Dynamics of Energy Balance Models for Planetary Climate

Alice Nadeau

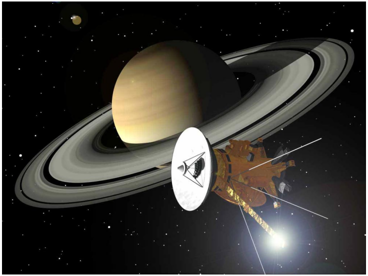
University of Minnesota

April 13, 2016

Motivation

- Low dimensional climate models are important for understanding the predominant forces affecting the climate of Earth

- Fly-bys of Pluto and other rocky celestial bodies in our solar system have raised interest in other climates



Photos from nasa.gov

Energy Balance Models in 1969

- Budyko and Sellers (independently) proposed energy balance models for the Earth (1, 14)
- Wanted to study if another glacial age was possible
- Both models had the same major components: incoming solar radiation, outgoing radiation, and energy transfer:

$$R\Delta T = Q(y)(1 - \alpha(y)) - (A + BT) + \Gamma(T)$$

Energy Balance Models Today

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + \Gamma(T)$$

where

$$\Gamma(T(y,\eta)) = -C\left(T(y,\eta) - \int_0^1 T(\gamma,\eta)d\gamma\right).$$

Widiasih introduced an equation for the dynamics of the ice line, η, in 2012 (18)

$$\frac{d}{dt}\eta = \epsilon(T(y,\eta) - T_c)$$

"Quadratic Approximation"

In (9) McGehee and Widiasih consider the Budyko-type equation for $y, \eta \in [0,1]$

$$\frac{\partial}{\partial t}T = \frac{1}{R}\left(Qs(y)(1-\alpha(\eta,y)) - (A + BT(y,\eta)) - C\left(T(y,\eta) - \overline{T}\right)\right)$$

with dynamic ice line

$$\dot{\eta} = \rho(T(y,\eta) - T_c)$$

and piecewise constant albedo function

$$\alpha(y,\eta) = \begin{cases} \alpha_w & y < \eta \\ \alpha_0 & y = \eta \\ \alpha_i & y > \eta \end{cases}$$

where

$$\alpha_0 = \frac{\alpha_w + \alpha_i}{2}.$$

Piecewise Function Space

The equilibrium temperature profile can be written

$$T^*_\eta(y) = \begin{cases} \frac{1}{B+C}(Qs(y)(1-\alpha_w) - A + C\overline{T^*_\eta}) & y < \eta \\ \frac{1}{B+C}(Qs(\eta)(1-\alpha_0) - A + C\overline{T^*_\eta}) & y = \eta \\ \frac{1}{B+C}(Qs(y)(1-\alpha_i) - A + C\overline{T^*_\eta}) & y > \eta \end{cases}$$

which motivates the four-dimensional function space X whose elements are of the form

$$T(y) = \begin{cases} w_0 + \frac{1}{2}z_0 + \left(w_2 + \frac{1}{2}z_2\right) p_2(y) & y < \eta \\ w_0 + w_2 p_2(\eta) & y = \eta \\ w_0 - \frac{1}{2}z_0 + \left(w_2 - \frac{1}{2}z_2\right) p_2(y) & y > \eta \end{cases}$$

New Budyko's "Equation"

Reformulating the $\partial_t T$ equation in this function space gives

$$\begin{aligned} R\dot{w}_0 &= Q(1-\alpha_0) - A - Bw_0 + C\left((\eta - \frac{1}{2})z_0 + z_2\int_0^\eta p_2(y)dy\right) \\ R\dot{z}_0 &= Q(\alpha_i - \alpha_w) - (B+C)z_0 \\ R\dot{w}_2 &= Qs_2(1-\alpha_0) - (B+C)w_2 \\ R\dot{z}_2 &= Qs_2(\alpha_i - \alpha_w) - (B+C)z_2 \end{aligned}$$

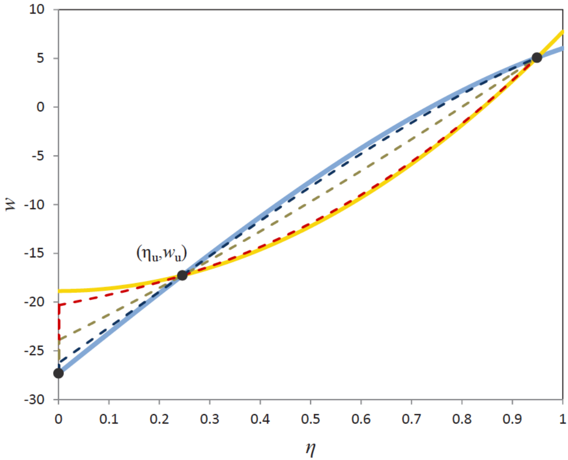
and the ice line equation becomes

$$R\dot{\eta} = \rho\left(w_0 - \frac{Q(1-\alpha_0)}{B+C}s_2 p_2(\eta) + T_c\right).$$

