Widiasih's Theorem and a Few of its Consequences

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University of Minnesota
Driven to Discover

MCRN
Mathematics and Climate Research Network

Budyko-Sellers Energy Balance Model

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T})$$

y =sine(latitude) ($y=1$ north pole; $y=0$ equator; $y=-1$ south pole)

$T(y)$ – annual average surface temperature at latitude y

$\overline{T}$ – global annual average temperature; $\overline{T} = \frac{1}{2}\int_{-1}^{1} T(y)dy$

Q – annual global average insolation for entire Earth

$s(y)$ – distribution of insolation over latitude; $\int_0^1 s(y)dy = 1$

R – heat capacity of Earth's surface

$\alpha_\eta(y) = \alpha(y,\eta) \in (0,1)$ – surface albedo at latitude y, parameter η

M. I. Budyko, The effect of solar radiation variation on the climate of the Earth, *Tellus* **21** (1969), 611-619.

W. D. Sellers, A global climatic model based on the energy balance of the Earth-atmosphere system, *Journal of Applied Meteorology* **8** (1969), 392-400.

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T})$$

Equilibria: $T^* = T^*(y,\eta)$

$$Qs(y)(1-\alpha(y,\eta)) - (A+BT^*) - C(T^*-\overline{T^*}) = 0 \quad \longleftarrow \frac{1}{2}\int_{-1}^{1}(\cdots)dy$$

$$Q(1-\overline{\alpha}(\eta)) - (A+B\overline{T^*}) - C(\overline{T^*}-\overline{T^*}) = 0, \quad \overline{\alpha}(\eta) = \frac{1}{2}\int_{-1}^{1}\alpha(y,\eta)s(y)dy$$

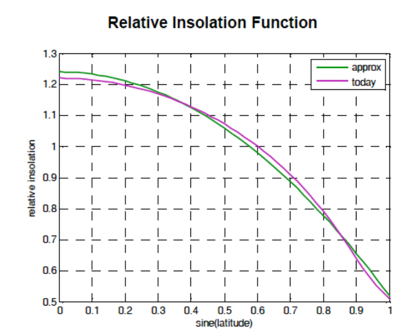
$$\overline{T^*} = \frac{1}{B}(Q(1-\overline{\alpha}(\eta)) - A)$$

$$T^* = T^*(y,\eta) = \frac{1}{B+C}(Qs(y)(1-\alpha(y,\eta)) - A + C\overline{T^*})$$

$$s(y) = 1 - (0.482)\frac{3y^2-1}{2}$$

green = quadratic approximation (Tung and North)

mauve = formula using obliquity of 23.5°



(slide by R. McGehee 11.09.11)

$$T^* = T^*(y,\eta) = \frac{1}{B+C}\left(Qs(y)(1-\alpha(y,\eta)) - A + C\overline{T^*}\right)$$

$$\overline{T^*} = \frac{1}{B}\left(Q(1-\overline{\alpha}(\eta)) - A\right)$$

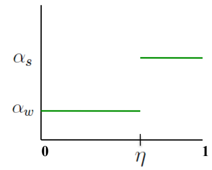
$$\overline{\alpha}(\eta) = \tfrac{1}{2}\int_{-1}^{1} \alpha(y,\eta)s(y)dy$$

Example 1

Assumptions:

- water world
- symmetry about the equator, so $y \in [0,1]$
- η = the iceline; ($y < \eta \rightarrow$ no ice, $y > \eta \rightarrow$ (snow covered) ice)
- $\alpha(y,\eta) = \begin{cases} \alpha_w, & y < \eta \\ \alpha_s, & y > \eta, \end{cases}$ $\quad \alpha_w$ – open water albedo, α_s – snow covered ice albedo, $\quad \alpha_w < \alpha_s$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T})$$



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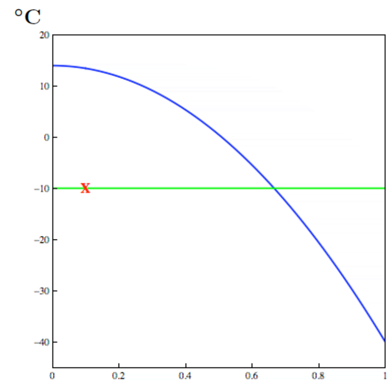

