Circle Maps Inspired By Glacial Cycles

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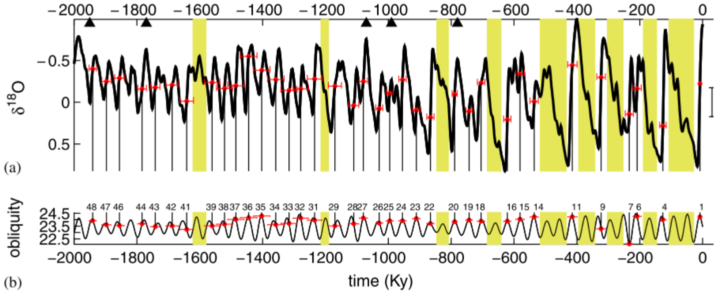
$\delta^{18}O$ content of the last 2Ma



Huybers' Discrete Model

$$
\begin{aligned}
V_t &= V_{t-1} + \eta_t \quad \text{and if } V_t \geq T_t \text{ terminate} \\
T_t &= at + b - c\theta'_t
\end{aligned}
$$

Upon termination, linearly reset V to 0 over 10 Ka

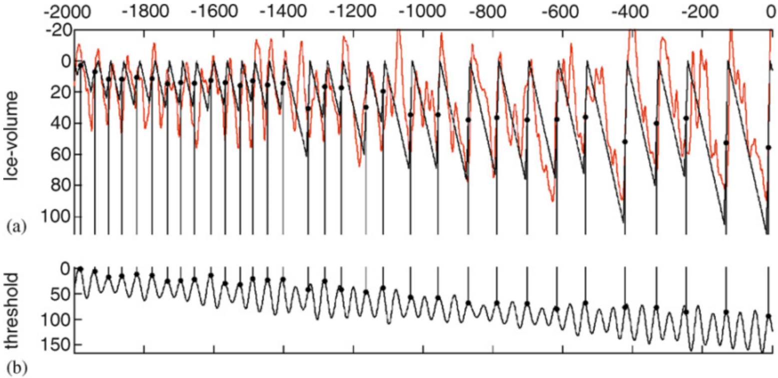
V : ice volume

T : deglaciation threshold

θ' : scaled obliquity

η : ice volume growth rate

Huybers, P. Glacial variability over the last two million years: an extended depth-derived agemodel, continuous obliquity pacing, and the Pleistocene progression. *Quaternary Science Reviews*. 2007.

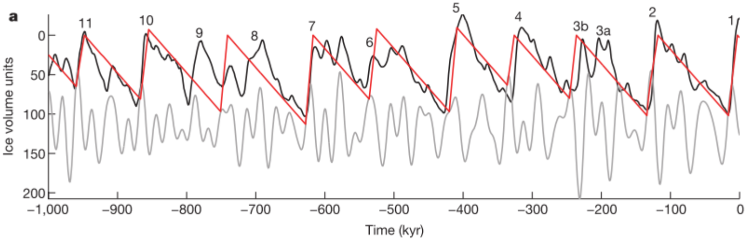


A deterministic run of the model

Discrete Model with Combined Forcing

$$
\begin{aligned}
V_t &= V_{t-1} + \eta_t \qquad \text{and if } V_t \geq T_t \text{ terminate} \\
T_t &= 110 - 25\mathcal{F}_t \\
\mathcal{F}_t &= \alpha^{1/2} e_t \sin(\omega_t - \phi) + (1-\alpha)^{1/2}\epsilon_t
\end{aligned}
$$

Huybers, P. Combined obliquity and precession pacing of late Pleistocene deglaciations. *Nature*. 2011.

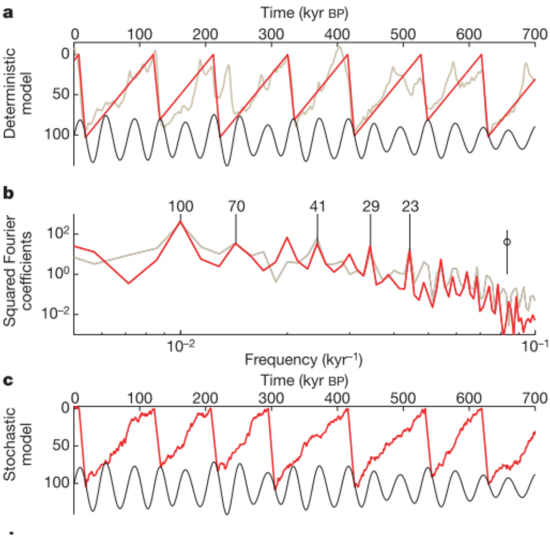


Discrete model with combined forcing

Wunsch and Huybers' original model

$$
\begin{aligned}
V_t &= V_{t-1} + \eta_t \qquad \text{and if } V_t \geq T_t \text{ terminate} \\
T_t &= 100 - \theta'_t
\end{aligned}
$$

Huybers, P. and Wunsch, C. Obliquity pacing of the late Pleistocene glacial terminations. *Nature*. 2005.



