Using multiple time-scales to understand Dansgaard-Oeschger events

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Outline

- Introduction
- Stommel
- Hysteresis vs. Relaxation Oscillations (and Canards?)
- Extending the Model
- Forcing and Amplitude Modulation (and Canards!)

Collaborators and Acknowledgments

- Collaborative work with Raj Saha (Bates—formerly MN)
- MCRN Paleoseminar: Dick McGehee, Esther Widiasih, Samantha (Oestreicher) Schumaker
- Nonsmooth Canards: Paul Glendinning (Manchester)
- Extending Stommel's model: Emanuel Coehlo (NRL)
- Dissertation advisor: Chris Jones
- Future work extending the theory: Anna Barry, Julie Leifeld

Dansgaard-Oeschger Events

- Pulses of abrupt warming over last 100 kyr
- Up to 10 °C warming over a few decades
- Slower “cooling phase”
- Average period 1.5 kyr $\rightarrow$ internal climate mechanism
- Most intense in North Atlantic, but effects felt globally (at least as far as China)
- Hypothesized mechanism: reversal of AMOC

The Data

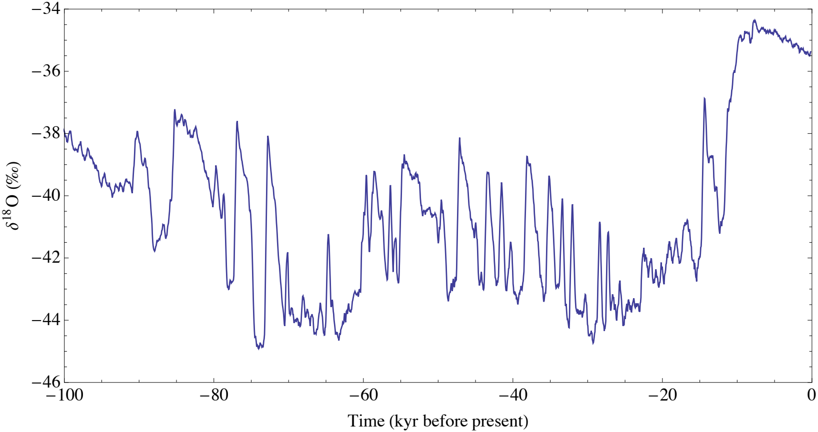


Figure: Oxygen isotope data from Greenland (NGRIP).

Bistability of the MOC

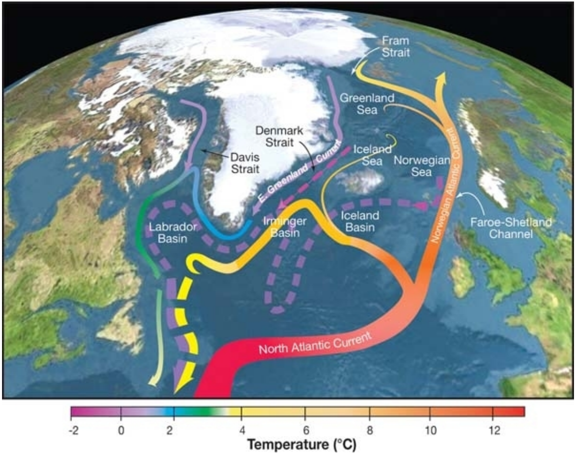


Figure: North Atlantic meridional overturning circulation (MOC).

Stommel

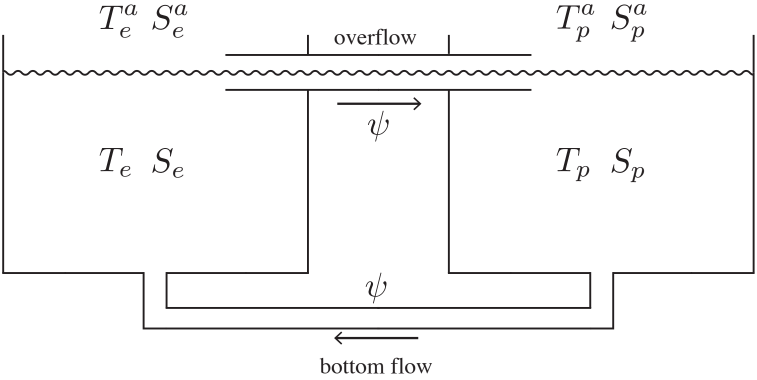


Figure: Schematic of Stommel's model.

