Interpreting Huybers' model as a nonsmooth dynamical system

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MCRN

Impose a vector field on Huybers' model

$$V_t = V_{t-1} + k_t$$
$$T_t = at + b + c\sin(2\pi t)$$
$$\text{If} \quad V_t \geq T_t, \text{ linearly reset over 10kyr to } V_t = 0$$

↓

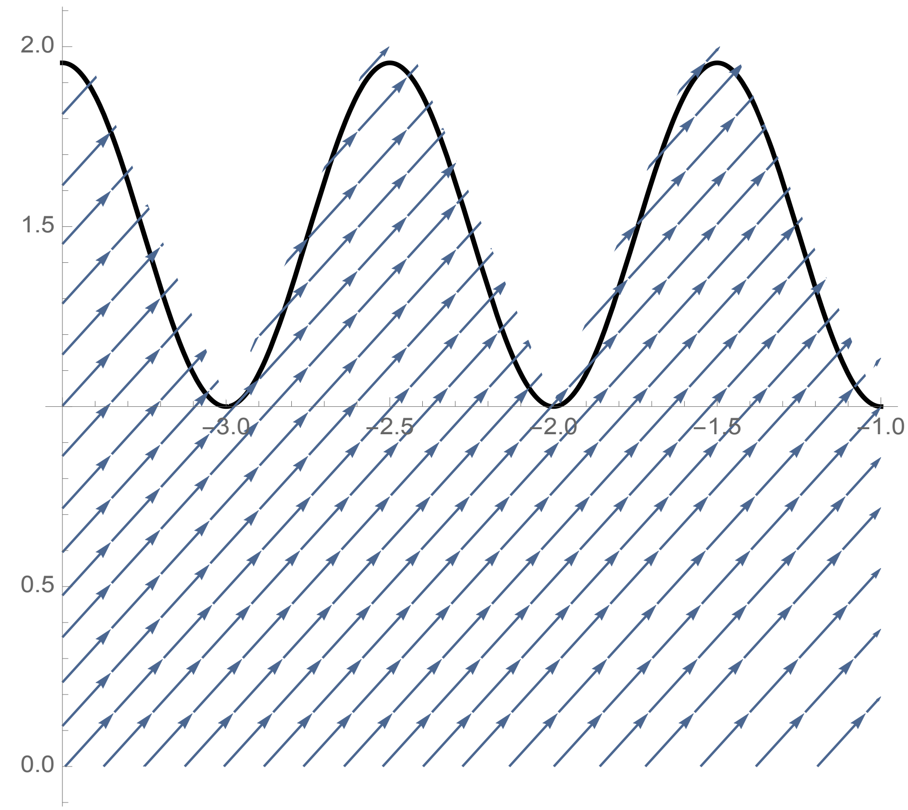
$$V_t = V_{t-\Delta t} + (\Delta t)k$$
$$T_t = at + b + c\sin(2\pi t)$$
$$\text{If } V_t \geq T_t, \text{ linearly reset over 10kyr to} V_t = 0$$

↓ $\Delta t \to 0$

$$\frac{dV}{dt} = k$$
$$T(t) = at + b + c\sin(2\pi t)$$
$$\text{If } V(t) \geq T(t), \text{ then reset to } V(t) = 0$$

$$\frac{d}{dt}\begin{bmatrix} t \\ V \end{bmatrix} = \begin{bmatrix} 1 \\ k \end{bmatrix}$$

