Nonlinear Sliding and its Role in Welander's Model

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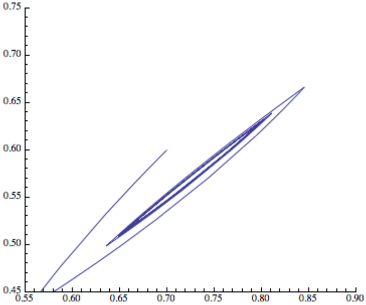
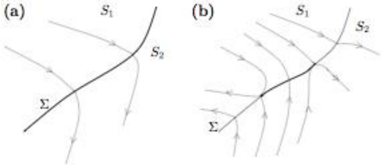
Motivation

Welander's Model has a nonsmooth Hopf bifurcation.



A Recap

There is an intuitive way to piece together dynamics in a nonsmooth model with a line of discontinuity, which was originally formulated by Filippov.



Di Bernardo, Mario, et al. "Bifurcations in nonsmooth dynamical systems." SIAM review (2008): 629-701.

Taking a convex combination of the systems on either side of the discontinuity gives a way to find a flow in the sliding region.

A Recap

The set up:
Consider the system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \lambda)$$

with discontinuity boundary given defined by the zero set of a scalar function $h(\mathbf{x})$.

$$\lambda = \begin{cases} 1 & h(\mathbf{x}) > 0 \\ 0 & h(\mathbf{x}) < 0 \end{cases} \quad \text{On } h(\mathbf{x}) = 0,\ \lambda \in [0, 1]$$

The standard Filippov formulation would be

$$\dot{\mathbf{x}} = \lambda f^{+}(\mathbf{x}) + (1 - \lambda) f^{-}(\mathbf{x})$$

A Recap

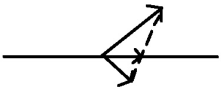
A sliding solution is defined as follows:

If

$$\begin{aligned} 0 &= f(\mathbf{x}, \lambda) \cdot \nabla h(\mathbf{x}) \\ 0 &= h(\mathbf{x}) \end{aligned}$$

can be solved for some $\lambda^* \in [0, 1]$, then $\dot{\mathbf{x}} = f(\mathbf{x}, \lambda^*)$ defines a sliding solution of the system.

Note that no sliding solutions of the Filippov formulation exist in crossing regions.



Nonlinear Sliding

One doesn't need to define the vector field on the boundary in terms of the convex combination. If $f(\mathbf{x}, \lambda)$ is already defined in terms of a nonsmooth parameter λ, then solving

$$\begin{aligned} 0 &= f(\mathbf{x}, \lambda) \cdot \nabla h(\mathbf{x}) \\ 0 &= h(\mathbf{x}) \end{aligned}$$

for $\lambda \in [0, 1]$ gives nonlinear sliding solutions. They don't need to be unique.

Nonlinear Sliding

$$\dot{x} = \begin{pmatrix} 1 \\ 2 - \lambda - x \end{pmatrix} - 2(1 - \lambda^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f^+ = \begin{pmatrix} 1 \\ 1 - x \end{pmatrix}$$

$$f^- = \begin{pmatrix} 1 \\ 3 - x \end{pmatrix}$$

so the Filippov sliding region is $1 \leq x \leq 3$
Nonlinear sliding solves $f \cdot \nabla h = 0$, giving a condition on λ

$$\lambda = \frac{1 \pm \sqrt{1 + 8x}}{4}$$

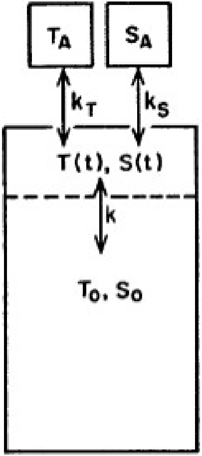
so a sliding solution exists on $x \geq -\frac{1}{8}$

Welander's Model

The nondimensionalized model:

$$\begin{aligned} \dot{T} &= 1 - T - k(\rho)T \\ \dot{S} &= \beta(1-S) - k(\rho)S \\ \rho &= -\alpha T + S \end{aligned}$$

$$k(\rho) = \frac{1}{\pi}\arctan\left(\frac{1}{a}(\rho - \varepsilon)\right) + \frac{1}{2}$$



Welander, Pierre. "A simple heat-salt oscillator." Dynamics of Atmospheres and Oceans 6.4 (1982): 233-242.