Stommel

The model is:

$$
\begin{aligned}
\frac{d}{dt}T_e &= R_T(T_e^a - T_e) + |\psi|(T_p - T_e) \\
\frac{d}{dt}T_p &= R_T(T_p^a - T_p) + |\psi|(T_e - T_p) \\
\frac{d}{dt}S_e &= R_S(S_e^a - S_e) + |\psi|(S_p - S_e) \\
\frac{d}{dt}S_p &= R_S(S_p^a - S_p) + |\psi|(S_e - S_p) \\
\psi &= \psi_0\left(\frac{\rho_p - \rho_e}{\rho_0}\right),
\end{aligned} \tag{1}
$$

where:

- T's denote temperatures, S's denote salinities
- Subscripts: a - atmosphere, e - equator, p - pole
- ψ - transport (advection, circulation strength)
- Density $\rho_i = \rho_0[1 - \alpha(T_i - T_0) + \beta(S_i - S_0)]$.

Stommel

The change of variables:

$$\begin{array}{cc} T = T_e - T_p, & S = S_e - S_p, \\ T^a = T_e^a - T_p^a, & S^a = S_e^a - S_p^a, \\ X = T_e + T_p & Y = S_e + S_p, \end{array}$$

turns the model into

$$\begin{aligned} \frac{d}{dt} T &= R_T(T^a - T) - 2|\psi| T \\ \frac{d}{dt} S &= R_S(S^a - S) - 2|\psi| S \\ \psi &= \psi_0(\alpha T - \beta S), \end{aligned} \quad (2)$$

where X, Y decouple.

Dimensionless Stommel

To non-dimensionalize the system, set

$$x = \frac{T}{T^a}, \quad y = \frac{\beta S}{\alpha T^a}, \quad \tau = R_S t, \quad \mu = \frac{\beta S^a}{\alpha T^a}, \quad A = \frac{2\psi_0 \alpha T^a}{R_S}.$$

Then the model becomes:

$$\begin{aligned} \varepsilon \dot{x} &= 1 - x - \varepsilon A|x - y|x \\ \dot{y} &= \mu - y - A|x - y|y, \end{aligned} \tag{3}$$

where

$$\varepsilon = \frac{R_S}{R_T} \ll 1.$$

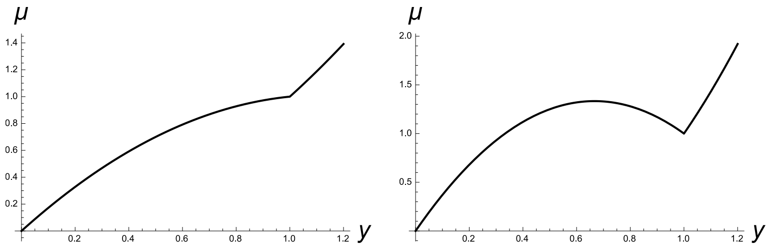
GSPT: The set $\{x = 1\}$ is attracting and normally hyperbolic.
For $\varepsilon \ll 1$, solutions will end up within $\mathcal{O}(\varepsilon)$ of $x = 1$.

Dynamics of the full system can be understood by the 1D system:

$$\dot{y} = \mu - y - A|1 - y|y. \tag{4}$$

Equilibria occur at

$$\mu = \begin{cases} (1+A)y - Ay^2 & \text{for } y < 1 \\ (1-A)y + Ay^2 & \text{for } y > 1 \end{cases} \tag{5}$$



(a) $A < 1$ (b) $A > 1$

Figure: Graphs of equilibria for (a) $A < 1$ and (b) $A > 1$.

Hysteresis

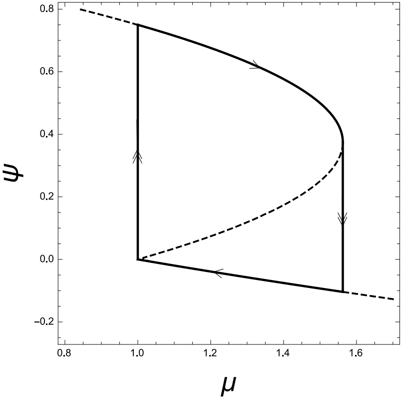


Figure: Bifurcation diagram for reduced equation (dashed), with a hysteresis loop (solid black) overlay. $\psi = \alpha\psi_0(1 - y)$.