For the purpose of this talk,
Set k = 1



Modify the growth terminating condition

$$V_t = V_{t-1} + k_t$$

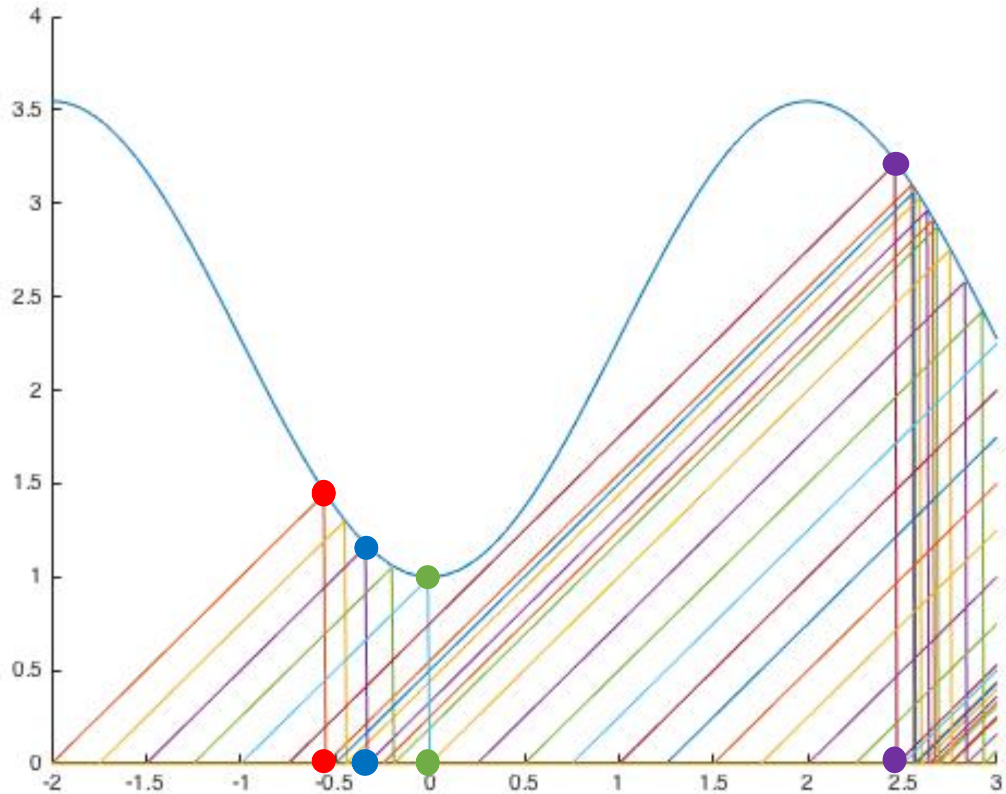
$$T_t = at + b + c\sin(2\pi t)$$

If $V_t \geq T_t$, linearly reset over 10kyr to $V_t = 0$

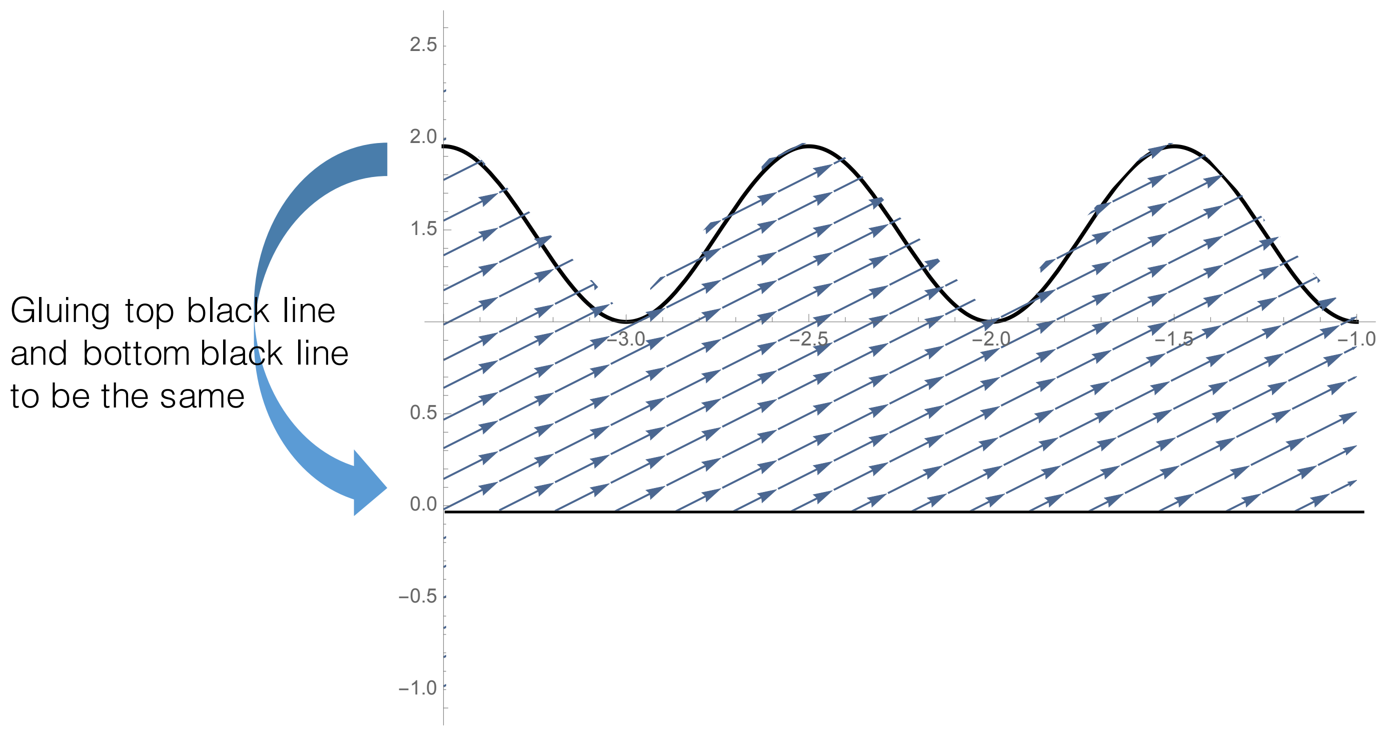
$$\frac{dV}{dt} = k$$

$$T(t) = at + b + c\sin(2\pi t)$$