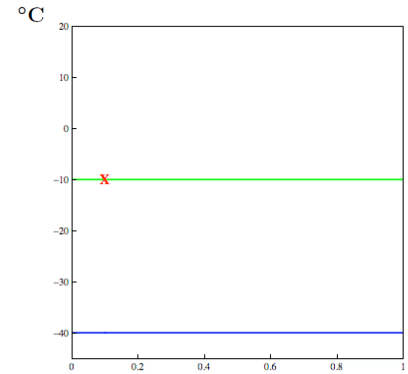

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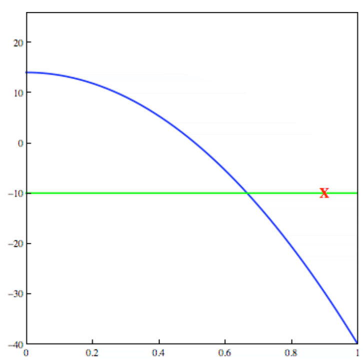
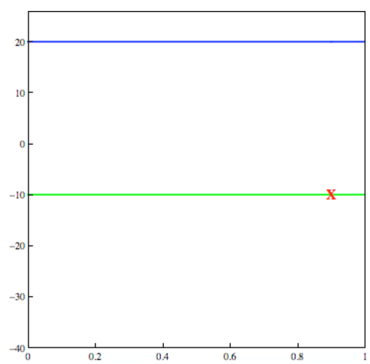




Esther Widiasih, *Dynamics of Budyko's Energy Balance Model*

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T}) \quad \longleftarrow \text{J/s/m}^2 \qquad K = \#\text{sec/yr}$$

Given $T_0 = T_0(y)$ set, for $n \geq 0$,

$$T_{n+1}(y) = T_n(y) + \frac{K}{R}\left(Qs(y)(1-\alpha(y,\eta)) - (A+BT_n(y)) - C\left(T_n(y) - \int_0^1 T_n(y)dy\right)\right)$$

$$\mathcal{B} = \{f : \mathbb{R} \rightarrow \mathbb{R} : \|f\|_\infty < \infty \text{ and } \mathrm{Lip}(f) < \infty\} \qquad \mathrm{Lip}(f) = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|}$$

$$\|f\|_{\mathcal{B}} = \max\{\ \|f\|_\infty, \mathrm{Lip}(f)\}$$

- $\alpha(y,\eta) \in \mathcal{B}$
- Embed domain of temperature profiles into $\mathbb{R}$

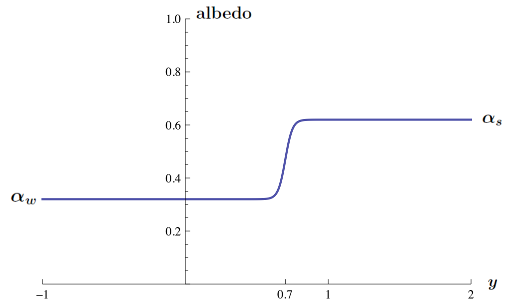
$$\frac{K}{R}\left(Qs(0)(1-\alpha(0,\eta)) - (A+BT(y)) - C\left(T(y) - \int_0^1 T(y)dy\right)\right), \quad y < 0$$

$$\frac{K}{R}\left(Qs(y)(1-\alpha(y,\eta)) - (A+BT(y)) - C\left(T(y) - \int_0^1 T(y)dy\right)\right), \quad 0 \leq y \leq 1$$

$$\frac{K}{R}\left(Qs(1)(1-\alpha(1,\eta)) - (A+BT(y)) - C\left(T(y) - \int_0^1 T(y)dy\right)\right), \quad y > 1$$

Esther Widiasih, *Dynamics of Budyko's Energy Balance Model*

$$\alpha(y,\eta) = \frac{\alpha_s + \alpha_w}{2} + \left(\frac{\alpha_s - \alpha_w}{2}\right)\tanh(M(y-\eta))$$



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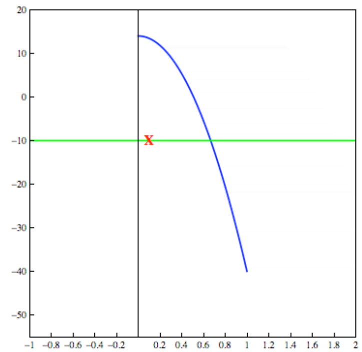

