Math and Climate Seminar — IMA

MCRN
Mathematics and Climate Research Network

Joint MCRN/IMA Math and Climate Seminar
Tuesdays 11:15 – 12:05
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Budyko's Model as an Infinite Dimensional Dynamical System

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Seminar on the Mathematics of Climate
IMA, MCRN, School of Mathematics
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Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

surface temperature; sin(latitude); $\overline{T}=\int_0^1 T(y)dy$; heat capacity; insolation; albedo; OLR; heat transport

Let X be the space of functions where T lives. (*e.g.* $L^1([0,1])$)
Let

$$L: X \to X : LT = C\overline{T} - (B+C)T,$$
$$f(y) = Qs(y)(1-\alpha(y)) - A$$

Budyko's equation can be written as a linear vector field on X.

$$R\frac{dT}{dt} = f + LT$$

Budyko's Model

Budyko's Equation

$$R\frac{dT}{dt} = f + LT$$

$$LT(y) = C\int_0^1 T(\xi)d\xi - (B+C)T(y)$$

If X is a decent space (*e.g.* $L^1([0,1])$), then L is a continuous linear operator on X.

Equilibrium solution:
$$T^* = -L^{-1}f$$
if L is invertible.

Stability depends on the spectrum of L.

Budyko's Model

Linear Operator

$$LT(y) = C\int_0^1 T(\xi)d\xi - (B+C)T(y)$$

Let

$$X_0 = \left\{T \in X : \int_0^1 T(y)dy = 0\right\}, \quad X_1 = \{T \in X : T \text{ is constant}\}$$

Then

$$X = X_0 \oplus X_1$$
$$LT = -(B+C)T, \text{ for } T \in X_0$$
$$LT = -BT, \text{ for } T \in X_1$$

L is invertible.

Spectrum

$-B$ is an eigenvalue, with eigenspace X_1 (dimension 1)
$-(B+C)$ is an eigenvalue, with eigenspace X_0 (codimension 1)

The equilibrium solution is stable.

Budyko's Model

Budyko's Equation

$$R\frac{dT}{dt} = f + LT$$

Choose coordinates
$u \in X_0$ and $\overline{T} \in X_1$

$$T = \overline{T} + u, \quad \overline{T} = \int_0^1 T(y)dy, \quad u = T - \overline{T}$$
$$f = \overline{f} + \varphi, \quad \overline{f} = \int_0^1 f(y)dy, \quad \varphi = f - \overline{f}$$

The equation becomes

$$R\frac{d}{dt}(\overline{T}+u) = \overline{f} + \varphi + L(\overline{T}+u)$$

or

$$R\frac{d\overline{T}}{dt} = \overline{f} - B\overline{T}$$
$$R\frac{du}{dt} = \varphi - (B+C)u$$

Budyko's Model

Solution

$$R\frac{d\overline{T}}{dt} = \overline{f} - B\overline{T}$$

$$R\frac{du}{dt} = \varphi - (B+C)u$$

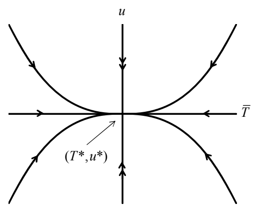
Equilibrium solution:

$$T^* = \overline{T}^* + u^*, \text{ where } \overline{T}^* = \overline{f}/B, \quad u^* = \varphi/(B+C)$$

Solution:

$$\overline{T}(t) = \overline{T}^* + \left(\overline{T}(0) - \overline{T}^*\right)e^{-(B/R)t}$$

$$u(y,t) = u^*(y) + \left(u(y,0) - u^*(y)\right)e^{-((B+C)/R)t}$$



Budyko's Model

Equilibrium

$$T^*(y) = \overline{T}^* + u^*,$$

$$\text{where } \overline{T}^* = \overline{f}/B, \quad u^* = \varphi/(B+C)$$

$$\text{Recall: } f(y) = Qs(y)\left(1-\alpha(y)\right) - A, \text{ so}$$

$$\overline{f} = Q\left(1-\overline{\alpha}\right) - A, \text{ where } \overline{\alpha} = \int_0^1 \alpha(y)s(y)dy, \text{ so}$$

