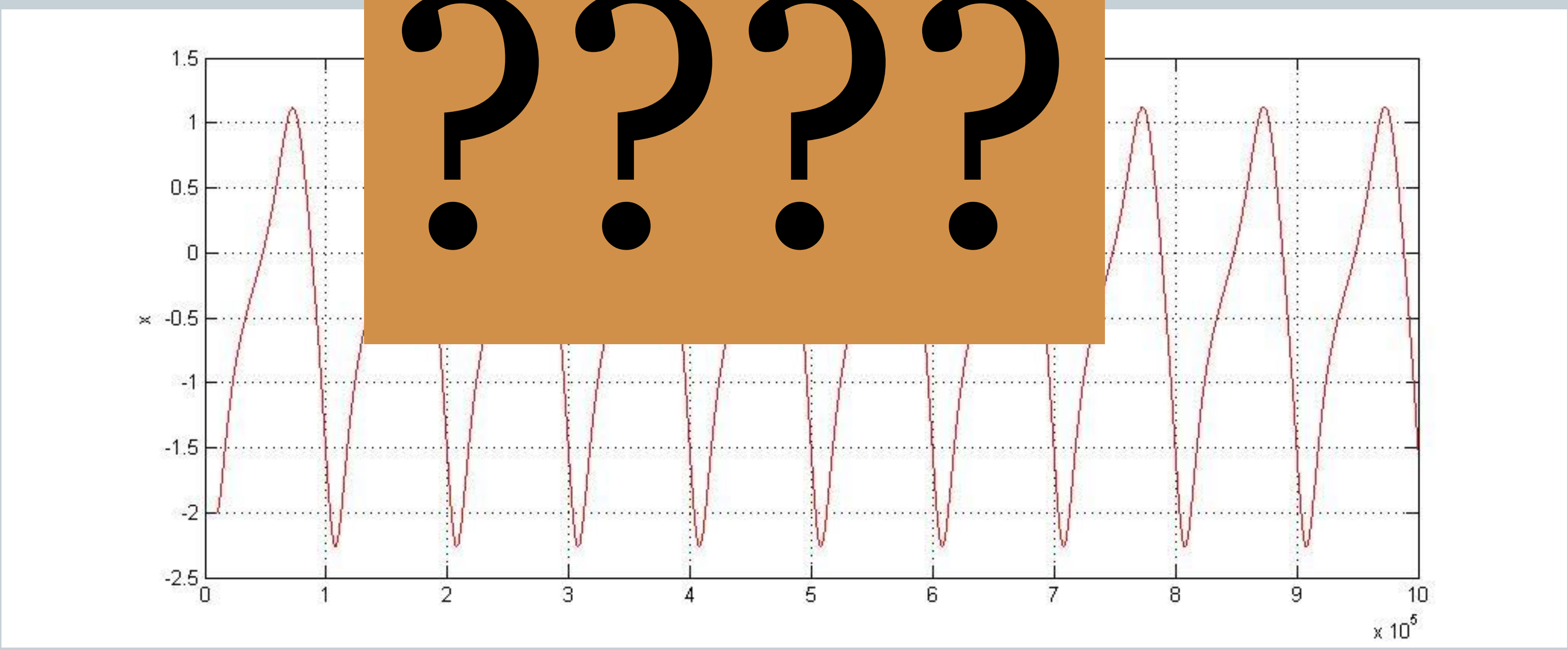


Varying Parameters

- We parameterize p and r:

$$p(\alpha) = 0.8 + 0.2\alpha \qquad r(\alpha) = 0.7 + 0.1\alpha$$

- We can view the system as continuous with respect to α.

$$Df = \begin{bmatrix} -1 & -1 & 0 \\ 0 & r(\alpha) - (X(\alpha))^2 & (-p(\alpha) - 2s(X(\alpha)) - 2(X(\alpha))^2) \\ -q & 0 & -q \end{bmatrix}$$