Deterministic and stochastic models with obliquity forcing

Idealized Model

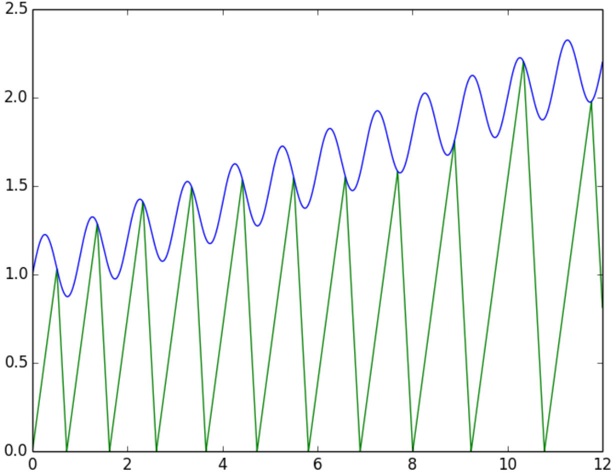
Discrete model:

$$\begin{aligned} V_{t_i} &= V_{t_{i-1}} + \eta_{t_i}\Delta_t \qquad \text{and if } V_{t_i} \geq T_{t_i} \text{ terminate} \\ T_{t_i} &= at_i + b + c\sin(2\pi t_i) \\ \Delta_t &= t_i - t_{i-1} \end{aligned}$$

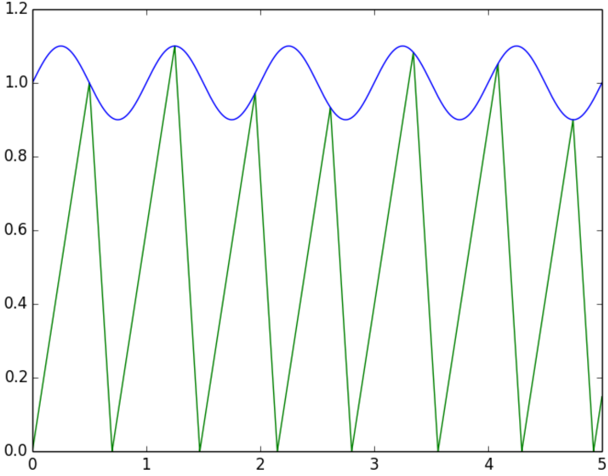
Continuous model: let $\Delta_t \to 0$.

Let $V_{t_0}(t)$ be the volume with initial condition $V_{t_0}(t_0) = 0$.

Numerical Simulations



Numerical Simulations



Reduction to a Periodic Map

Suppose the threshold $T(x)$ is periodic: $T(x+1) = T(x)$.

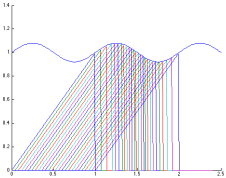
Let $g : \mathbb{R} \to \mathbb{R}$ be the map sending a termination time t to the next termination time.

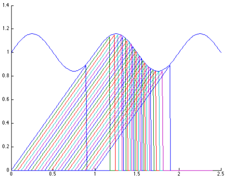
$$g(t) = min\{t' > t : V_t(t') = 0\}$$

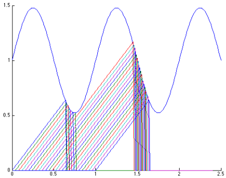
Then $g(t)$ is also periodic: $g(t+1) = g(t)$.

Reduction to a Periodic Map

The map g can be smooth, continuous, or discontinuous.







Circle Maps

A function $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a circle map.

Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be defined as

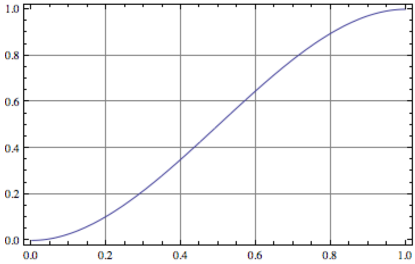
$$\pi(x) = e^{2\pi i x}$$

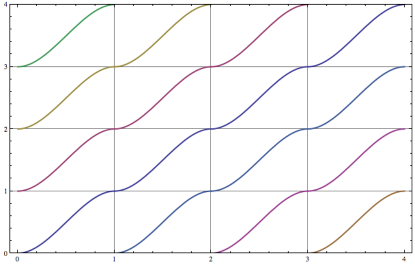
A lift of a circle map is a map $F : \mathbb{R} \to \mathbb{R}$ such that

$$\pi \circ F = f \circ \pi$$

Circle Maps

- There are infinitely many lifts of any circle map f.
- If f is continuous, any two continuous lifts differ by an integer.
- We say a continuous circle map f is orientation preserving if a lift F has the property $F(x) \leq F(y)$ if $x < y$.