$$\overline{T}^* = \left(Q\left(1-\overline{\alpha}\right) - A\right)/B$$

$$\text{Also, } \varphi = f - \overline{f} = Q\left(s(y) - 1 - \alpha(y) + \overline{\alpha}\right)$$

$$u^*(y) = \left(Q\left(s(y) - 1 - \alpha(y) + \overline{\alpha}\right)\right)/(B+C)$$

$$T^*(y) = \frac{Q\left(1-\overline{\alpha}\right) - A}{B} + \frac{Q\left(s(y) - 1 - \alpha(y) + \overline{\alpha}\right)}{B+C}$$

approached with exponential rate $-B/R$

approached with exponential rate $-(B+C)/R$

Budyko's Model

Ice Albedo Feedback

So far, we have assumed that the albedo function $\alpha(y)$ is a fixed function. What if the albedo changes as the temperature changes?

Standard assumption: Permanent ice forms if the annual average temperature is below T_c=-10 °C and melts if the annual average temperature is above T_c.

Possible albedo function:

$$\alpha : [0,1] \times X \to [0,1]: \quad \alpha(y,T) = \begin{cases} \alpha_1 & \text{if } T(y) > T_c \\ \alpha_2 & \text{if } T(y) < T_c \end{cases}$$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,T)) - (A+BT) + C(\overline{T} - T)$$

Big Problem: This equation is no longer linear.
Bigger Problem: Without additional assumptions, the solutions are ridiculous.

Budyko's Model

Ridiculous Solutions

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,T)) - (A+BT) + C(\overline{T} - T)$$

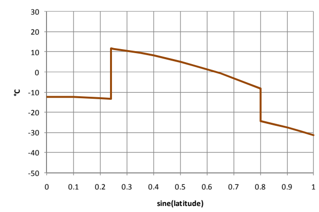
$$\alpha : [0,1] \times X \to [0,1]: \quad \alpha(y,T) = \begin{cases} \alpha_1 & \text{if } T(y) > T_c \\ \alpha_2 & \text{if } T(y) < T_c \end{cases}$$

There exits a large set of equilibrium solutions with the property:

$$T(y) < T_c \quad \text{for } y \in E$$

$$T(y) > T_c \quad \text{for } y \in E^c$$

Exercise: Can you construct such a solution with a totally disconnected E ?



Budyko's Model

Ice Albedo Feedback

What if the albedo changes as the temperature changes?

Better assumption: There is a single ice line at $y=\eta$ between the equator and the pole. The albedo is α_1 below the ice line and α_2 above it.

$$\alpha(y,\eta) = \begin{cases} \alpha_1 & y < \eta \\ \alpha_2 & y > \eta \end{cases}$$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T} - T)$$

For fixed η, this is just the linear equation we already analyzed.

How do we determine η ?

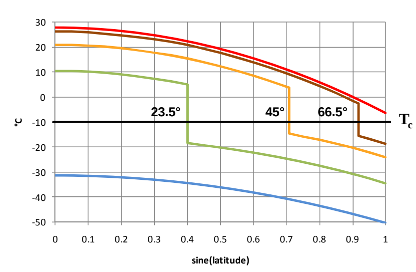
Budyko's Model

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T} - T)$$

For each fixed η, there is a stable equilibrium for Budyko's equation.

How to pick η ?



Budyko's Model

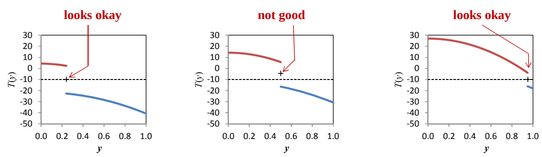
Ice Albedo Feedback

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

For each fixed η, there is a stable equilibrium for Budyko's equation.