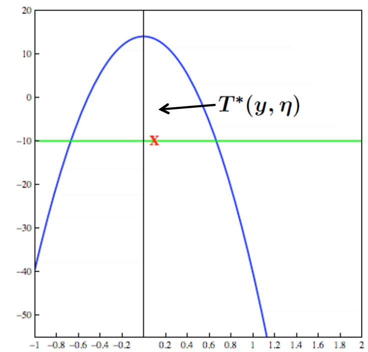

Let $\mathcal{T}^* = \{\ (T^*(y,\eta),\eta) : \eta \in \mathbb{R}\ \}$

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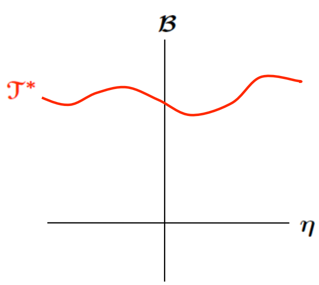
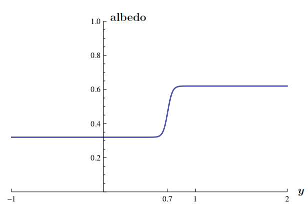




Widiasih: There exists $\widetilde{\mathcal{B}} \subset \mathcal{B}$ such that, given $\eta \in \mathbb{R}$, given $T_0(y,\eta) \in \widetilde{\mathcal{B}}$,

$$\|T_n(y,\eta) - T^*(y,\eta)\|_{\mathcal{B}} \to 0 \text{ as } n \to \infty.$$

The ice line should move as the temperature profile evolves.

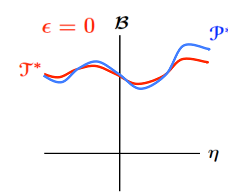
Esther Widiasih, *Dynamics of Budyko's Energy Balance Model*

$T_c = -10°\text{C}, \quad \epsilon \ll 1$

Given (T_0,η_0), set

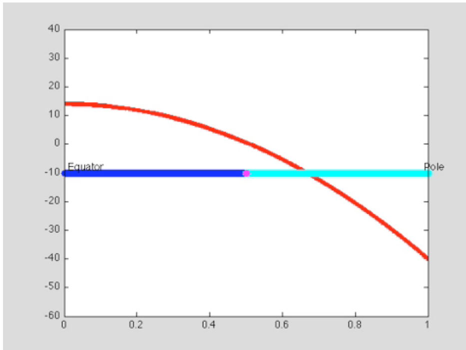
$$\begin{cases} T_{n+1}(y) = T_n(y) + \dfrac{K}{R}\left(Qs(y)(1-\alpha(y,\eta_n)) - (A+BT_n(y)) - C\left(T_n(y) - \displaystyle\int_0^1 T_n(y)dy\right)\right) \\ \eta_{n+1} = \eta_n + \epsilon(T_n(\eta_n) - T_c) \end{cases}$$

$M : \mathcal{B}\times\mathbb{R} \to \mathcal{B}\times\mathbb{R},\ (T_n,\eta_n) \mapsto (T_{n+1},\eta_{n+1})$



For sufficiently small ϵ,

1. There exists a Lipschitz continuous $\Phi^* : \mathbb{R} \to \mathcal{B}$ such that $\mathcal{P}^* = \{(\Phi^*(\eta),\eta) : \eta \in \mathbb{R}\}$ is invariant under M.
2. There is a closed subset $\widetilde{\mathcal{B}} \subset \mathcal{B}$ such that for any $(T_0,\eta_0) \in \widetilde{\mathcal{B}} \times \mathbb{R}$, $\|M^k(T_0,\eta_0) - \mathcal{P}^*\|_{\mathcal{B}\times\mathbb{R}} \to 0$ (exponentially fast) as $k \to \infty$.
3. $\mathcal{P}^*$ is within $O(\epsilon)$ of $\mathcal{T}^*$.



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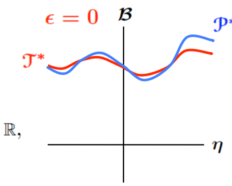
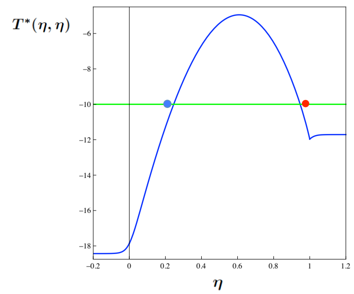




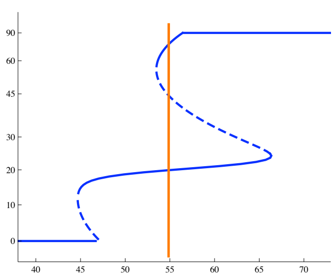
small ice cap - *stable*

large ice cap - *unstable*

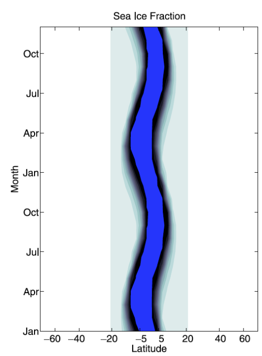
Example 2: The Jormungand Model

$$R\frac{\partial T}{\partial t} = Qs(y)(1 - \alpha(y,\eta)) - (A + BT) - C(T - \overline{T})$$

η at equilibrium



decreasing A ⟶



Dorian S. Abbot, Aiko Voigt, and Daniel Koll, The Jormungand global climate state and implications for Neoproterozoic glaciations, *Journal of Geophysical Research* **116** (2011).

