# Catastrophes and Resilience of a Zero-Dim Climate System

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# Motivation: exhibit catastrophes in realistic situations

K. Fraedrich, "Catastrophes and resilience of a zero dim'l climate system w/ ice-albedo and greenhouse feedback" *Quart. J. R. Met. Soc.* (1979)

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*A catastrophe is a singularity (of a smooth map), exhibited by (small) changes of an external parameter.*

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- The author's goal is to exhibit fold catastrophes and cusp catastrophes.

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3. Along the way: bifurcation diagrams

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- Most important factor: incoming and outgoing radiation

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# Assumptions

$$c\frac{dT}{dt} = R\downarrow - R\uparrow,$$

where

- $T$ is temperature,
- $t$ is time, and
- $R\downarrow$ and $R\uparrow$ are incoming and outgoing radiation.

Here, $c$ is a positive constant, determined by external measurements.

# Assumptions: incoming radiation

$$R \downarrow = \frac{1}{4}\mu I_0(1 - \alpha_p).$$

- $I_0$ and $\mu$ are used to account for variations in the solar constant, planetary orbit (von Woerkom 53).

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  - For Fraedrich:

    $$\alpha_p = a_2 - b_2 T^2$$

# Assumptions: outgoing radiation

$$R \uparrow = \epsilon_{sa} \sigma T^4 = \epsilon_s \sigma T^4 - \epsilon_a \sigma T^4,$$

(Stefan-Boltzmann)

- $\epsilon_{sa} = \epsilon_s - \epsilon_a$

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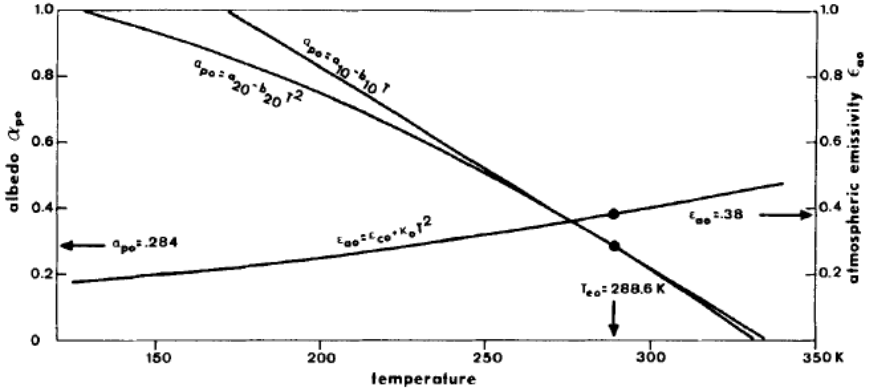
# Picture of temp feedbacks (from p. 149)



# First model

Use equations

$$c\frac{dT}{dt} = R\downarrow - R\uparrow; \tag{1}$$

$$R\downarrow = \frac{1}{4}\mu I_0(1-\alpha_p), \tag{2}$$

where $\alpha_p$ is a constant;

$$R\uparrow = \epsilon_{sa}\sigma T^4, \tag{3}$$

where $\epsilon_{sa} = \epsilon_s - \epsilon_a$, which are constants.
Let $x = (\mu, \alpha_p, \epsilon_s, \epsilon_a, c)$.

# First model - solving

Let $\frac{dT}{dt} = f(T; x)$, where $x$ is the vector of parameter values.
Let $T_e(x)$ stand for the temperature(s) at which $\frac{dT}{dt}\Big|_{T_e} = f(T_e; x) = 0$.

