MCRN Mathematics and Climate Research Network

Diffusive heat transport in the Budyko-Widiasih climate model

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Mathematics of Climate Seminar
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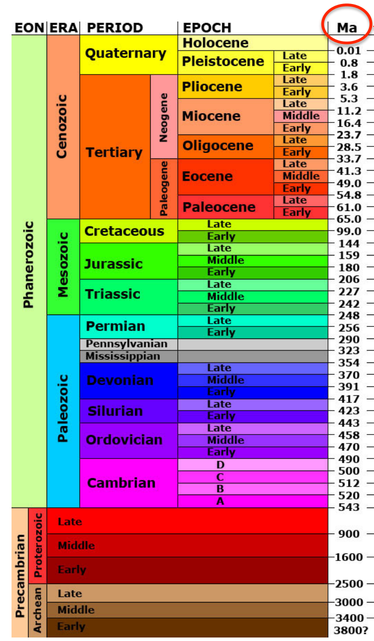
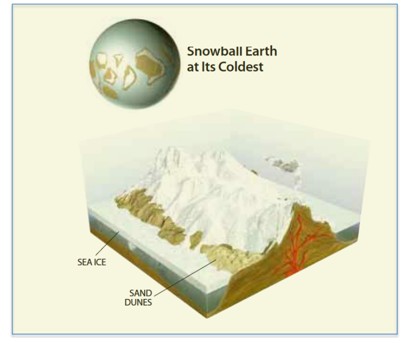
University of Minnesota
Driven to Discover℠



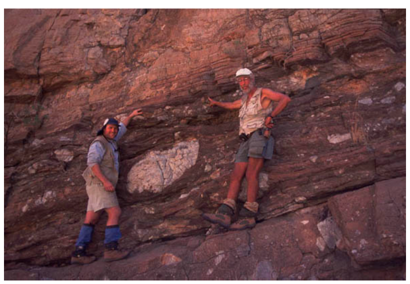
Million years ago!



Geological and paleomagnetic evidence indicate that during at least two Neoproterozoic glacial periods (~630 Ma and ~715 Ma) continental ice sheets flowed into the ocean near the equator.

Geological evidence: An example

- Occurrence of **glacial debris near sea level in the tropics**



Hoffman, P.F. & Schrag, D.P., 2000. Snowball Earth. *Scientific American* **282**, 68-75

Lots more evidence!
P. Hoffman & D. Schrag, The snowball Earth hypothesis: testing the limits of global change, *Terra Nova*, Vol 14, No. 3, 129–155.

Concern about the survival of life

--evidence that photosynthetic eukaryotes thrived both before and immediately after the Snowball episodes (organism whose cells contain complex structures enclosed within membranes)

--evidence that multiple lineages of sponges may have survived these glaciations (more complex marine animals)

Two critical points remain controversial:

--extent of the ice cover

--Marinoan glaciation (~630 Ma): static event vs. repeated glaciations

M. Bender, *Paleoclimate*, Princeton University Press (2013)

Points of widespread agreement

--glaciation extended to sea level in the tropics

--glaciation was a global event

--much of the ocean was ice covered

--deglaciation was rapid

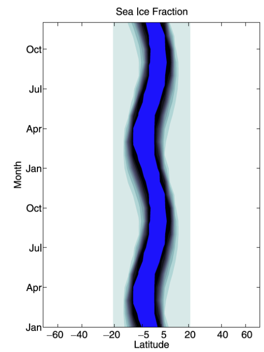
--there were large changes in the local or global carbon cycles

M. Bender, *Paleoclimate*, Princeton University Press (2013)

An alternative Neoproterozoic glaciation model

- **Jormungand climate state**:

Ocean is very nearly globally ice-covered, down to 5-15° latitude, with a thin strip of open ocean near the equator



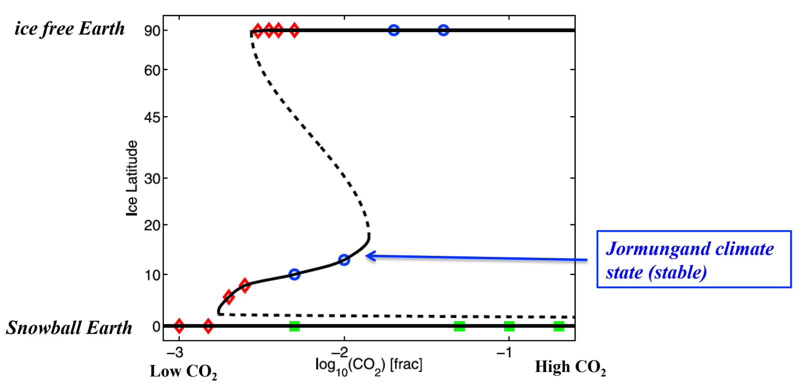
Henry Fuseli (1788)

https://www.pinterest.com/pin/83316661832493156/

D. Abbot, A. Viogt and D. Koll, The Jormungand global climate state and implications for Neoproterozoic glaciations, *J. Geophys. Res.*, **116** (2011).