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- water world
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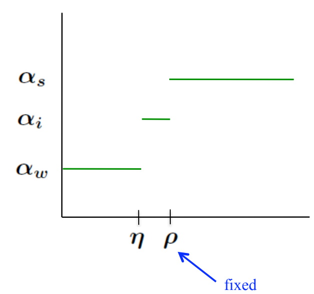
- η = the iceline
- new albedo function $\alpha(y, \eta)$



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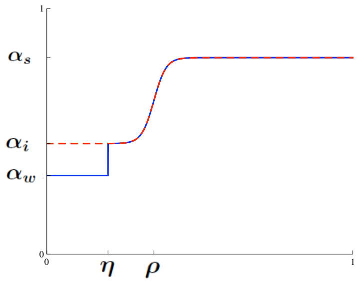
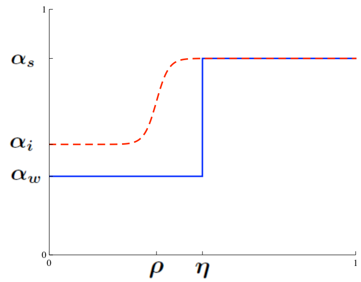
$$\alpha(y,\eta) = \begin{cases} \alpha_w, & y < \eta \\ 0.5(\alpha_s+\alpha_i) + 0.5(\alpha_s-\alpha_i)\tanh(M(y-\rho)), & \eta \le y \end{cases}$$





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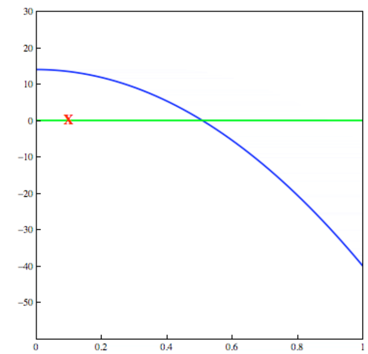

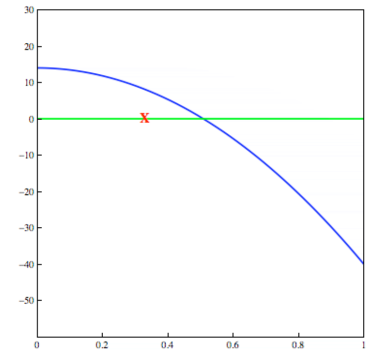

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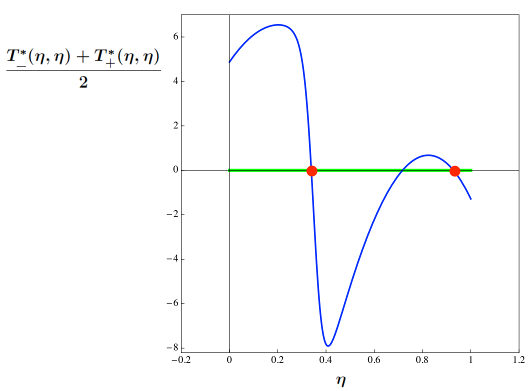
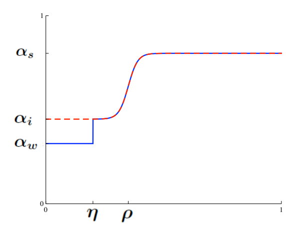
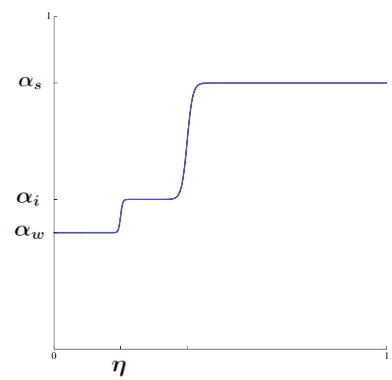