- Linearization about $T = T_e$:

$$dT/dt \approx f(T_e; x) + \frac{df}{dT}\Big|_{T_e} (T - T_e) = -\lambda(T - T_e).$$

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$$-\lambda = \frac{df}{dT}\Big|_{T_e} < 0,$$

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  - Unstable:

$$-\lambda = \frac{df}{dT}\Big|_{T_e} > 0.$$

# First model - solving

Want:

$$f(T_e; x) = 0,$$
$$\frac{df}{dT}\Big|_{T_e} = \frac{d}{dT}\frac{1}{c}\left(-\epsilon_{sa}\sigma T^4 + \frac{1}{4}\mu I_0(1-\alpha_p)\right)\Big|_{T_e}$$
$$= -4(\epsilon_{sa}\sigma/c)T_e^3 < 0.$$

We have a single, stable equilibrium:

$$T_e = \sqrt[4]{\frac{\mu I_0}{4\epsilon_{sa}\sigma}(1-\alpha_p)}.$$

# First model - solving

Want:

$$\begin{aligned} f(T_e; x) &= 0, \\ \frac{df}{dT}\Big|_{T_e} &= \frac{d}{dT}\frac{1}{c}\left(-\epsilon_{sa}\sigma T^4 + \frac{1}{4}\mu I_0(1-\alpha_p)\right)\Big|_{T_e} \\ &= -4(\epsilon_{sa}\sigma/c)T_e^3 < 0. \end{aligned}$$

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# First model - solution

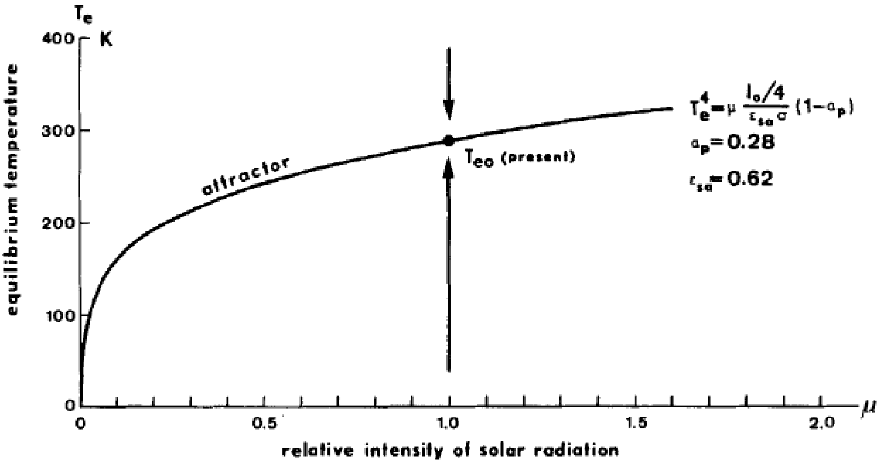
$$T_e = \sqrt[4]{\frac{\mu I_0}{4\epsilon_{sa}\sigma}(1-\alpha_p)}.$$

Use

$$x_0 = \left(\alpha_{p0} = 0284,\ \epsilon_{sa0} = 0.62,\ \mu_0 = 1,\ c_0 = 10^8 kgK^{-1}s^{-2}\right),$$

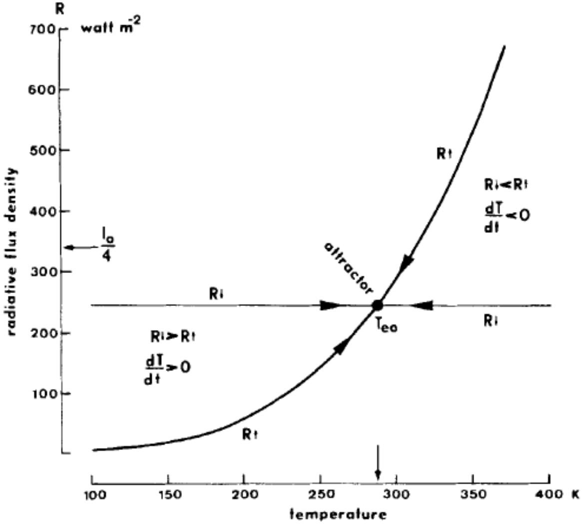
and to get the "present day" (1979) averaged equilibrium: $T_{e0} = 288.6K$.