Problem: what causes μ to vary?
Mathematics? Climate?

Hysteresis vs. Relaxation Oscillations

Hysteresis:

$$\dot{y} = \mu(t) - y - A|1 - y|y$$

Oscillations result from slowly varying parameter μ. Period and amplitude determined by external forces.

Relaxation oscillation:

$$\dot{y} = \mu - y - A|1 - y|y$$
$$\dot{\mu} = \delta g(x, y, \mu; \lambda)$$

Oscillations result from periodic orbit, μ changes based on state variables. Period and amplitude sensitive to parameters of the system.

Hopf Bifurcation

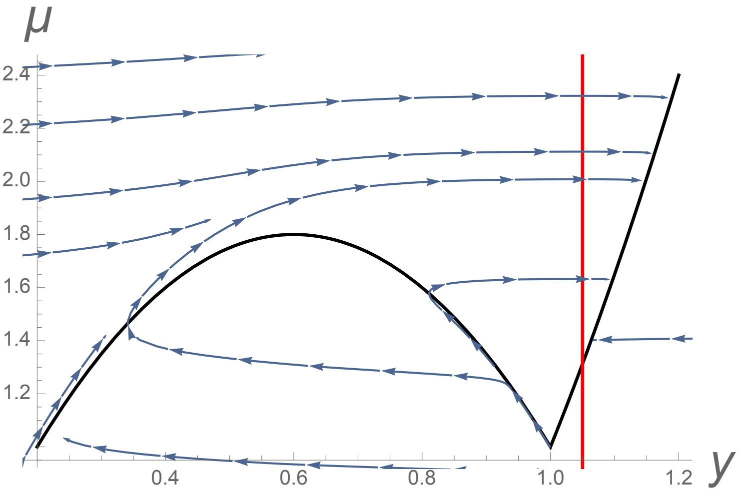


Figure: Critical manifold $\mu = A|1 - y|y$ when $A > 1$. Globally attracting equilibrium when $\lambda = 1.05$.

Hopf Bifurcation

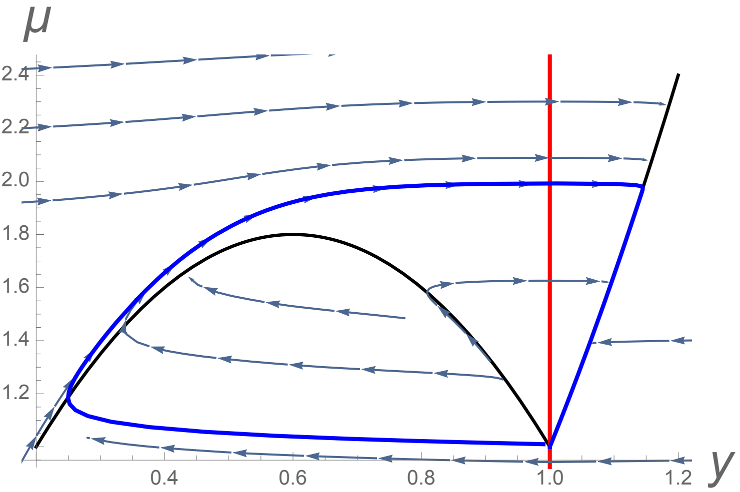


Figure: Critical manifold $\mu = A|1-y|y$ when $A > 1$. Continuum of homoclinic orbits when $\lambda = 1$.

Hopf Bifurcation

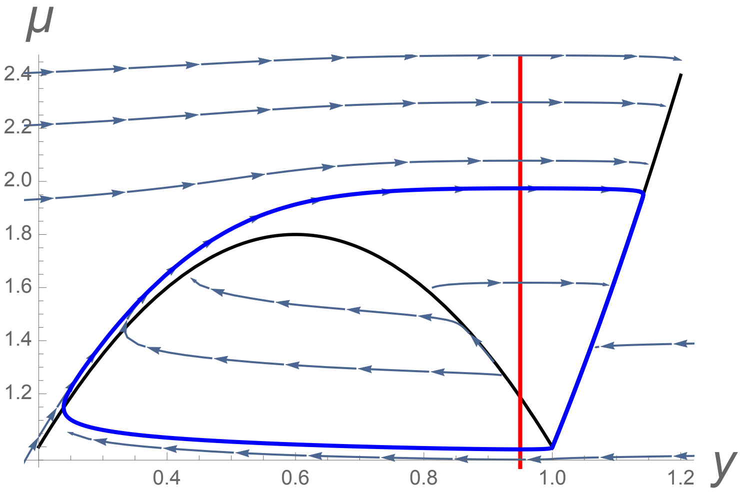
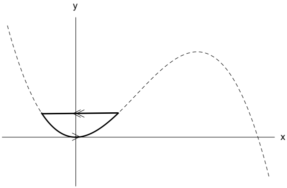
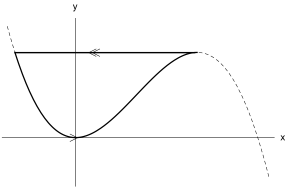