Write

$$\begin{aligned} R\dot{w}_0 &= -B(w_0 - F(\eta)) \\ R\dot{\eta} &= -\rho(w_0 - G(\eta)) \end{aligned}$$

(η, w_0) Phase Space



$$R\dot{w}_0 = -B(w_0 - F(\eta))$$

$$R\dot{\eta} = -\rho(w_0 - G(\eta))$$

The Jormungand Model

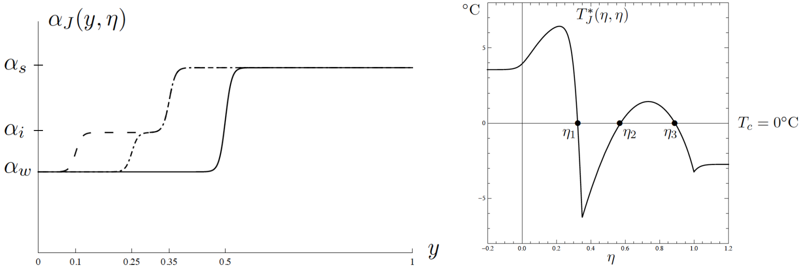
Define the function

$$\delta(\eta) = \begin{cases} -\eta + .35, & \eta < .35 \\ 0, & \eta \geq .35 \end{cases}$$

which represents the extent of the bare ice and the Jormungand albedo function

$$\alpha_J(y, \eta) = \frac{\alpha_s + \alpha_w}{2} + \frac{\alpha_i - \alpha_w}{2} \tanh(M(y - \eta)) + \frac{\alpha_s - \alpha_i}{2} \tanh(M(y - (\eta + \delta(\eta))))$$

Jormungand albedo function and Temperature Profile



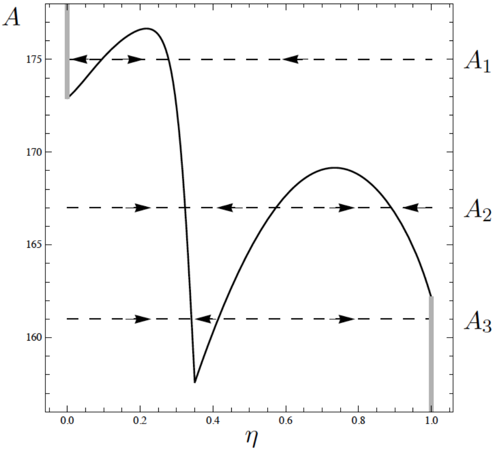
- Widiasih's Theorem still applies with this albedo function
- There is a locally attracting invariant manifold

$$\mathcal{P}_J^* = \{(\Phi_J^*(\eta), \eta) : \eta \in \mathbb{R}\}$$

within $\mathcal{O}(\epsilon)$ of the manifold of fixed points

$$\mathcal{T}_J^* = \{(T_J^*(y,\eta), \eta) : \eta \in \mathbb{R}\}$$

Jormungand Bifurcation in the Greenhouse Gas Parameter



Insolation Distribution on Rapidly Spinning Planets

- Actual distribution can be found using orbital parameters (as seen in (5, 8, 17)):

$$s(y,\beta) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\sin\gamma - y\cos\beta\right)^2}\,d\gamma$$

- We use Legendre approximations in EBMs because the above integral doesn't have a closed form expression. Instead use