# First model - equilibrium diagram




p. 152

# First model - phase portrait




p. 152

# Second model - ice albedo

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- Fraedrich uses

$$\alpha_{p0} = a_{20} - b_{20}T^2,$$

where $a_{20}$ and $b_{20}$ are chosen to match present day albedo, and

$$\alpha_{p0} = \frac{d(a_{10} - b_{10}T)}{dt}$$

at present day.

# Second model - ice albedo feedback

This time,

$$\frac{dT}{dt} = (1/c)\left(-\epsilon_{sa}\sigma T^4 + \frac{1}{4}\mu I_0 b_2 T^2 + \frac{1}{4}\mu I_0(1 - a_2)\right) = f(T; x).$$

The equilibria $T_e$ are again where $f(T_e; x) = 0$, where

$$x = (a_2, b_2, \epsilon_{sa}, \mu, c).$$

# Second model - equilibria

Equilibria are (positive) solutions to

$$T_e^4 - mT_e^2 + n = 0,$$

$$\text{where } m = \frac{\mu I_0}{4\epsilon_{sa}\sigma} b_2,$$

$$n = -\frac{\mu I_0}{4\epsilon_{sa}\sigma}(1 - a_2).$$

Allowing $\mu$ to vary:

- $T_e^4 - mT_e^2 + n = 0$

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# Second model - ice albedo - new parameters

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- There are 0, 1, or 2 solutions, depending on the values of $m$, $n$.

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- There are 0, 1, or 2 solutions, depending on the values of $m$, $n$.
  - What is the sign of $\frac{1}{4}m^2 - n$?

# Second model - stability of equilibria

Equilibria:

$$T_e^{\pm} = \sqrt{\frac{1}{2}m \pm \sqrt{\frac{1}{4}m^2 - n}},$$

$$\frac{dT}{dt} = \frac{1}{c}f(T;x),$$

Assume at least one solution $(\frac{1}{4}m^2 - n \geq 0)$ Determine the stability the same way as in the trivial model.

- Linearize about $T_e$: $T(t;x) \approx \frac{df}{dT}\Big|_{T_e}(T - T_e)$

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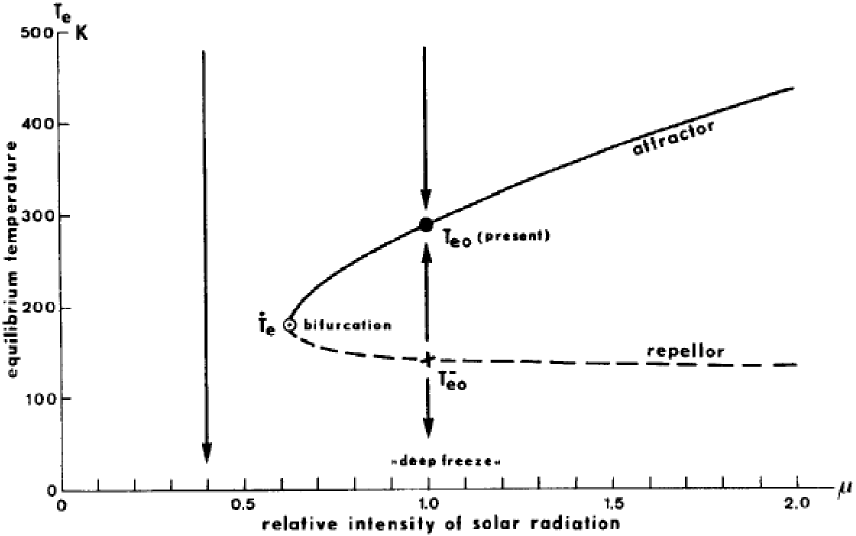
- Unstable:

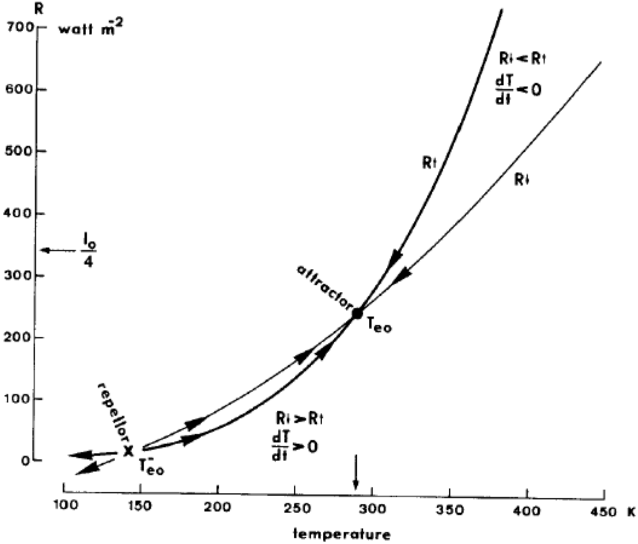
$$-\frac{4\epsilon_{sa}}{c}\left(T_e^2 - m/2\right)T_e > 0$$

- $T_e^+$ stable, $T_e^-$ unstable

# Second model - ice albedo - equilibria

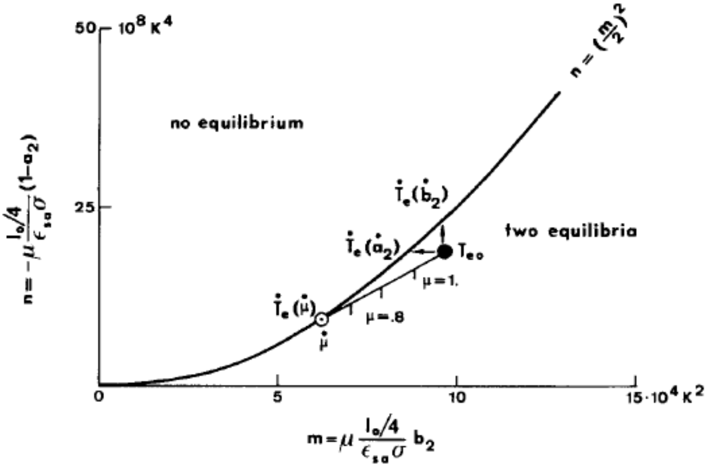


# Second model - ice albedo - phase portrait



# Second model - bifurcation diagram

$$\text{Equilibria: } T_e^{\pm} = \sqrt{\frac{1}{2}m \pm \sqrt{\frac{1}{4}m^2 - n}}$$



# Third model

Greenhouse feedback, no ice albedo:

$$c\frac{dT}{dt} = R\downarrow - R\uparrow$$

with

$$R\downarrow = \frac{1}{4}\mu I_0(1-\alpha_p),$$

where $\alpha_p$ is held constant;

$$R\uparrow = \epsilon_s \sigma T^4 - \epsilon_a \sigma T^4,$$

where $\epsilon_a = \epsilon_c + \kappa T^2$, ($\epsilon_c$ is $CO_2$ emittance), so

$$R\uparrow = \epsilon_s \sigma T^4 - \epsilon_c \sigma T^4 - \kappa \sigma T^6.$$

# Third model - equilibria

Combining the previous equations gives

$$\frac{dT}{dt} = \frac{1}{c}\left(\kappa\sigma T^6 - \epsilon_{sc}\sigma T^4 + \frac{1}{4}\mu I_0(1-\alpha_p)\right),$$

where $\epsilon_{sc} = \epsilon_s - \epsilon_c$.
Equilibria satisfy

$$T_e^6 - \frac{\epsilon_{sc}}{\kappa}T_e^4 + \frac{\mu I_0}{4\kappa\sigma}(1-\alpha_p) = 0.$$

