Math and Climate Seminar — IMA

Joint MCRN/IMA Math and Climate Seminar

Tuesdays 11:15 – 12:05

streaming video available at

www.ima.umn.edu

MCRN www.mathclimate.org

Budyko's Model as a Dynamical System

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Seminar on the Mathematics of Climate

IMA, MCRN, School of Mathematics

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Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

surface temperature: T; heat capacity: R; insolation: $Qs(y)$; sin(latitude): y; albedo: $\alpha(y)$; OLR: $A+BT$; heat transport: $C(\overline{T}-T)$, where $\overline{T} = \int_0^1 T(y)dy$

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241(3y^2-1)$$

Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

equilibrium solution: $T = T^*(y)$

$$Qs(y)(1-\alpha(y)) - (A+BT^*(y)) + C(\overline{T}^* - T^*(y)) = 0$$

Integrate:

$$\int_0^1 \left(Qs(y)(1-\alpha(y)) - (A+BT^*(y)) + C(\overline{T}^* - T^*(y))\right)dy = 0$$

$$Q(1-\overline{\alpha}) - (A+B\overline{T}^*) = 0$$

where $\overline{\alpha} = \int_0^1 \alpha(y)s(y)dy$

Global mean temperature at equilibrium

$$\overline{T}^* = \frac{1}{B}\left(Q(1-\overline{\alpha}) - A\right)$$

Budyko's Model

Budyko's Equilibrium

$$Qs(y)(1-\alpha(y)) - (A+BT^*(y)) + C(\overline{T}^* - T^*(y)) = 0$$

Global mean temperature at equilibrium:

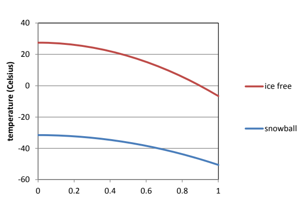
$$\overline{T}^* = \frac{1}{B}\left(Q(1-\overline{\alpha}) - A\right) \qquad \left(\overline{\alpha} = \int_0^1 \alpha(y)s(y)dy\right)$$

Equilibrium temperature profile:

$$T^*(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha(y)) - A + C\overline{T}^*\right)$$

Example:

$C = 3.04$
$\alpha = 0.32$: ice free
$\alpha = 0.62$: snowball



Budyko's Model

Ice Albedo Feedback

What if the albedo is not constant?

Ice Line Assumption: There is a single ice line at $y=\eta$ between the equator and the pole. The albedo is α_1 below the ice line and α_2 above it.

$$\alpha(y,\eta) = \begin{cases} \alpha_1 & y < \eta \\ \alpha_2 & y > \eta \end{cases}$$

Equilibrium condition:

$$Qs(y)(1-\alpha(y,\eta)) - (A+BT_\eta^*(y)) + C(\overline{T}^* - T_\eta^*(y)) = 0$$

Equilibrium solution:

$$T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha(y,\eta)) - A + C\overline{T}_\eta^*\right)$$

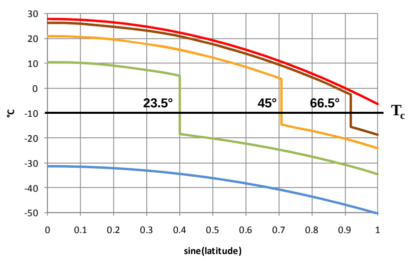
where $\overline{T}_\eta^* = \frac{1}{B}\left(Q(1-\overline{\alpha}(\eta)) - A\right) \qquad \left(\overline{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy\right)$

Budyko's Model

Ice Albedo Feedback

$$T_\eta^*(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right)-A+C\overline{T}_\eta^*\right)$$

For each fixed η, there is an equilibrium solution for Budyko's equation.



Budyko's Model

Dynamics

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y))-(A+BT)+C(\overline{T}-T)$$

surface temperature; sin(latitude); $\overline{T}=\int_0^1 T(y)dy$; heat capacity; insolation; albedo; OLR; heat transport

Let X be the space of functions where T lives. (*e.g.* $L^1([0,1])$)

Let

$$L:X\to X: LT=C\overline{T}-(B+C)T,$$
$$f(y)=Qs(y)\left(1-\alpha(y)\right)-A$$

Budyko's equation can be written as a linear vector field on X.

$$R\frac{dT}{dt}=f+LT$$

Budyko's Model

Budyko's Equation

$$R\frac{dT}{dt}=f+LT$$

$$LT(y)=C\int_0^1 T(\xi)d\xi-(B+C)T(y)$$

If X is a decent space (*e.g.* $L^1([0,1])$), then L is a continuous linear operator on X.

Equilibrium solution:

$$T^*=-L^{-1}f$$

if L is invertible.

Stability depends on the spectrum of L.

Budyko's Model

Linear Operator

$$LT(y)=C\int_0^1 T(\xi)d\xi-(B+C)T(y)$$

Let

$$X_0=\left\{T\in X:\int_0^1 T(y)dy=0\right\},\quad X_1=\{T\in X: T \text{ is constant}\}$$

Then

$$X=X_0\oplus X_1$$
$$LT=-(B+C)T,\ \text{for } T\in X_0$$
$$LT=-BT,\ \text{for } T\in X_1$$

L is invertible.