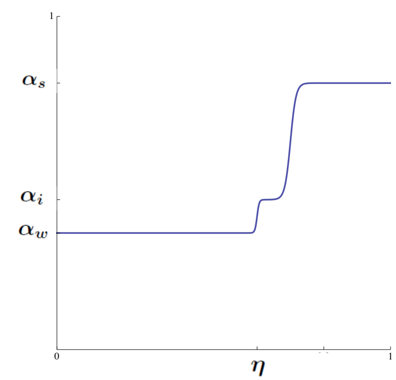


stable equilibria

Example 2: The Jormungand Model

To do – repeat with $\alpha(y,\eta) \in \mathcal{B}$





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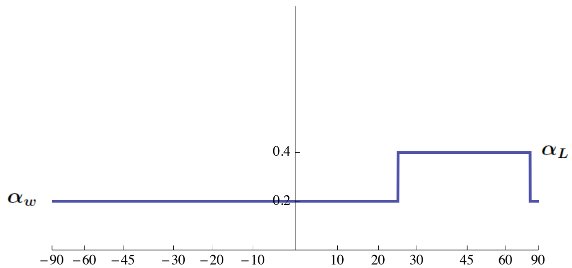
Example 3:

Assumptions:

- annulus of land: $\text{lat}_1 \leq y \leq \text{lat}_2$
- albedo function $\alpha(y, \text{lat}_1)$



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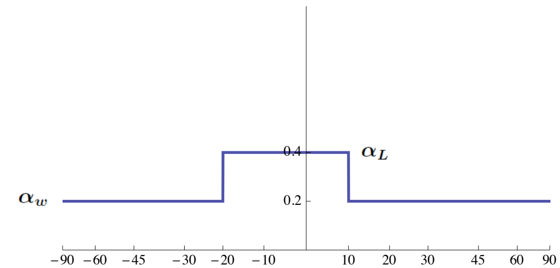

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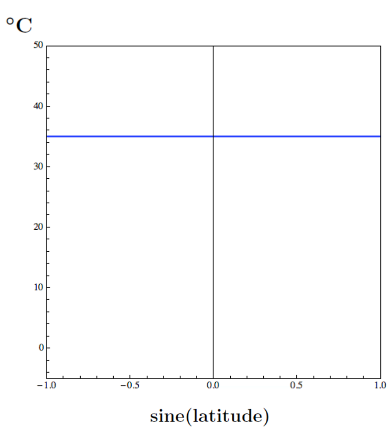

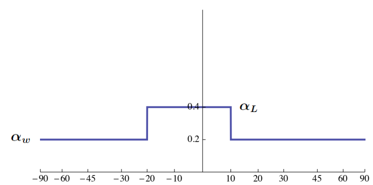

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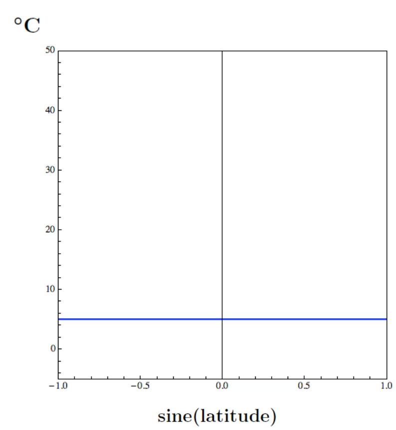

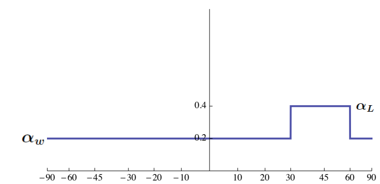

Interesting Problem (suggested by R. McGehee): Assume symmetry about equator

- Occurrence of glacial debris near sea level in the tropics

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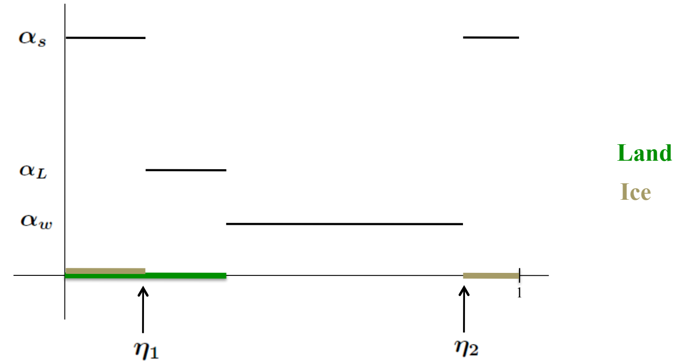

Snowball Earth? Jormungand State?