If $V(t) \geq T(t)$, then reset instantaneously to $V(t) = 0$



4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3


Gluing top black line
and bottom black line
to be the same
2.5
2.0
1.5
0.5
0.0
−0.5
−1.0
−3.0
−2.5
−2.0
−1.5
−1.0

Making a cylinder by defining an equivalence relation

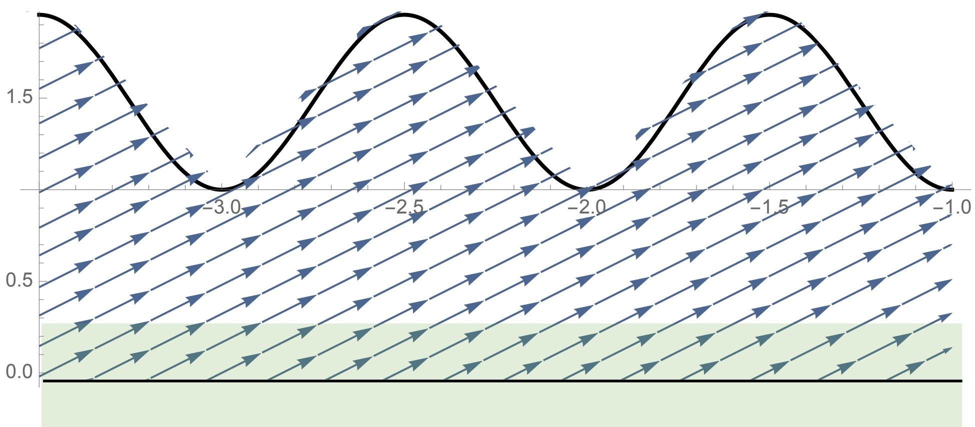
Given $X = \{(t, V) : 0 \leq V \leq T(t)\}$