Spectrum

$-B$ is an eigenvalue, with eigenspace X_1 (dimension 1)

$-(B+C)$ is an eigenvalue, with eigenspace X_0 (codimension 1)

The equilibrium solution is stable.

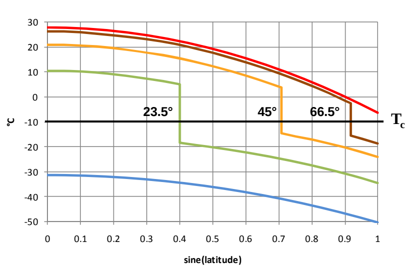
Budyko's Model

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\overline{T}-T)$$

For each fixed η, there is a **stable** equilibrium solution for Budyko's equation.

How to pick one?



Budyko's Model

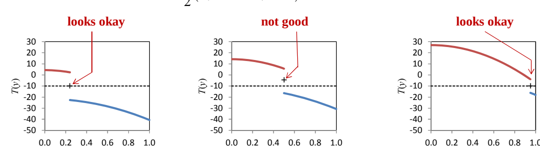
Ice Albedo Feedback

For each fixed η, there is a stable equilibrium solution for Budyko's equation.

Standard assumption: Permanent ice forms if the annual average temperature is below T_c=-10 °C and melts if the annual average temperature is above T_c.

Additional condition: The average temperature across the ice boundary is the critical temperature T_c.

$$\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=T_c=-10$$



Budyko's Model

Ice Albedo Feedback

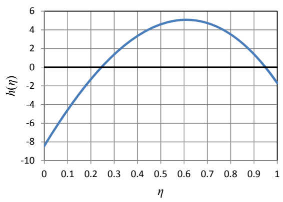
$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

The additional condition: $\frac{1}{2}\left(T_\eta^*(\eta+) + T_\eta^*(\eta-)\right) = T_c = -10$

can be written:

$$h(\eta) \equiv \frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2 + (\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy\right)\right) - \frac{A}{B} - T_c = 0$$

Two equilibria (zeros of h) satisfy the additional condition.



Budyko's Model

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

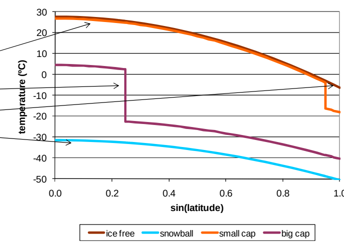
Interesting Solutions:

small cap

large cap

ice free

snowball

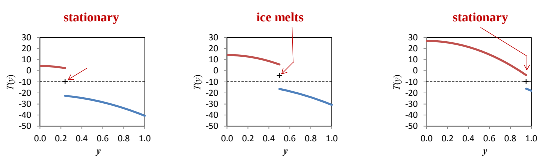


Budyko's Model

Dynamics of the Ice Line

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

Idea:
If the average temperature across the ice line is above the critical temperature, some ice will melt, moving the ice line toward the pole. If it is below the critical temperature, the ice will advance toward the equator.



Widiasih's equation: $\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$

Budyko's Model

Dynamics of the Ice Line

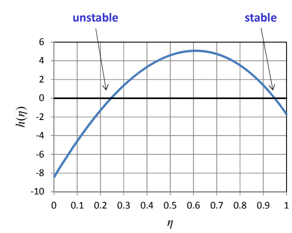
$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

State space: $[0,1]\times X$

Widiasih's Theorem. For sufficiently small ε, the system has an attracting invariant curve given by the graph of a function $\Phi_\varepsilon : [0,1] \to X$. On this curve, the dynamics are approximated by the equation

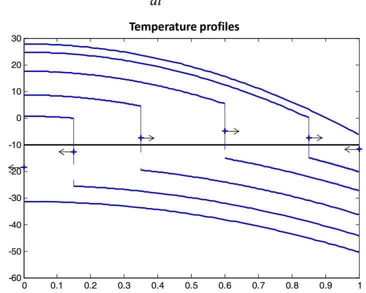
$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



Budyko's Model

Summary

surface temperature — sin(latitude) — $\overline{T} = \int_0^1 T(y)\,dy$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

heat capacity — insolation — albedo — OLR — heat transport

reduces to

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

$$\frac{d\eta}{dt} = \varepsilon h(\eta) = \varepsilon\left(\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2 + (\alpha_2-\alpha_1)\int_0^\eta s(y)\,dy\right)\right) - \frac{A}{B} - T_c\right)$$

Budyko's Model

Budyko-Widiasih Model

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

surface temperature; sin(latitude); $\overline{T} = \int_0^1 T(y)dy$; heat capacity; insolation; albedo; OLR; heat transport

What about the greenhouse effect?

$A+BT$ is the outgoing long wave radiation. This term decreases if the greenhouse gases increase.
We view A as a parameter.