$$s(y,\beta) = \sum_{n=0}^{\infty} s_{2n}(\beta)p_{2n}(y)$$

Insolation Distribution on Rapidly Spinning Planets

Write the $2n$-th degree Legendre polynomial as

$$p_{2n}(y) = \sum_{k=0}^{n} a_{2k} y^{2k}$$

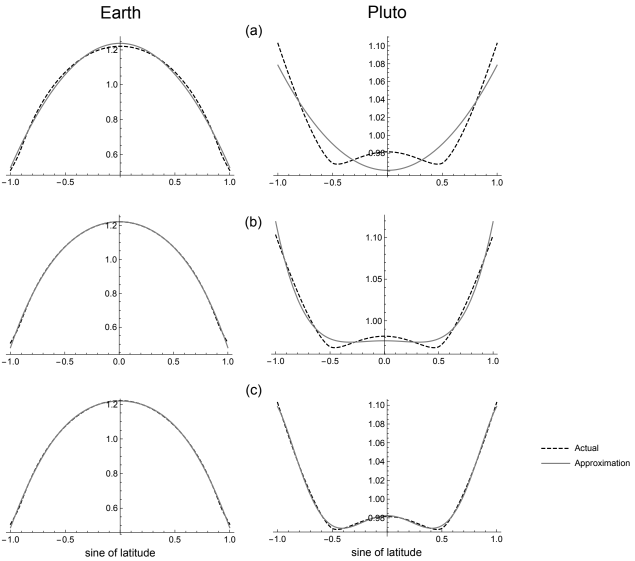
and

$$s_{2n}(\beta) = P_{2n} \sum_{k=0}^{n} a_{2k} c_{2k}(\beta).$$

where

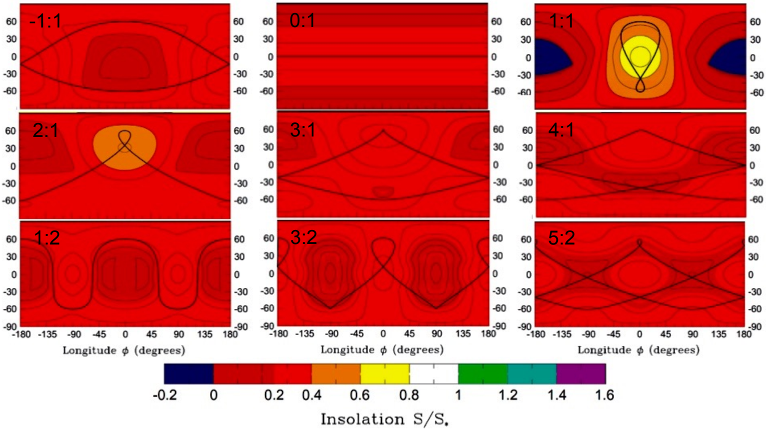
$$c_{2k}(\beta) = \sum_{j=0}^{k} \binom{2k}{2j} \frac{(\sin\beta)^{2(k-j)}(\cos\beta)^{2j}}{\pi^2} \cdot \left(\int_{-\pi/2}^{\pi/2} (\cos\hat{\phi})^{2(k+1-j)}(\sin\hat{\phi})^{2j}\, d\hat{\phi}\right)\left(\int_{0}^{2\pi} (\cos\hat{\theta})^{2(k-j)}\, d\hat{\theta}\right)$$

Insolation Distribution on Rapidly Spinning Planets



Small Integer Spin-Orbit Resonances

Mean annual insolation distributions for obliquity $\beta = 60^\circ$ and eccentricity $e = .2$:



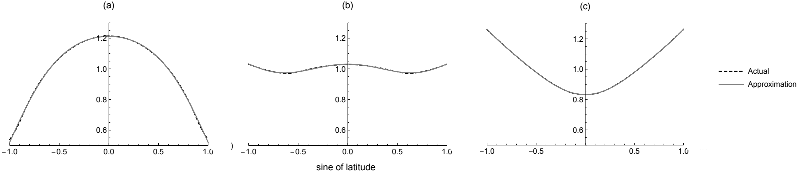
A. Dobrovolskis, "Insolation on exoplanets with eccentricity and obliquity."

Open Questions about Insolation

- Can we quantify when we can use the "rapidly spinning planet" method/formula and still have error less than τ in our approximation?
- Can we find a closed form expression for insolation on planets with small integer resonances?

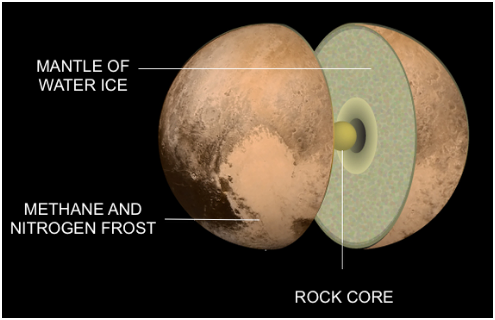
Open Questions about Insolation

- For a reasonable range of parameter values, the Budyko map for Pluto doesn't have any nontrivial stable fixed points. Why?
  - Could Pluto's "upside down" insolation be playing a factor?
  - Is it because Pluto's insolation is relatively flat?