Additional condition: The average temperature across the ice boundary is the critical temperature T_c.

$$\frac{1}{2}\left(T_\eta^*(\eta+) + T_\eta^*(\eta-)\right) = T_c = -10$$



Budyko's Model

Ice Albedo Feedback

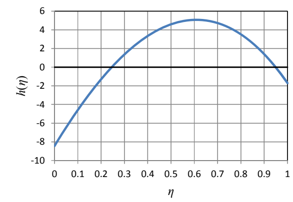
$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

The additional condition: $\frac{1}{2}\left(T_\eta^*(\eta+) + T_\eta^*(\eta-)\right) = T_c = -10$

can be written:

$$h(\eta) \equiv \frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2 + (\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy\right)\right) - \frac{A}{B} - T_c = 0$$

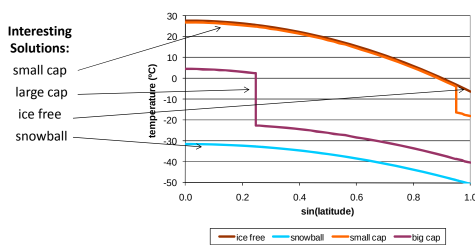
Two equilibria (zeros of h) satisfy the additional condition.



Budyko's Model

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

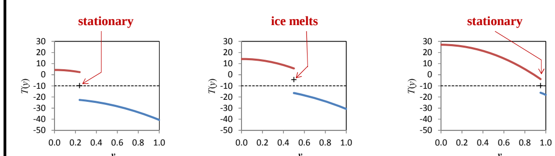


Budyko's Model

Dynamics of the Ice Line

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

Idea:
If the average temperature across the ice line is above the critical temperature, some ice will melt, moving the ice line toward the pole. If it is below the critical temperature, the ice will advance toward the equator.



Widiasih's equation: $\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$

Budyko's Model

Dynamics of the Ice Line

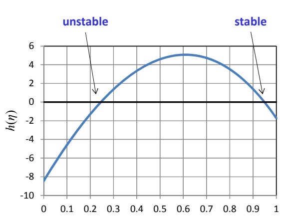
$$\frac{d\eta}{dt} = \varepsilon(T(\eta) - T_c)$$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

State space: $[0,1]\times X$

Widiasih's Theorem. For sufficiently small ε, the system has an attracting invariant curve given by the graph of a function $\Phi_\varepsilon : [0,1] \to X$. On this curve, the dynamics are approximated by the equation

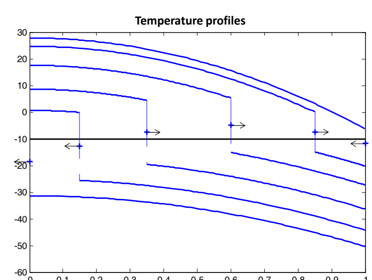
$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



Budyko's Model

Summary

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

surface temperature; sin(latitude); $\overline{T} = \int_0^1 T(y)dy$

heat capacity; insolation; albedo; OLR; heat transport

reduces to

$$\frac{d\eta}{dt} = \varepsilon h(\eta) \qquad \boxed{\infty \to 1}$$

Next Week:
An approximation yielding a simplified proof of Widiasih's Theorem.

But First:
Fun with Budyko!

Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta) = \varepsilon\left(\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\int_0^\eta s(y)dy\right)\right) - \frac{A}{B} - T_c\right)$$

What about the greenhouse effect?

$A+BT$ is the outgoing long wave radiation. This term decreases if the greenhouse gases increase.
We view A as a parameter.

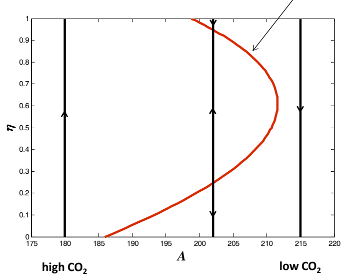
$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

isocline $h(\eta, A) = 0$



Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

What if A is a dynamical variable?

Simple equation:

$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

$$0 < \tilde{\varepsilon} \ll \varepsilon \ll 1$$

MCRN Paleocarbon equation

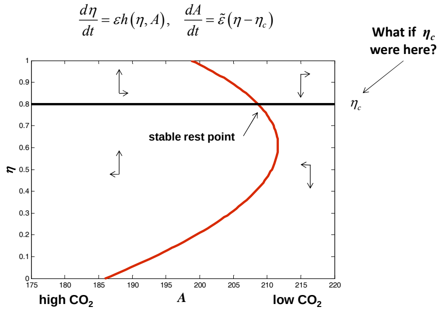
New system:

$$\boxed{\begin{aligned}\frac{dA}{dt} &= \tilde{\varepsilon}(\eta-\eta_c)\\ \frac{d\eta}{dt} &= \varepsilon h(\eta, A)\end{aligned}}$$