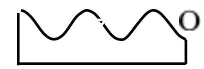
Define the equivalence relation and the quotient space to be the following:

$$M = X/\sim$$

$$\sim := (t, V(t)) \sim (t, 0) \text{ if } V(t) = T(t)$$

Making a cylinder by defining an equivalence relation

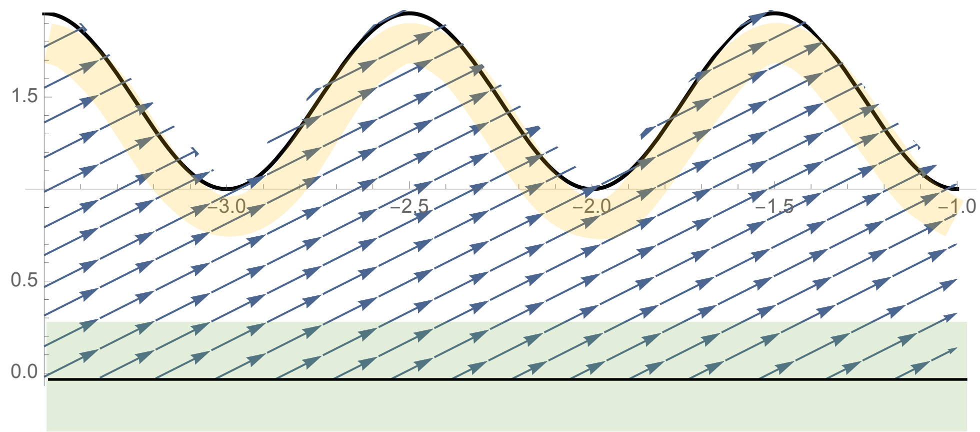


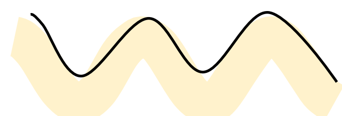
$u_1 = X^{\mathrm{o}} =$ 

$u_2 = \mathbb{R} \times (-\epsilon, \epsilon) =$

$$u_1 = X^{\mathrm{o}} \qquad \gamma_1 \begin{pmatrix} \nu \\ \eta \end{pmatrix} = \begin{pmatrix} \nu \\ \eta \end{pmatrix}$$

$$u_2 = \mathbb{R} \times (-\epsilon, \epsilon) \qquad \gamma_2 \begin{pmatrix} \nu \\ \eta \end{pmatrix} = \begin{cases} (\nu, \eta) & 0 \leq \eta < \epsilon \\ (\nu, \eta + T(\nu)) & -\epsilon < \nu \leq 0 \end{cases}$$



Note,  and  are the same strip after gluing.

Question: The first strip has the constant vector field. How does this vector field get deformed by the gluing?

$u_1 = X^{\circ}$ $\qquad \gamma_1 \begin{pmatrix} \nu \\ \eta \end{pmatrix} = \begin{pmatrix} \nu \\ \eta \end{pmatrix}$