# Third model - equilibria

Equilibria:

$$T_e^6 - \frac{\epsilon_{sc}}{\kappa} T_e^4 + \frac{\mu I_0}{4\kappa\sigma}(1 - \alpha_p) = 0.$$

To solve, let $y = -\frac{\epsilon_{sc}}{3\kappa} + T_e^2$ and get

$$y^3 - uy + v = 0,$$

where

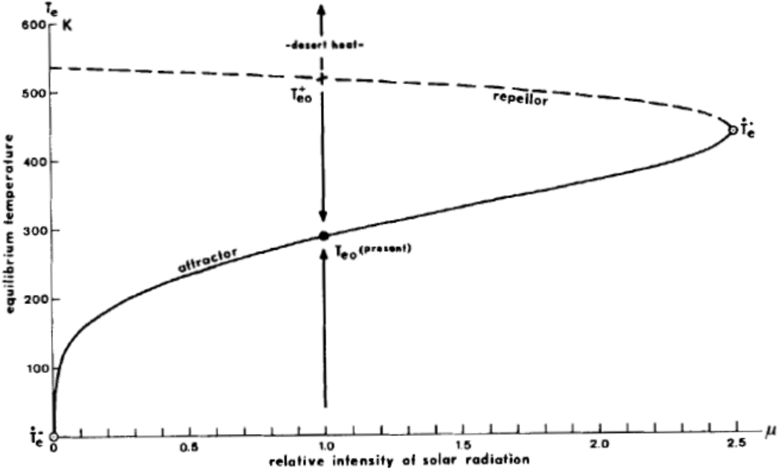
$$u = 3\left(\frac{\epsilon_{sc}}{3\kappa}\right)^2, \; v = -2\left(\frac{\epsilon_{sc}}{3\kappa}\right)^3 + \frac{\mu I_0}{4\kappa\sigma}(1 - \alpha_p).$$

We get

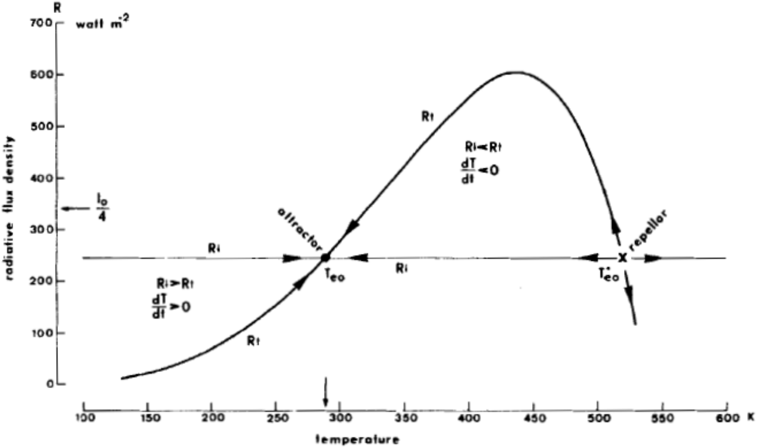
$$T_e = \sqrt{\frac{\epsilon_{sc}}{3\kappa} + 2\sqrt{u/3}A},$$

where $A$ depends on $u$ and $v$.

