Homoclinic Bifurcation in a Climate Application

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Welander's Model

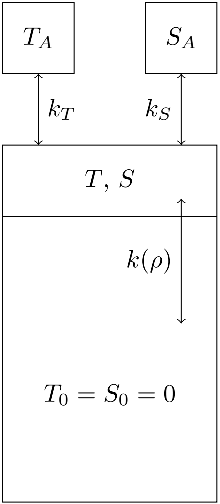
T =Temperature, S =Salinity, ρ =Density

The Model:

$$
\begin{aligned}
\dot{T} &= 1 - T - k(\rho)T \\
\dot{S} &= \beta(1 - S) - k(\rho)S \\
\rho &= -\alpha T + S \\
k(\rho) &= \tfrac{1}{\pi}\tan^{-1}\left(\frac{\rho - \varepsilon}{a}\right) + \tfrac{1}{2}
\end{aligned}
$$

Parameters:

$$\alpha = \frac{4}{5}, \beta = \frac{1}{2}$$



Preliminary Coordinate Change

$$x = T,\ y = \rho - \varepsilon$$

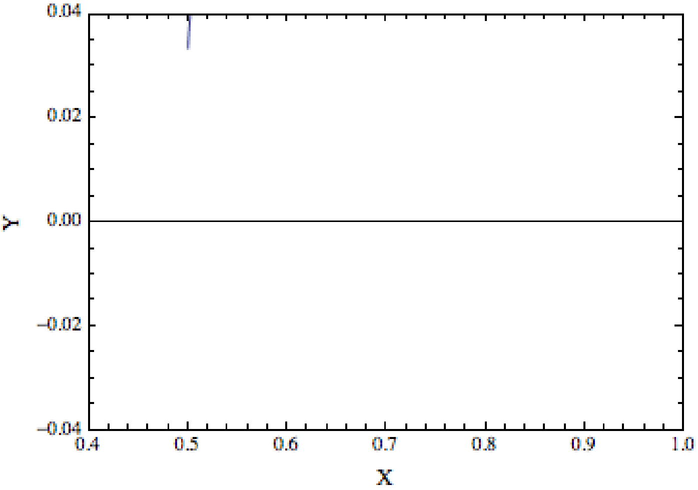
The new system:

$$\begin{aligned} \dot{x} &= 1 - x - k(y)x \\ \dot{y} &= \beta - \beta\varepsilon - \varepsilon k(y) - \alpha - (\beta + k(y))y - (\alpha\beta - \alpha)x \end{aligned}$$

$$k = \frac{1}{\pi}\tan^{-1}\left(\frac{y}{a}\right) + \frac{1}{2} \to \begin{cases} 1 & y > 0 \\ 0 & y < 0 \end{cases}$$

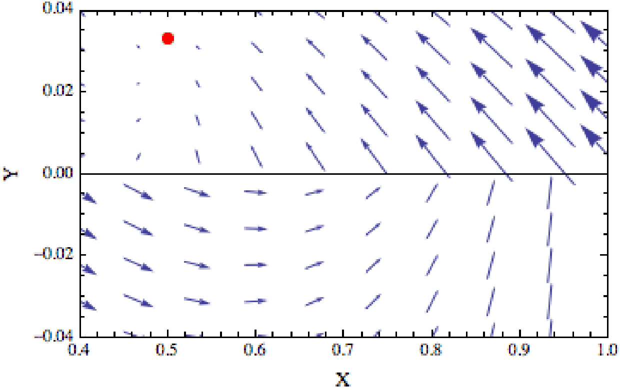
Oscillations In Welander's Model

An interesting bifurcation happens as $\varepsilon = -\frac{1}{15}$.

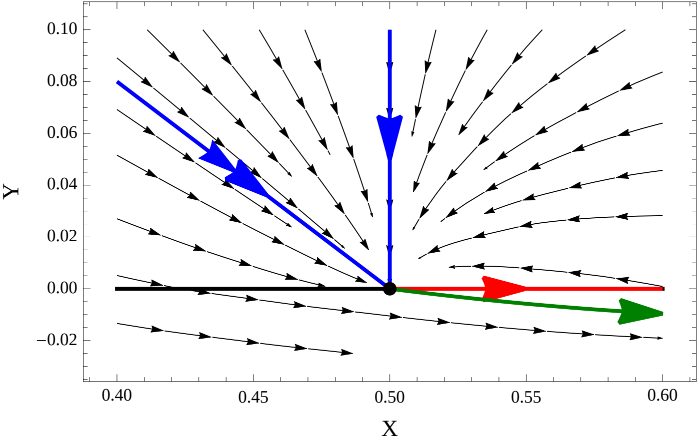


A Local Picture

The boundary collision can be thought of as a stability transition as a pseudoequilibrium leaves the splitting manifold.