Rotation Number

Choose a basepoint $x \in \mathbb{S}^1$ and $x' \in \mathbb{R}$ with $\pi(x') = x$.
Then for f with lift F define

$$\rho(x, f) = \rho(x', F) = \lim_{n \to \infty} \frac{F^n(x') - x'}{n}$$

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"Average" amount of rotation from one iteration of f

Rotation Number

Define the rotation set

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Rotation Number

Define the rotation set

$$\rho(f) = \{\rho(x, f) : x \in \mathbb{S}^1\}$$

- If f is a diffeomorphism and orientation-preserving, $\rho(f)$ exists uniquely. (Poincaré)
- If f is degree one and continuous, $\rho(f)$ is an interval $[\rho_1(f), \rho_2(f)]$. (Ito, 1981)

Rotation Number

- For a degree one, continuous circle map f,

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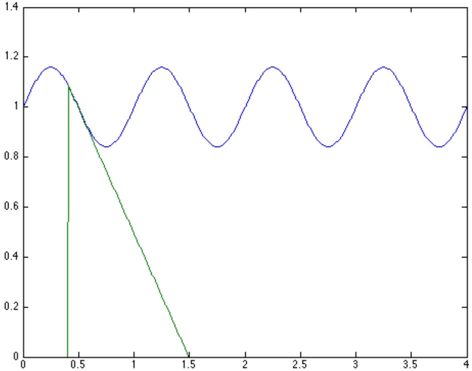
- If $\rho(f)$ is irrational, F is semi-conjugate to a rigid rotation.

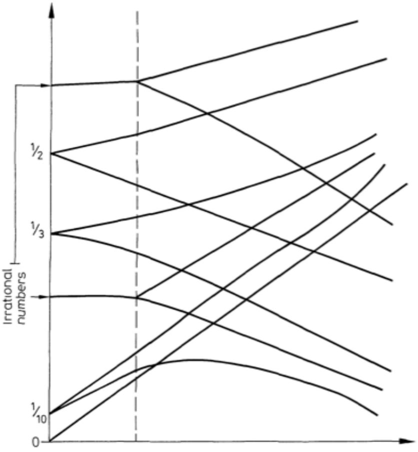
$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \downarrow H & & \downarrow H \\ \mathbb{R} & \xrightarrow{R_{\rho(F)}} & \mathbb{R} \end{array}$$

Canonical family of circle maps

$$f(x) = x + b + \frac{\omega}{2\pi}\sin(2\pi x) \quad \text{mod } 1$$



Arnold Tongues for canonical maps



Boyland, P. Bifurcations of circle maps: Arnol'd tongues, bistability and rotation intervals, Comm. Math. Phys. 106 (1986), 353-381.

Discontinuous Rotations

What holds true for discontinuous rotations?

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- Existence and uniqueness if f is orientation preserving. (Brette, 2003; Kozaykin, 2005)
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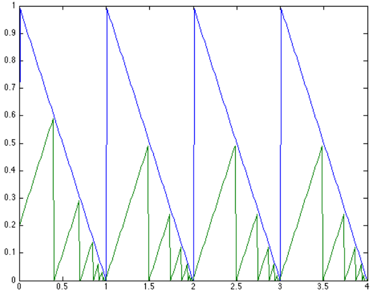
Discontinuous Rotations

What holds true for discontinuous rotations?

- Existence and uniqueness if f is orientation preserving. (Brette, 2003; Kozaykin, 2005)
- If there exists point z with $f^q(z) = z$, $p/q \in \rho(f)$
- However, $p/q \in \rho(f)$ does not imply the existence of a periodic point: $f(x) = (1/2)x + 1/2$



Relations on $\mathbb{S}^1$

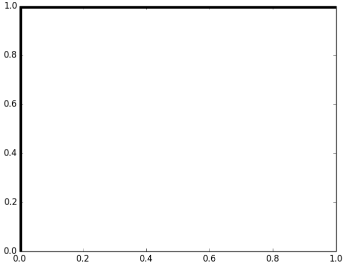
A relation on $\mathbb{S}^1$ is a subset of $\mathbb{S}^1 \times \mathbb{S}^1$.
The analogue of an iteration is an *orbit* of a relation f:

$$\{...x_{-1}, x_0, x_1, x_2, ...\} \text{ such that } (x_i, x_{i+1}) \in f.$$

We may be able to prove more general statements about relations on $\mathbb{S}^1$.

Relations on $\mathbb{S}^1$

f



f^2