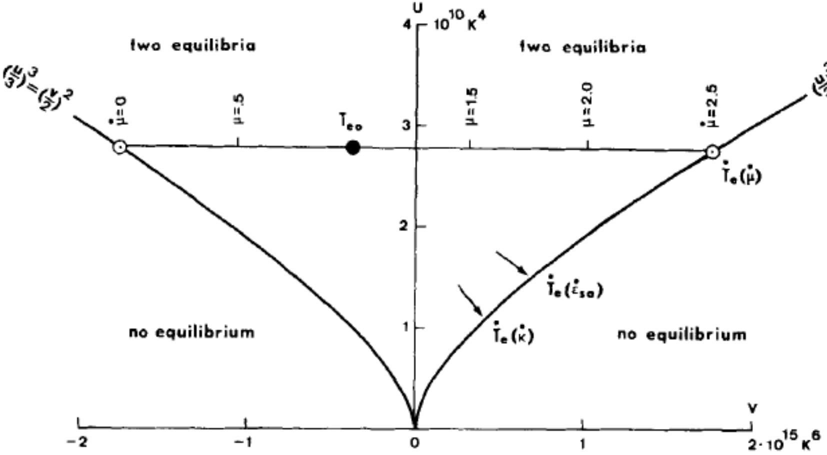
# Third model - equilibria diagram



# Third model - phase portrait



# Third model - bifurcation diagram



# Forth model - $CO_2$ and ice

Combining greenhouse temperature feedback and ice temperature feedback:

$$\alpha_p = a_2 - b_2 T^2, \; \epsilon_a = \epsilon_c + \kappa T^2,$$

we get

$$\frac{dT}{dt} = \frac{1}{c}\left(\kappa\sigma T^6 - \epsilon_{sc}\sigma T^4 + \frac{1}{4}\mu I_0 b_2 T^2 + \frac{1}{4}\mu I_0 (1 - a_2)\right).$$

The equilibria are solutions to

$$T_e^6 - T_e^4 \frac{\epsilon_{sc}}{\kappa} + T_e^2 \left(\frac{\mu I_0}{4\kappa\sigma}\right) b_2 + \frac{\mu I_0}{4\kappa\sigma}(1 - a_2) = 0.$$

# Fourth model - solving

$$T_e^6 - T_e^4 \frac{\epsilon_{sc}}{\kappa} + T_e^2 \left( \frac{\mu I_0}{4\kappa\sigma} \right) b_2 + \frac{\mu I_0}{4\kappa\sigma}(1 - a_2) = 0$$

Let $y = -\frac{\epsilon_{sc}}{3\kappa} + T_e^2$, and this becomes

$$y^3 - py + q = 0,$$

where

$$p = 3 \left( \frac{\epsilon_{sc}}{3\kappa} \right)^2 - \frac{\mu I_0}{4\kappa\sigma} b_2$$
$$q = -2 \left( \frac{\epsilon_{sc}}{3\kappa} \right)^3 + \frac{\epsilon_{sc}\mu I_0}{3 \cdot 4\kappa^2\sigma} b_2 + \frac{\mu I_0}{4\kappa\sigma}(1 - a_2).$$

The parameters are $x = (a_2,\ b_2,\ \epsilon_s,\ \epsilon_c,\ \kappa,\ \mu)$.

# Fourth model - stability

At $T = T_e$, stability is determined by

$$-\lambda = \frac{df}{dT}\Big|_{T_e} = \frac{\kappa\sigma}{c}\left(T_e^4 - \frac{2\epsilon_{sc}}{3\kappa}T_e^2 + \frac{\mu I_0 b_2}{4\cdot 3\kappa\sigma}\right)6T_e$$

- Stable: $-\lambda < 0$

With certain parameter values, there are three equilibrium branches: one attractor, two repellers.

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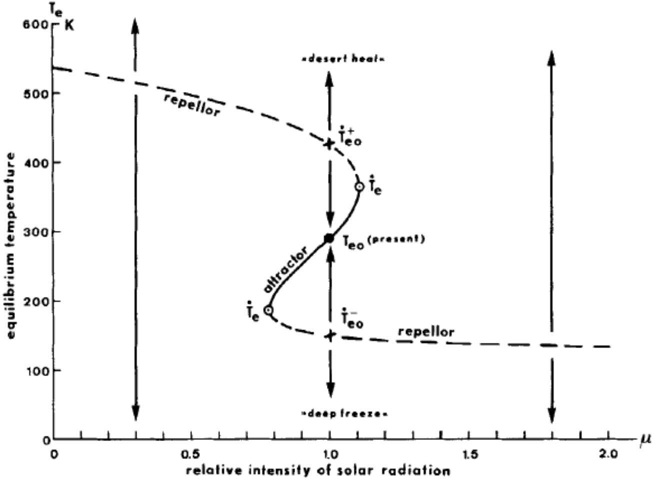