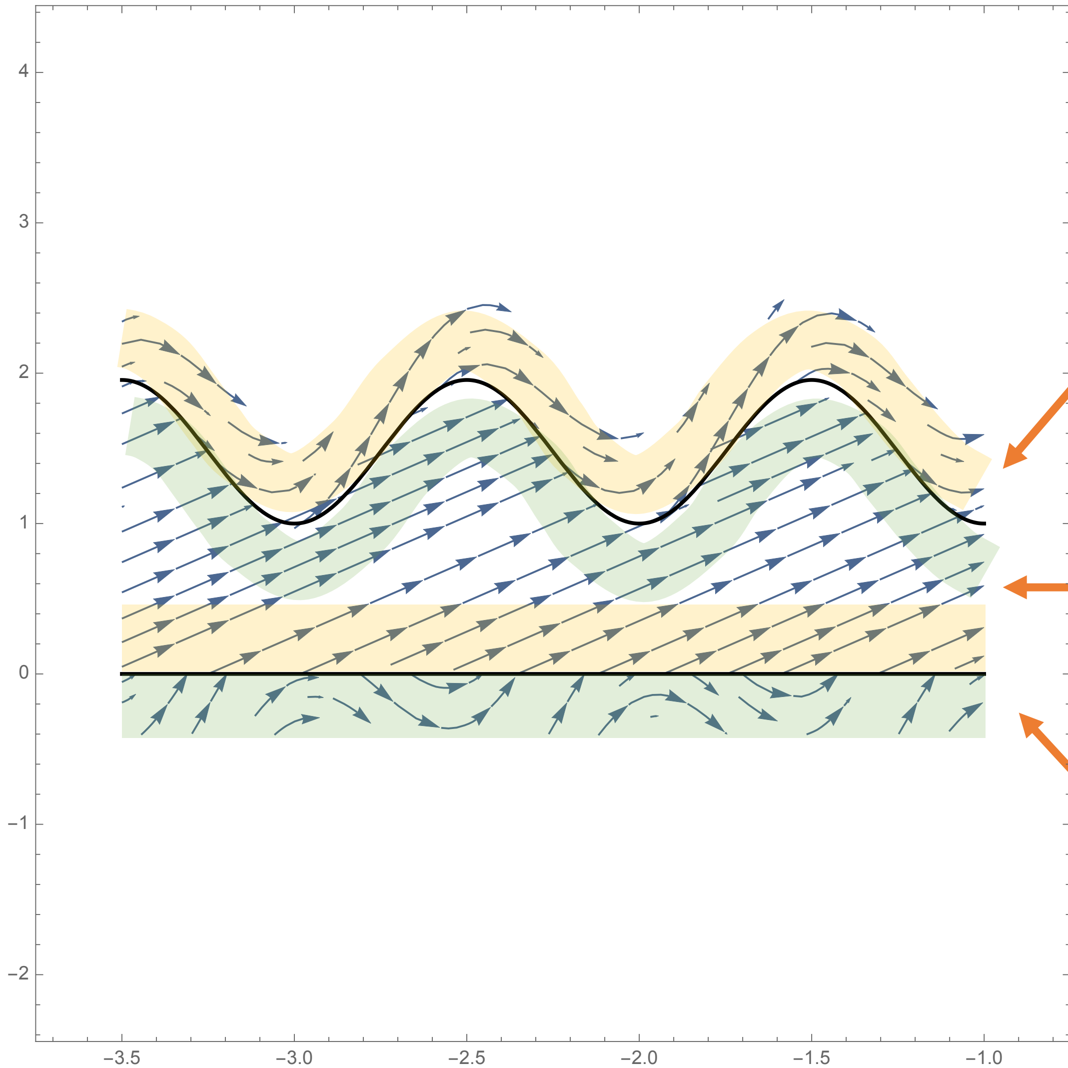
$u_2 = \mathbb{R} \times (-\epsilon, \epsilon)$ $\qquad \gamma_2 \begin{pmatrix} \nu \\ \eta \end{pmatrix} = \begin{cases} (\nu, \eta) & 0 \leq \eta < \epsilon \\ (\nu, \eta + T(\nu)) & -\epsilon < \nu \leq 0 \end{cases}$

Observe that in $\mathbb{R} \times (-\epsilon, 0)$ we have:

$$\gamma_1^{-1} \circ \gamma_2 \ (\nu, \eta) = (\nu, \eta + T(\nu))$$

We can calculate the resulting vector field by calculating the Jacobian of this transformation:

$$D^{-1}(\gamma_1^{-1} \circ \gamma_2(\nu, \eta)) \cdot \begin{bmatrix} 1 \\ k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -T'(\nu) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ k - T'(\nu) \end{bmatrix}$$



$$\frac{d}{dt}\begin{bmatrix} t \\ V \end{bmatrix} = \begin{bmatrix} 1 \\ k + T'(t) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} t \\ V \end{bmatrix} = \begin{bmatrix} 1 \\ k \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} t \\ V \end{bmatrix} = \begin{bmatrix} 1 \\ k - T'(t) \end{bmatrix}$$