Preliminary coordinate change

A coordinate change transforms the splitting manifold into the x (T) axis.
Let $x = T$, $y = \rho - \varepsilon$. Then the system is

$$\begin{aligned}\dot{x} &= 1 - x - kx \\ \dot{y} &= \beta - \beta\varepsilon - k\varepsilon - \alpha - (\beta + k)y - (\alpha\beta - \alpha)x\end{aligned}$$

with $k = \frac{1}{\pi}\tan^{-1}\left(\frac{y}{a}\right) + \frac{1}{2}$
Note that this coordinate change preserves the Filippov formulation.

The Blow Up Method

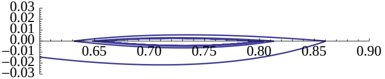
One can make a coordinate change to focus on what happens on the splitting manifold.

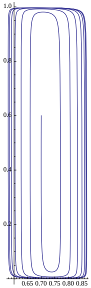
$$k = \Phi(y) \to \lambda = \begin{cases} 1 & y > 0 \\ 0 & y < 0 \end{cases}$$

where Φ is a bijection between $\mathbb{R}$ and $[0, 1]$. Looking at the system in the x, k coordinate system "blows up" the region around $y = 0$. As $a \to 0$, the whole space corresponds to the original region $y = 0$.

The Blow Up System in Welander's Model

$$
\begin{aligned}
\dot{x} &= 1 - x - kx \\
\dot{k} &= \frac{1}{a}\Phi'(k)\left(\beta - \beta\varepsilon - k\varepsilon - \alpha + (\beta + k)\left(a\cot(\pi k)\right) - (\alpha\beta - \alpha)x\right)
\end{aligned}
$$





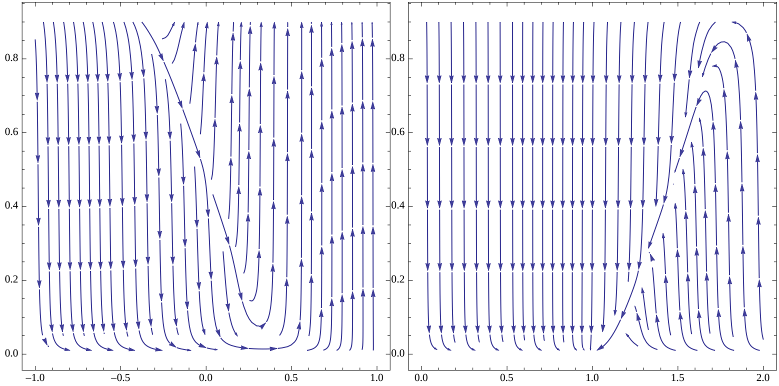
Fast/slow Analysis of the Blow Up system

Because $a << 1$, this is a fast slow system. The fast system is

$$\begin{aligned} x' &= \frac{a\pi}{\sin^2(\pi k)}(1 - x - kx) \\ k' &= \beta - \beta\varepsilon - k\varepsilon - \alpha + (\beta + k)(a\cot(\pi k)) - (\alpha\beta - \alpha)x \end{aligned}$$

Nonlinear sliding solutions are k nullclines, i.e. places where solutions go into the splitting manifold and stick.
The k nullcline is a line, which intersects $k = 0$ at $x = \frac{\beta - \beta\varepsilon - \alpha}{\alpha\beta - \alpha}$, and intersects $k = 1$ at $x = \frac{\beta - \beta\varepsilon - \alpha - \varepsilon}{\alpha\beta - \alpha}$.

Sliding solutions



Filippov Analysis of the system

Alternatively, the boundaries of the sliding region under a Filippov analysis are points where $\dot{y} = 0$ in the original system. The boundaries are the same as the intersections of the nullcline in the blow up system: $x = \frac{\beta - \beta\varepsilon - \alpha}{\alpha\beta - \alpha}$, and $x = \frac{\beta - \beta\varepsilon - \alpha - \varepsilon}{\alpha\beta - \alpha}$. So there is no nonlinear sliding in this model!

Considerations for a normal form

- Nonlinear sliding can destroy a periodic orbit
- If given a nonsmooth system, nothing can be determined about the behavior in the smooth system
- Which kinds of transformations are allowed? Do they need to preserve the Filippov vector field?
- If a transformation doesn't preserve the Filippov vector field, will it introduce nonlinear sliding which preserves the flow on the splitting manifold?