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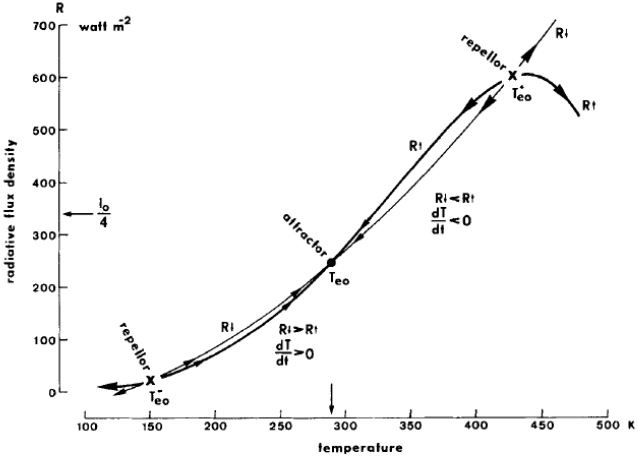
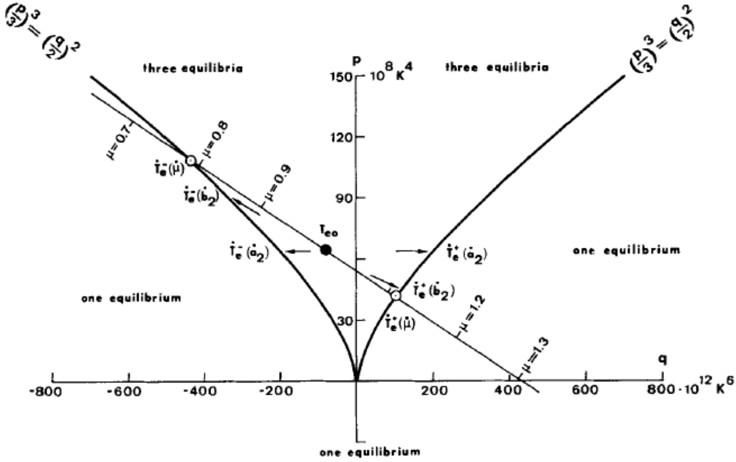
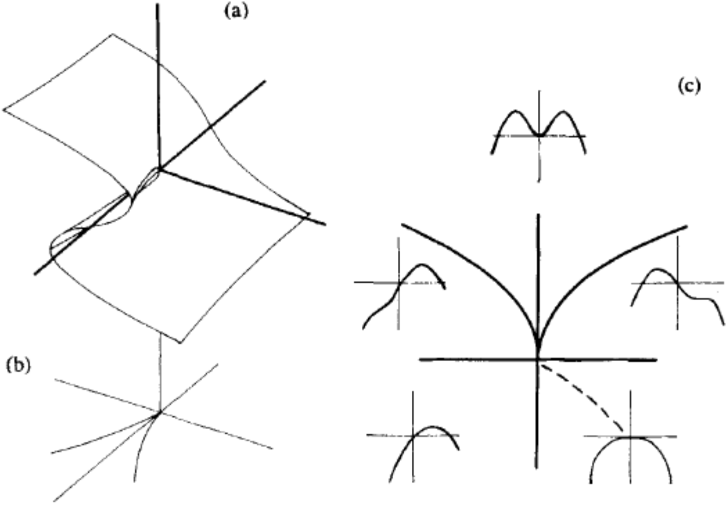


# Fourth model - bifurcation diagram



# Fourth model - some perspective



# References

1. K. Fraedrich. "Catastrophes and resilience of a zero-dimensional climate system with ice-albedo and greenhouse feedback." *Quart. J. R. Met. Soc.* **105**, 147-167 (1979).