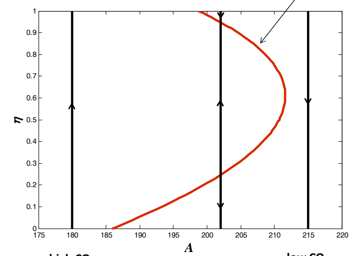
$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

isocline $h(\eta, A) = 0$



Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

What if A is a dynamical variable?

Simple equation:

$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

MCRN Paleocarbon equation

$$0 < \tilde{\varepsilon} \ll \varepsilon \ll 1$$

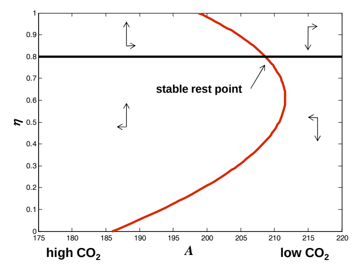
New system:

$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$
$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

Budyko's Model

Budyko-Widiasih-Paleocarbon Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A), \quad \frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

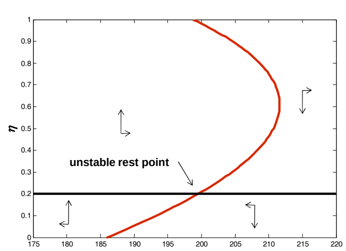


What if η_c were here? η_c

Budyko's Model

Budyko-Widiasih-Paleocarbon Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A), \quad \frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

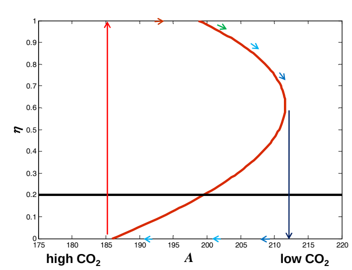


What if η_c were here? η_c

Budyko's Model

Budyko-Widiasih-Paleocarbon Model

Snowball – Hothouse Oscillations



Budyko's Model

Water Belt Model

The snowball Earth hypothesis is not universally accepted.

Complex ocean-dwelling photosynthetic organisms appear to have survived through the time when the snowball would have occurred.

Perhaps the equator never fully froze over, leaving a belt of open water covering the equator.

Hence "water belt" planet, aka "Jormungand".

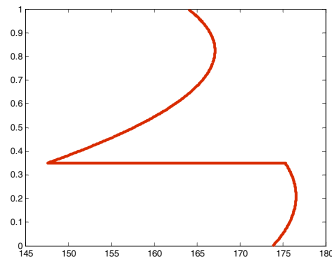
D.S. Abbot, A. Voigt, and D. Koll (2011), The Jormungand global climate state and implications for Neoproterozoic glaciations, *J. Geophys. Res.* **116.**

Budyko's Model

Water Belt Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A),$$
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

h is modified, so the isocline looks like this:

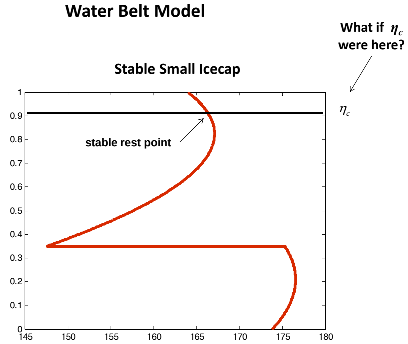


Budyko's Model

Water Belt Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A),$$
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

Stable Small Icecap

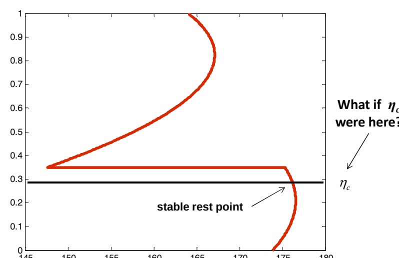


Budyko's Model

Water Belt Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A),$$
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

Stable Water Belt

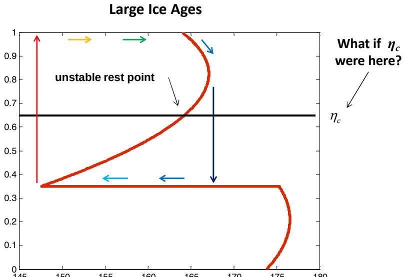


Budyko's Model

Water Belt Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A),$$
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

Large Ice Ages

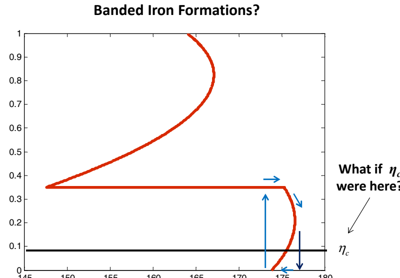


Budyko's Model

Water Belt Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A),$$
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

Banded Iron Formations?



Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

$$\overline{T} = \int_0^1 T(y)\,dy$$

surface temperature · **sin(latitude)** · **heat capacity** · **insolation** · **albedo** · **OLR** · **heat transport**

Next Time

Glacial Cycles