Open Questions about Ice Lines

- Reformulate model to accommodate ice planets
  - Is more than one ice present and how do we account for different albedos?
  - How are the ices situated on the surface? Are ices mixed? Are they layered?



From nasa.gov

Open Questions about Ice Lines

- Remove symmetry assumptions?
  - Investigate four ice lines $(\eta_{SP}, \eta_{SE}, \eta_{NE}, \eta_{NP})$ with the properties

    (i) $\eta_{SP}, \eta_{SE}, \eta_{NE}, \eta_{NP} \in [-1, 1]$.
    (ii) $-1 \leq \eta_{SP} \leq \eta_{SE} \leq \eta_{NE} \leq \eta_{NP} \leq 1$.
    (iii) Ice is located between η_{SP} and η_{SE} and between η_{NE} and η_{NP}.

  - Initial investigations into this case show potential oscillations in the ice line dynamics

Other Open Questions Concerning EBMs

- How do we accurately model a planet whose atmosphere "freezes out" (as Pluto's might)?
- Is the diffusion model more accurate for a planet with no oceans? For a planet without an atmosphere?
  - In which cases do we get the same results with both models?
  - In which cases do we get different results?

(1) M. Budyko. "The effect of solar radiation variations on the climate of the Earth." *Tellus*, **21**(5): 1969. 611-619.

(2) R. Cahalan and G. North. "A Stability Theorem for Energy-Balance Climate Models." *Journal of the Atmospheric Sciences*, **36**: 1979. 1178-1188.

(3) P Chýlek and J. Coakley. "Analytical Analysis of a Budyko-Type Climate Model." *Journal of the Atmospheric Sciences*, **32**: 1975. 675-679.

(4) A. Dobrovolskis."Spin States and Climates of Eccentric Exoplanets." *Icarus* **192**: 2007. 1-23.

(5) A. Dobrovolskis."Insolation patterns on synchronous exoplanets with obliquity." *Icarus* **204**: 2009. 1-10.

(6) A. Dobrovolskis."Insolation on exoplanets with eccentricity and obliquity." *Icarus* **226**: 2013. 760-776.

(7) H. Kaper and H. Engler. *Mathematics and Climate*, **Society for Industrial and Applied Mathematics**: 2013.

(8) R. McGehee and C. Lehman. "A Paleoclimate model of Ice Albedo Feedback Forced by Variations in Earth's Orbit." *SIAM J. Applied Dym. Sys.*, **11** (2): 2012.

(9) R. McGehee and E. Widiasih. "A Quadratic Approximation to Budyko's Ice-Albedo Feedback Model with Ice Line Dynamics." *SIAM J. Applied Dym. Sys.*, **13** (1): 2014.

(10) G. North. "Analytical Solution to a Simple Climate Model with Diffusive Heat Transport." *Journal of the Atmospheric Sciences*, **32** (7): 1975. 1301-1307.

(11) G. North. "Theory of Energy Balance Climate Models." *Journal of the Atmospheric Sciences*, **32** (11): 1975. 2033-2043.

(12) G. North. "The Small Ice Cap Instability in diffusive Climate Models." *Journal of the Atmospheric Sciences*, **41** (23): 1984. 3390-3395.

(13) G. North. "Multiple Solutions in Energy Balance Climate Models." *Palaeogeography, Palaeocliamtology, Palaeoecology*, **82**: 1990. 225-235.

(14) W. Sellers. "A Global Climatic Model Based on the Energy Balance of the Earth-Atmosphere System." *Journal of Applied Meteorology*, **8**: 1969. 392-400.

(15) J. Walsh and E. Widiasih. "A Dynamics Approach to a Low-Order Climate Model." *Discrete and Continuous Dynamical Systems Series B*, **19**(1): 2014. 257-279.

(16) J. Walsh and C. Rackauckas. "On the Budyko-Sellers Energy Balance Climate Model with Ice Line Coupling." *Discrete and Continuous Dynamical Systems Series B*, **20**(7): 2015. 1-29.

(17) W. Ward. "Climatic Variations on Mars: Astronomical Theory of Insolation." *Journal of Geophysical Reseach*, **79**(24): 1974. 3375-3386.

(18) E. Widiasih. "Dynamics of the Budyko Energy Balance Model." *SIAM J. Applied Dym. Sys.*, **12** (4): 2013