Simulation of Neoproterozoic glaciation: Idealized GCM

Bare sea ice albedo ~0.45, snow covered sea ice albedo ~0.79



D. Abbot, A. Viogt and D. Koll (2011)

Simulation of Neoproterozoic glaciation: Idealized GCM

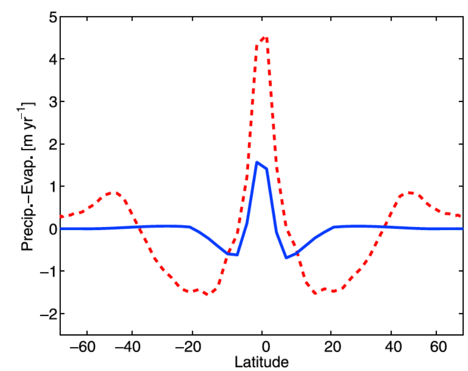
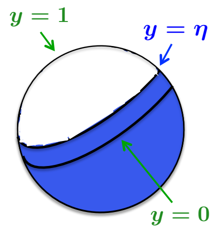


Figure 4. Annual and zonal mean precipitation minus evaporation for the ice-free state (red dashed) and the Jormungand state (blue)

D. Abbot, A. Viogt and D. Koll (2011)

Budyko's energy balance model: At equilibrium

- symmetry across equator
- latitude $= \theta$, $y = \sin\theta$ (all functions even in y)
- $T(y,t) =$ mean annual surf. temp at latitude y (°C)
- ice line at $y = \eta$



$$R\frac{\partial T}{\partial t} = E_{\text{in}} - E_{\text{out}} - E_{\text{transport}} \quad (\text{W/m}^2)$$

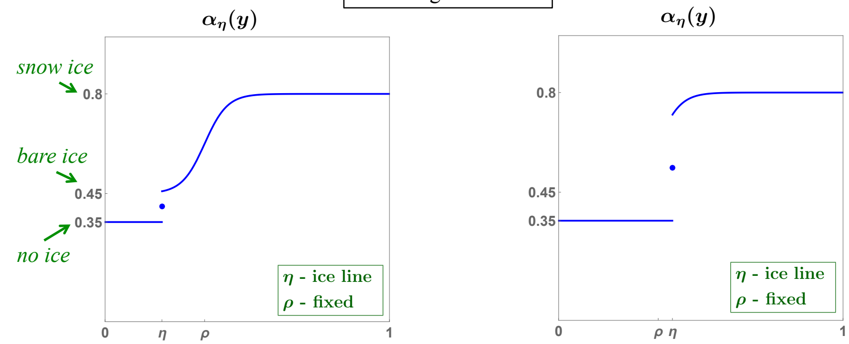
$$= Qs(y)(1-\alpha_\eta(y)) - (A + BT(y,t)) - C\left(T(y,t) - \underbrace{\int_0^1 T(y,t)dy}_{\overline{T}}\right)$$

M. I. Budyko, The effect of solar radiation variation on the climate of the Earth, *Tellus* **21** (1969), 611-619.

Budyko's energy balance model: At equilibrium

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha_\eta(y)) - (A + BT(y,t)) - C\left(T(y,t) - \int_0^1 T(y,t)dy\right)$$

Jorumungand Albedo

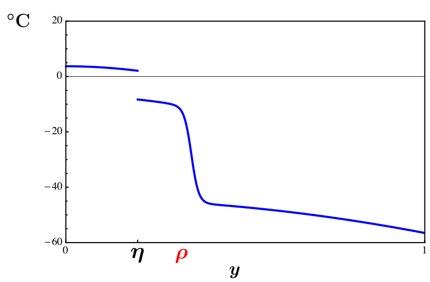


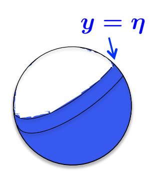
D. Abbot, A. Viogt and D. Koll (2011) – more or less!

Budyko's energy balance model: At equilibrium

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha_\eta(y)) - (A + BT(y,t)) - C\left(T(y,t) - \int_0^1 T(y,t)dy\right)$$

Equilibrium temp profile $T^*(y)$

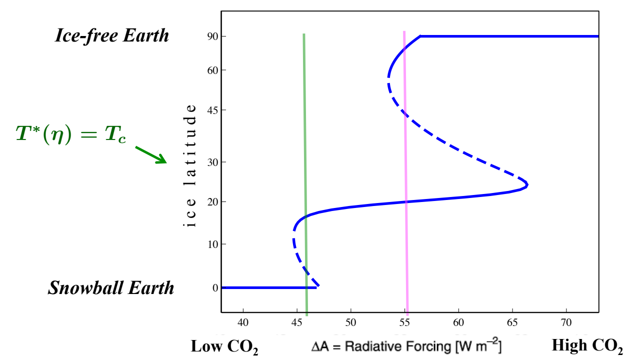




Do any satisfy $T^*(\eta) = T_c = 0$°C?