Varying Parameters

$Re(\lambda_0) < 0$,

$Re(\lambda_{1,2}) > 0$,

$\lambda_{1,2}$ in **C**,

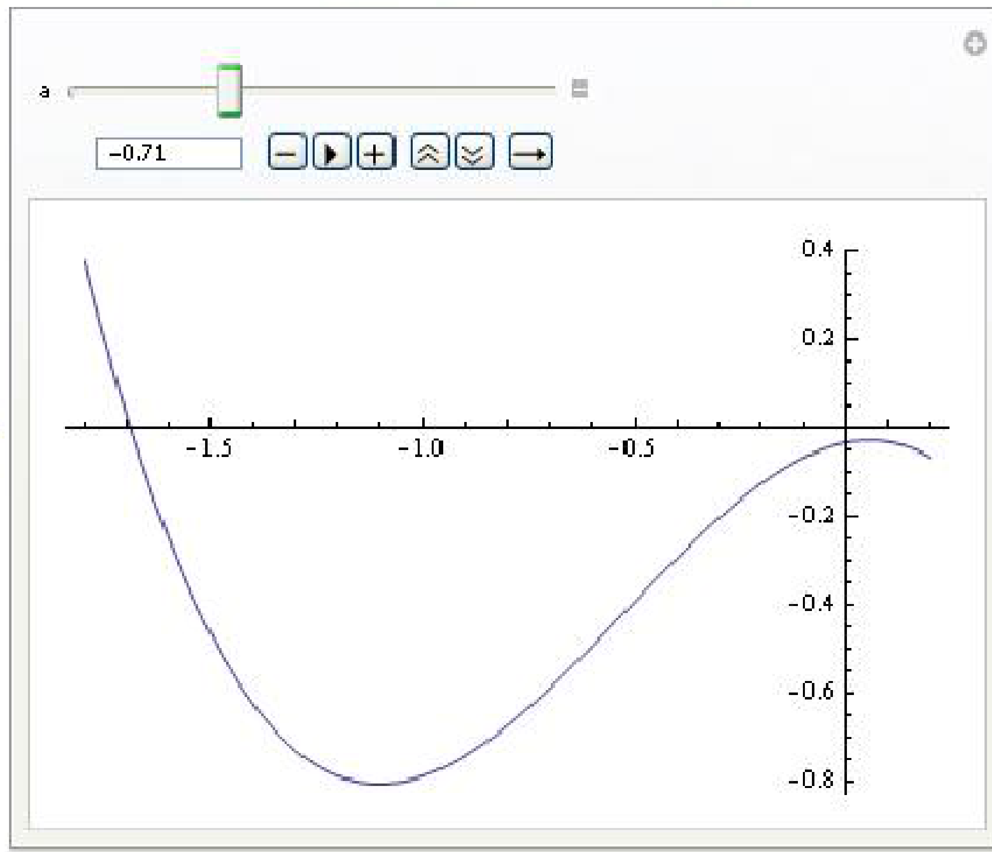
for all α in (0,1)

No interesting dynamics due to eigenvalues.

```
In[224]:= g[a_, λ_] := -λ^3 + (-1-q+r[a])*λ^2 + (-q+r[a]+q*r[a])*λ -
            p[a]*q + q*r[a]

In[255]:= Manipulate[Plot[g[a, λ], {λ, -1.8, .2}], {a, -2, 2}]
```