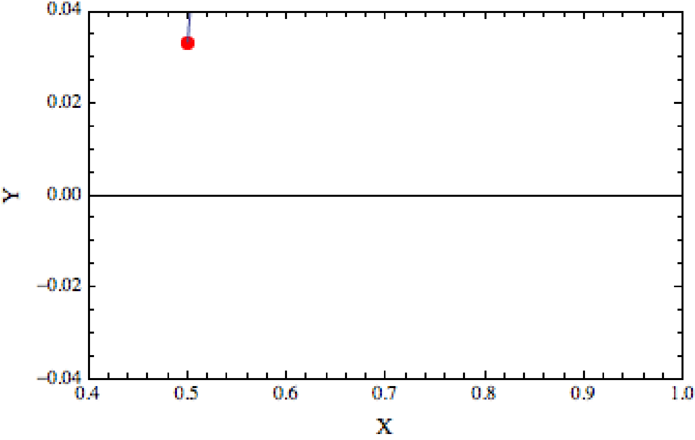


A Local Picture



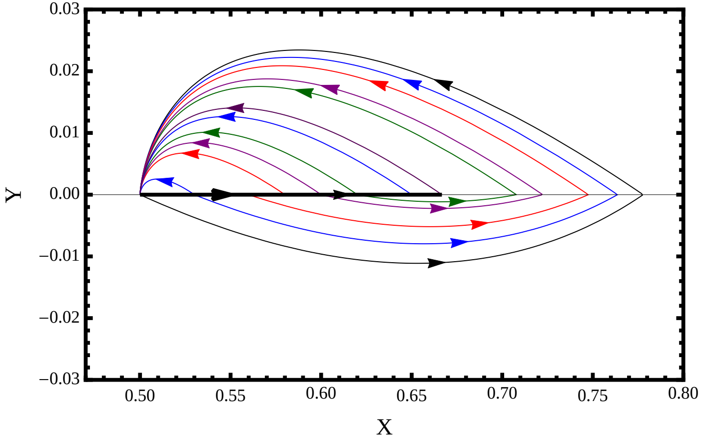
A Global Picture

The periodic orbit is destroyed through a homoclinic bifurcation. It is easy to show that the periodic orbit limits to a homoclinic orbit through the point $\left(\frac{1}{2}, 0\right)$.

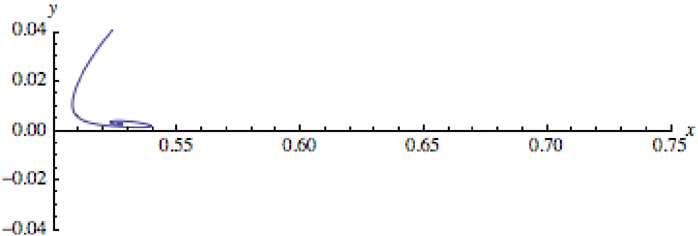


A Global Picture

Infinitely many homoclinic orbits go through the bifurcating equilibrium. There is no equilibrium point inside any of the orbits.

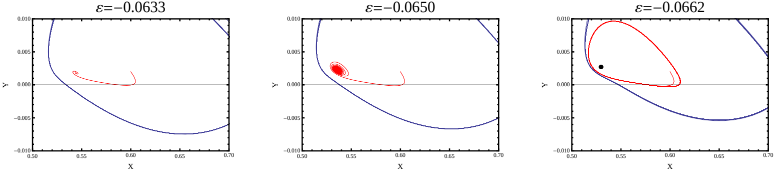


The Smooth Model

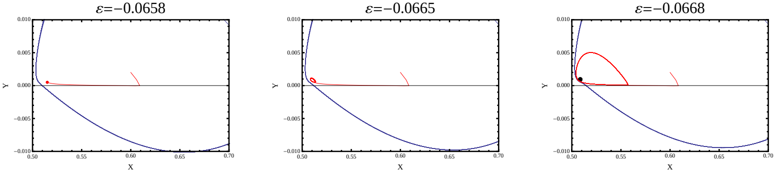


The smooth model

$$a = 10^{-3}$$



$$a = 10^{-4}$$



Remarks and Questions

- In Welander's Model, the periodic orbit is destroyed through a nonsmooth homoclinic bifurcation.
- This bifurcation is a limit of two phenomena in the smooth system, a subcritical Hopf bifurcation, and a periodic orbit saddle node.
- Near the bifurcation point, the nonsmooth system is not qualitatively similar to the perturbed smooth system, even though the transition function is monotonic.
- Is the homoclinic bifurcation always the limit of a subcritical Hopf bifurcation and a periodic orbit saddle node bifurcation?

References

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