Figure: Critical manifold $\mu = A|1 - y|y$ when $A > 1$. Unique periodic orbit when $\lambda = 0.95$.

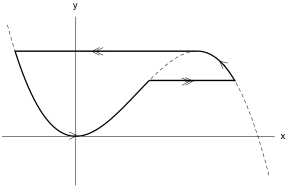
Canard cycles



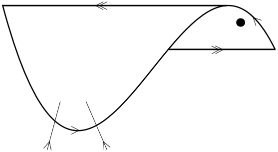
(a) Singular canard cycle.



(b) Singular maximal canard.



(c) Singular canard with head.



(d) A duck!

Figure: Singular Canards

Canard Explosion in Smooth Systems

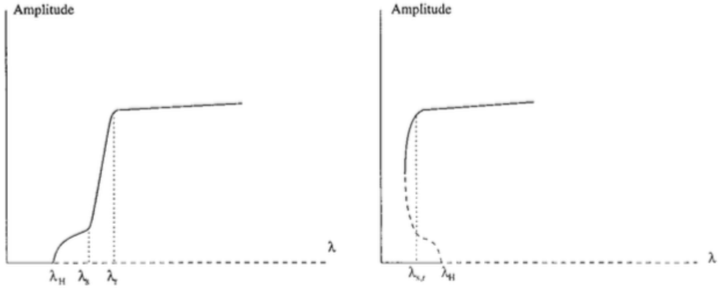
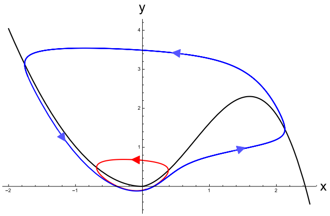
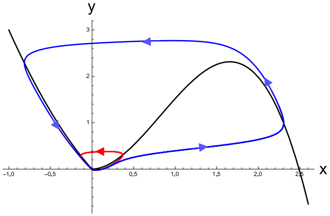


Figure: Canard explosion for fixed $\varepsilon > 0$. Figure from Krupa and Szmolyan (2001)

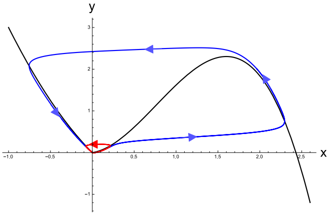
Canards in PWSC Systems?



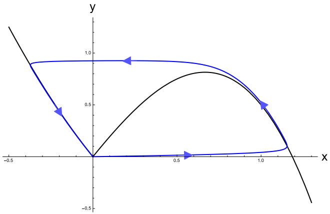
(a) Canards when Hopf bifurcation for $\lambda < 0$.



(b) Canards when Hopf bifurcation occurs for $\lambda > 0$.



(c) Canards when Hopf bifurcation occurs at $\lambda = 0$.



(d) Super-explosion.

Extending Stommel's Model

Including μ as slow state variable:

$$\begin{aligned} x' &= 1 - x - \varepsilon A|x-y|x \\ y' &= \varepsilon(\mu - y - A|x-y|y) \\ \mu' &= \varepsilon\delta(1 + ax - by). \end{aligned} \tag{6}$$

Question: what climate component is modeled by μ?

GSPT and GSPT

Critical manifold $\{x = 1\}$ is still attracting. However, the reduced problem,

$$\begin{aligned} \dot{y} &= \mu - y - A|1 - y|y \\ \dot{\mu} &= \delta(1 + a - by), \end{aligned} \tag{7}$$

is now itself a fast/slow system which is analyzed using GSPT. The critical manifold of (7) is given by

$$M_0 = \{\mu = y + A|1 - y|y\}. \tag{8}$$

Note that this is precisely the curve of equilibria from (3) the dimensionless Stommel model earlier.