Out[255]=



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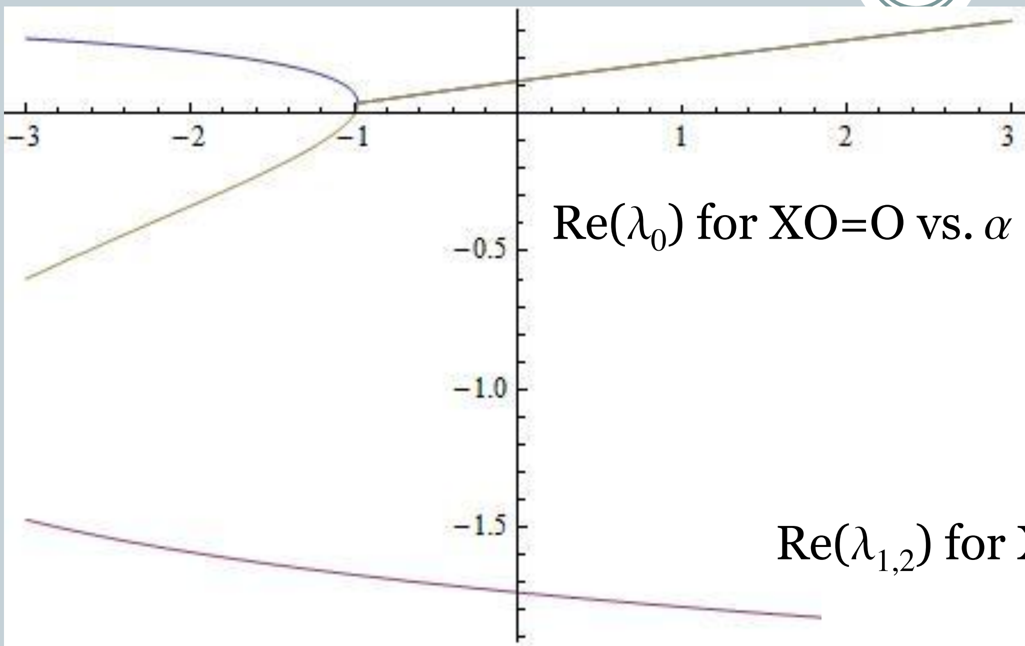
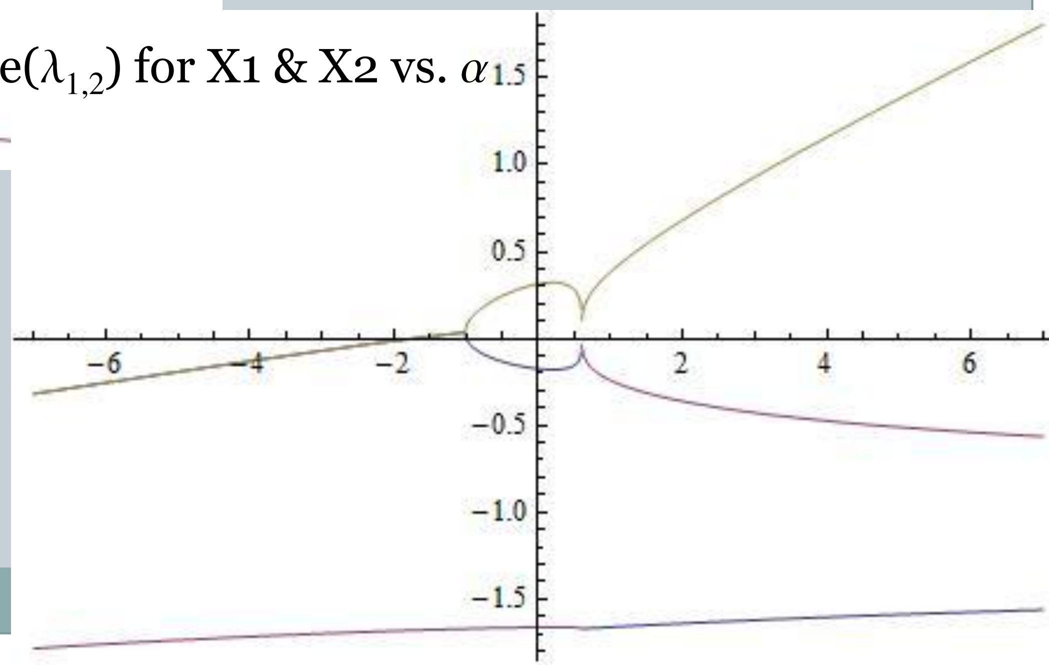
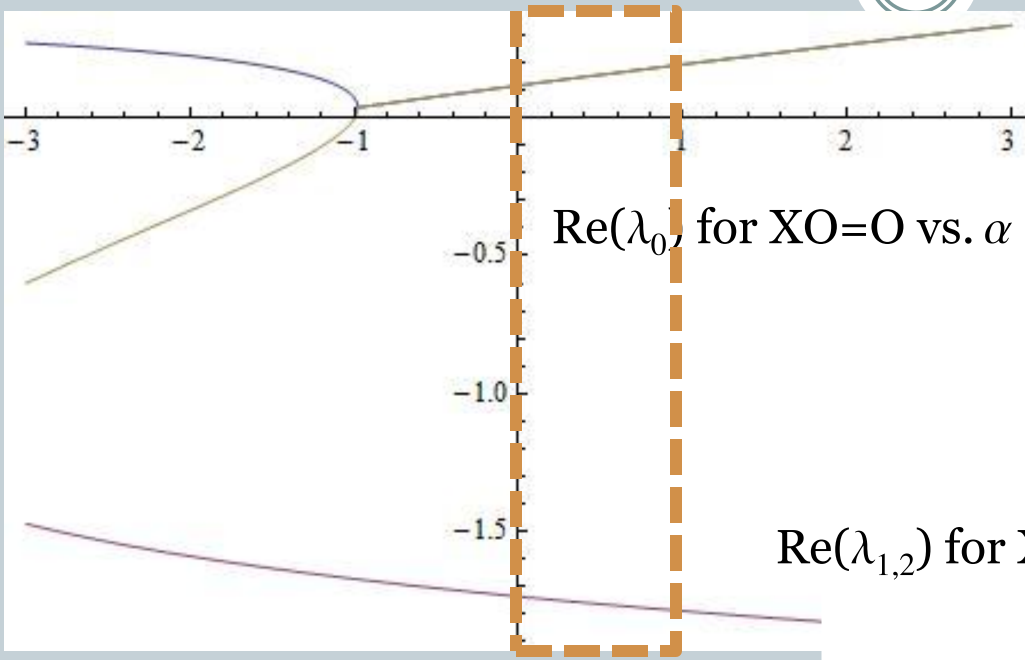