Reducing the Filippov system to a circle relation

Define the termination time map

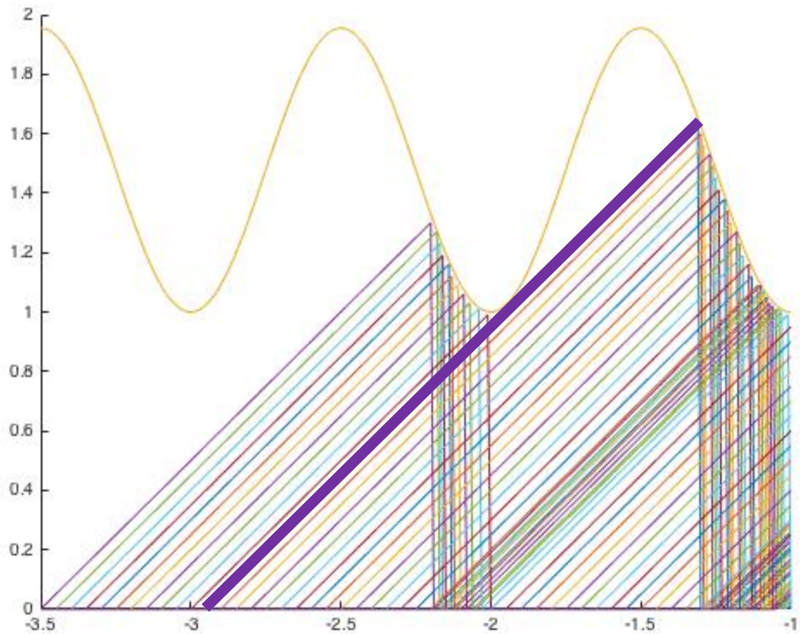
$$n(\tau) = \min\{t > \tau : V_\tau(t) = T(t)\}$$