Budyko's Model

Budyko-Widiasih-Paleocarbon Model

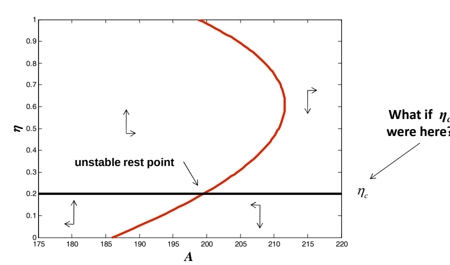
$$\frac{d\eta}{dt} = \varepsilon h(\eta, A), \quad \frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$



Budyko's Model

Budyko-Widiasih-Paleocarbon Model

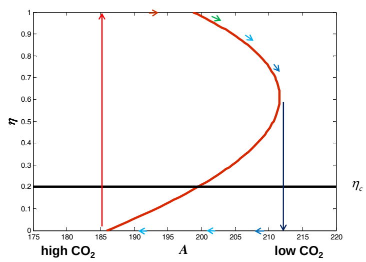
$$\frac{d\eta}{dt} = \varepsilon h(\eta, A), \quad \frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$



Budyko's Model

Budyko-Widiasih-Paleocarbon Model

Snowball – Hothouse Oscillations



Budyko's Model

Water Belt Model

The snowball Earth hypothesis is not universally accepted.

Complex ocean-dwelling photosynthetic organisms appear to have survived through the time when the snowball would have occurred.

Perhaps the equator never fully froze over, leaving a belt of open water covering the equator.

Hence "water belt" planet, aka "Jormungand".

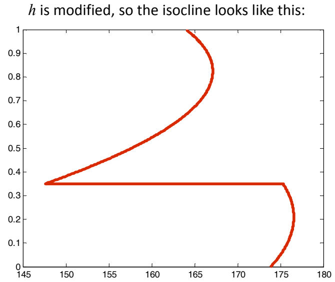
D.S. Abbot, A. Voigt, and D. Koll (2011), The Jormungand global climate state and implications for Neoproterozoic glaciations, *J. Geophys. Res.* **116.**

Budyko's Model

Water Belt Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A),$$
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

h is modified, so the isocline looks like this:



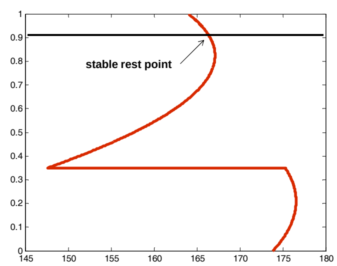
Budyko's Model

Water Belt Model

Stable Small Icecap

What if η_c were here?

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A),$$
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

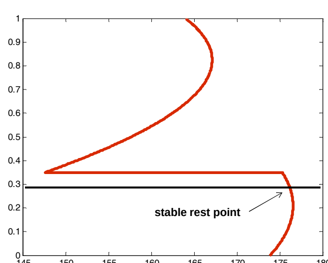


Budyko's Model

Water Belt Model

Stable Water Belt

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A),$$
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$



What if η_c were here?

η_c

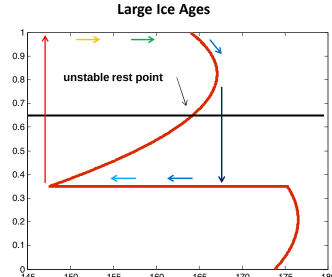
Budyko's Model

Water Belt Model

Large Ice Ages

What if η_c were here?

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A),$$
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

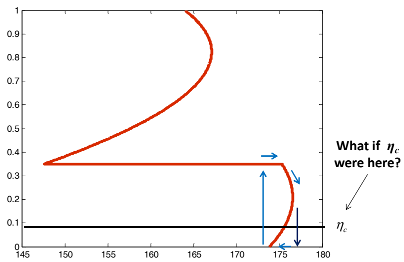


Budyko's Model

Water Belt Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A),$$
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

Banded Iron Formations?



Budyko's Model

Budyko's Equation

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$$\overline{T} = \int_0^1 T(y)dy$$

surface temperature; sin(latitude); heat capacity; insolation; albedo; OLR; heat transport

Next Time

An approximation yielding a simplified proof of Widiasih's Theorem.