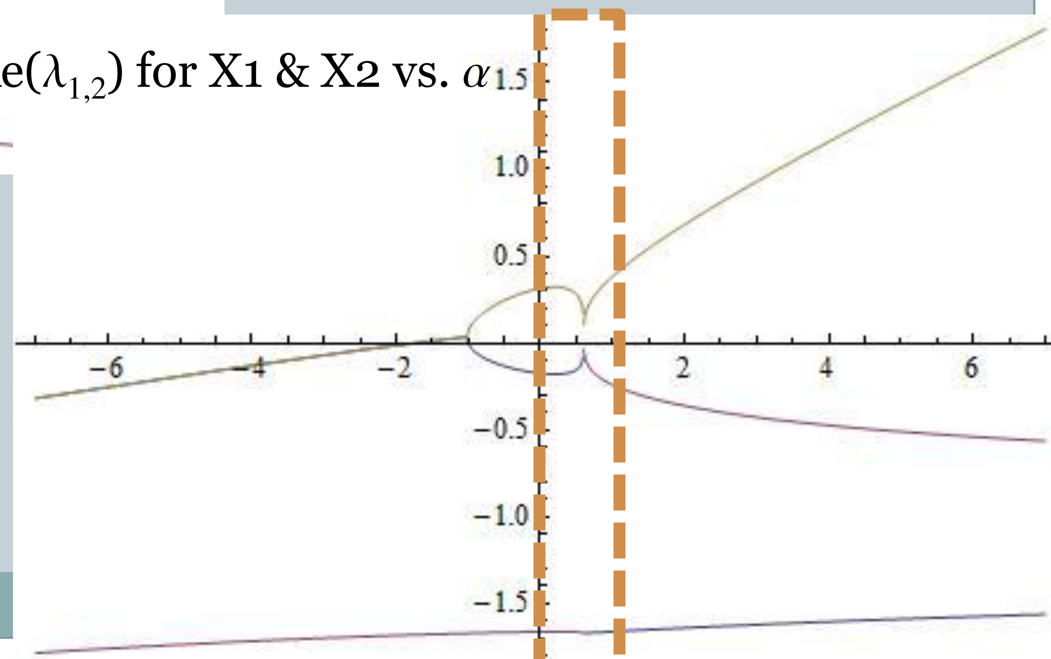


The only possible stable eq. pts. Are when α < -2. This is well outside the published range of [0,1].

Varying Parameters



Re(λ_0) for XO=O vs. α



Re($\lambda_{1,2}$) for X1 & X2 vs. α

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Conclusions

- Changing α seems to cause global system changes which can not be captured in the standard local bifurcation approach.

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- Changing α seems to cause global system changes which can not be captured in the standard local bifurcation approach.
- Despite any errors, the main concept that Maasch and Saltzman present with respect to bifurcation values is still valid.
- It is likely that there exists a small parameter shift that would cause a large change in the oscillations of the system.