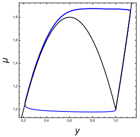
$A > 1$

To simplify the analysis, rewrite (7) as

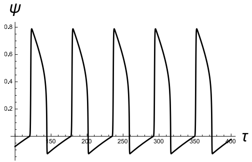
$$\begin{aligned} \dot{y} &= \mu - y - A|1 - y|y \\ \dot{\mu} &= \delta_0(\lambda - y), \end{aligned} \tag{9}$$

where $\delta_0 = \delta b$ and $\lambda = (1 + a)/b$.

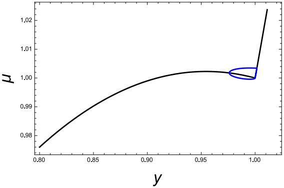
Oscillations in the Extended Stommel Model



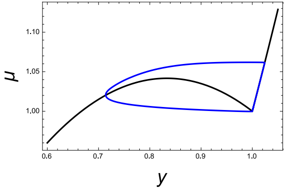
(a) Stable periodic orbit when $A = 5$, $\lambda = 0.8$, and $\delta = 0.1$



(b) Time series for ψ for the trajectory in (a)



(c) Canard trajectory when $A = 1.1$, $\lambda = 0.995$, and $\delta_0 = 0.01$.



(d) Super-explosion when $A = 1.5$, $\lambda = 0.995$, and $\delta_0 = 0.01$.

Figure: Oscillatory behavior in (9).

Astronomical Forcing

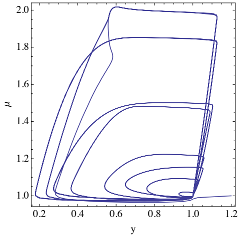
A variation in zonal insolation gradients will naturally affect the atmospheric temperature (T^a) and salinity (S^a) gradients. In system (9), the inclusion of orbital forcing implies that parameters A and λ become time-dependent. The new system becomes

$$\begin{aligned} \dot{y} = &\quad \mu - y - A(\tau)|1-y|y \\ \dot{\mu} = &\quad \delta_0(\lambda(\tau) - y), \end{aligned} \tag{10}$$

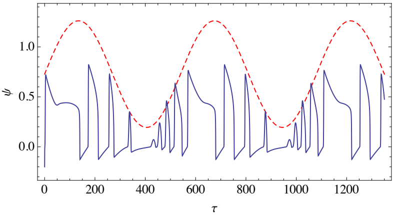
where

$$\begin{aligned} A(\tau) &= \bar{A} + p \sin \omega\tau \\ \lambda(\tau) &= \bar{\lambda} + q \sin \omega(\tau - \theta). \end{aligned}$$

Amplitude Modulation



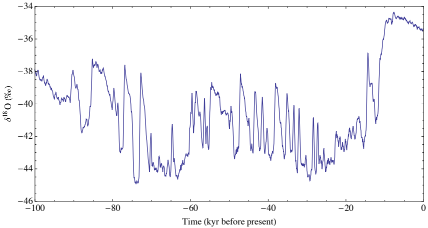
(a) Trajectory of the forced system in phase space.



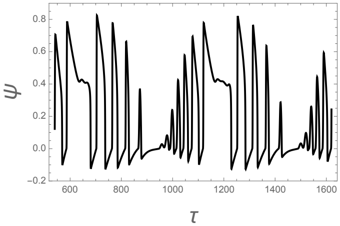
(b) Time series for ψ (solid) with the scaled obliquity (dashed) variations from the last 100 kyr. The units of time are arbitrary.

Figure: Oscillations in the forced system (10) when $\delta_0 = 0.07$, $\bar{A} = 3.5$, $p = 2.4$, $\bar{\lambda} = 0.8$, $q = 1.99$, $\omega = \pi/270$, and $\theta = 250$.

Comparison with data



(a) Oxygen isotope data from Greenland (NGRIP).



(b) Time series for ψ (solid) in (7).

Figure: Dansgaard-Oeschger cycles in (a) data and (b) a conceptual model (7). The parameters used to generate the times series in (b) are $\delta_0 = 0.1$, $\bar{A} = 3.5$, $p = 2.4$, $\bar{\lambda} = 0.8$, $q = 1.99$, $\omega = \pi/270$, and $\theta = 250$.