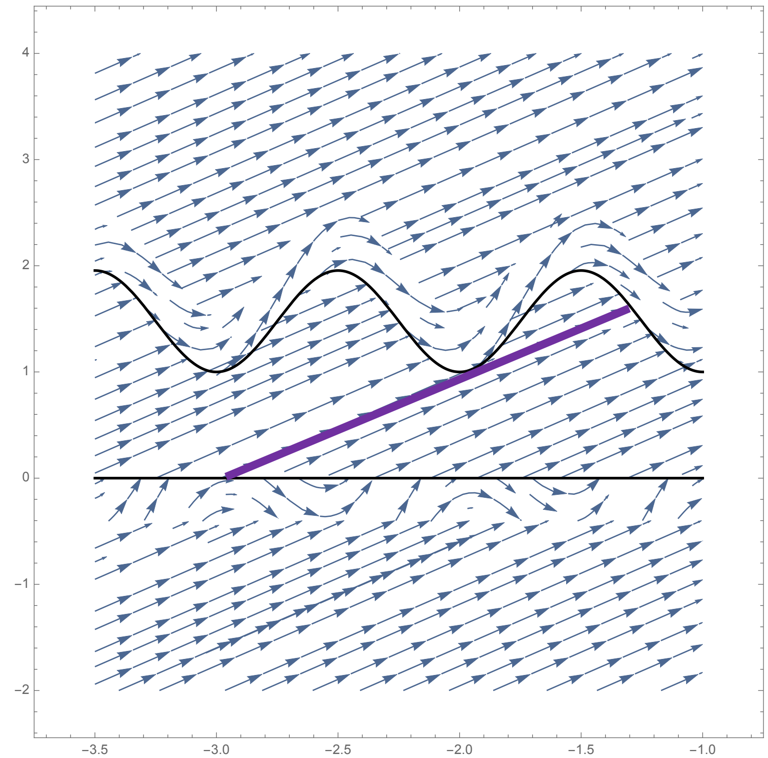
Then we get that

$$n(\tau + 1) = n(\tau) + 1$$

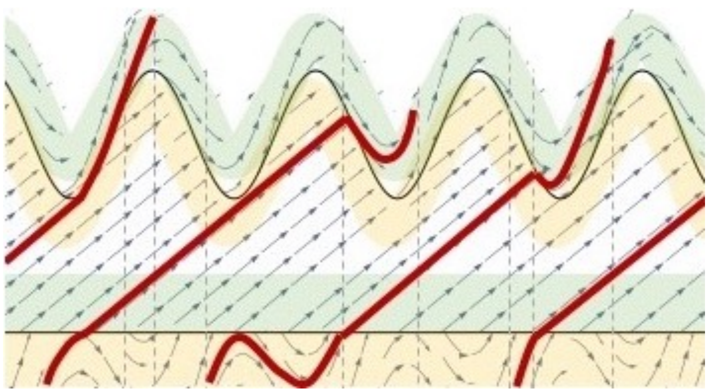
$$\begin{aligned} n(\tau+1) &= \min\{t > \tau+1 : V_{\tau+1}(t) = T(t)\} \\ &= \min\{t+1 > \tau+1 : V_{\tau+1}(t+1) = T(t+1)\} \\ &= \min\{t > \tau : V_{\tau+1}(t+1) = T(t+1)\} + 1 \\ &= \min\{t > \tau : V_{\tau+1}(t+1) = T(t+1)\} + 1 \\ &= \min\{t > \tau : V_\tau(t) = T(t)\} + 1 \\ &= n(\tau) + 1 \end{aligned}$$

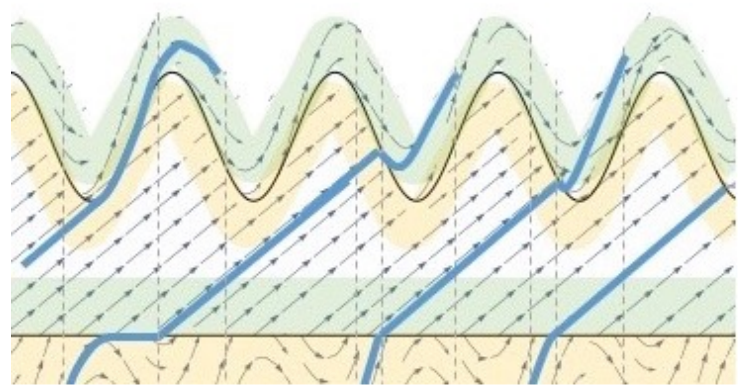
Let's follow one trajectory and see what happens..



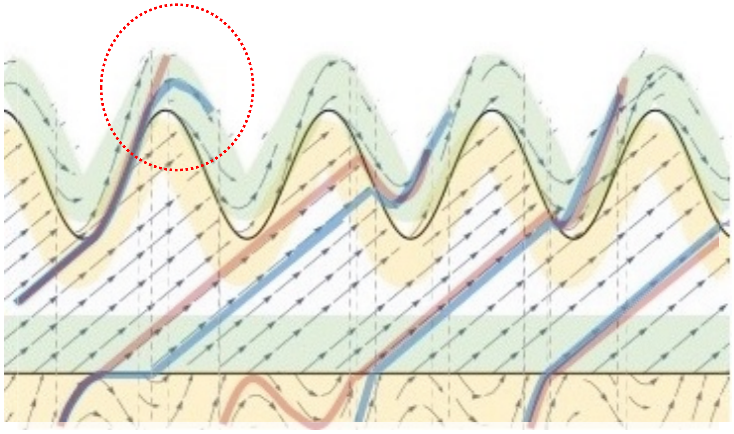


Example Trajectories





Two example trajectories overlaid:



- In the beginning they were the same trajectory, then they branched into two trajectories
- The initial trajectory can branch out into multiple trajectories, depending on how much time it spends along the sliding region

To be continued…

- Calculate a vector field again to accommodate 10kyr delay
- Circle Map approach
  - We have a circle relation -- can we relate these multiple termination times to real life?
- Nonsmooth system (Filippov system) approach
  - Study the behavior at the discontinuity boundary
  - Sliding region