Budyko's energy balance model: At equilibrium

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha_\eta(y)) - (A + BT(y,t)) - C\left(T(y,t) - \int_0^1 T(y,t)dy\right)$$

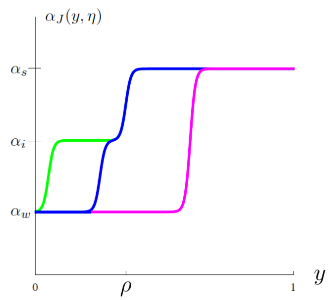


After D. Abbot, A. Viogt and D. Koll (2011)

Budyko--Widiasih model: Dynamics

$$\begin{cases} R\dfrac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T}) \\ \dfrac{d\eta}{dt} = \epsilon(T(\eta,t) - T_c), \quad \epsilon > 0 \end{cases}$$

Smooth Jormungand albedo function



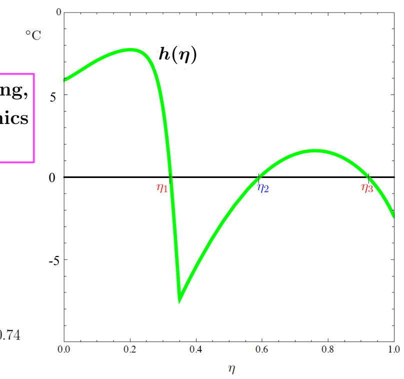
Green: $\eta = 0.05$. Blue: $\eta = 0.25$. Magenta: $\eta = 0.6$

J. A. Walsh & E. Widiasih, A dynamics approach to a low-order climate model, *Disc. Cont. Dyn. Syst. B* **19** (2014), 257--279

Budyko--Widiasih model: Dynamics

$$\begin{cases} R\dfrac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T}) \\ \dfrac{d\eta}{dt} = \epsilon(T(\eta,t) - T_c), \quad \epsilon > 0 \end{cases}$$

For sufficiently small ϵ, $\exists$ a locally attracting, invariant 1-D manifold on which the dynamics are described by the ODE $\dot{\eta} = \epsilon h(\eta)$

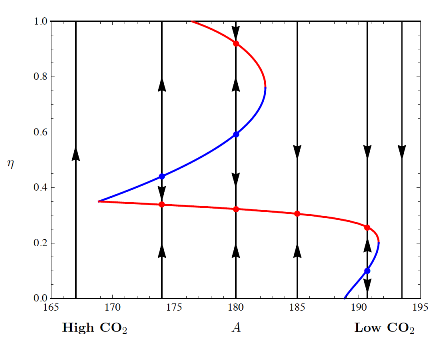


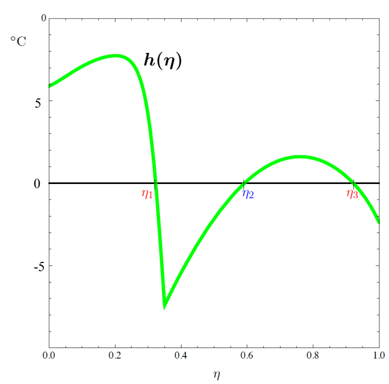
$A = 180, B = 1.5, C = 2.25, Q = 321, \alpha_w = 0.32, \alpha_i = 0.44, \alpha_s = 0.74$

J. A. Walsh & E. Widiasih, A dynamics approach to a low-order climate model, *Disc. Cont. Dyn. Syst. B* **19** (2014), 257-279

Budyko--Widiasih model: Dynamics

$$\begin{cases} R\dfrac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T}) \\ \dfrac{d\eta}{dt} = \epsilon(T(\eta,t) - T_c), \quad \epsilon > 0 \end{cases}$$





Budyko--Widiasih model: Diffusive heat transport

$$R\frac{\partial T}{\partial t} = E_{\text{in}} - E_{\text{out}} - E_{\text{transport}}$$

- $E_{\text{transport}} = C(T - \overline{T})$ *(relaxation to the mean)*
- $E_{\text{transport}} = D\nabla^2 T = D\dfrac{\partial}{\partial y}(1-y^2)\dfrac{\partial T}{\partial y}$ *(diffusion process)*

(fix radius, no longitudinal dependence, $y = \sin\theta$)

An advantage: $\dfrac{d}{dy}(1-y^2)\dfrac{d}{dy}p_n(y) = -n(n+1)p_n(y)$,

$p_n(y)$ – n^{th} Legendre polynomial

Budyko's equation: Diffusive heat transport

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + D\frac{\partial}{\partial y}(1-y^2)\frac{\partial T}{\partial y}$$

$$\boxed{T(y,t) = \sum_{n=0}^{N} T_{2n}(t)p_{2n}(y)}$$

- $\frac{\partial T}{\partial t} = \sum_{n=0}^{N} \dot{T}_{2n}p_{2n}(y)$
- $s(y) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-(\sqrt{1-y^2}\,\sin\beta\cos\gamma - y\cos\beta)^2}\,d\gamma,$

  $s(y) = \sum_{n=0}^{N} s_{2n}p_{2n}(y), \quad s_{2n} = (4n+1)\int_0^1 s(y)p_{2n}(y)dy$
- $s(y)\alpha(y,\eta) = \sum_{n=0}^{N}\overline{\alpha}_{2n}p_{2n}(y), \ \overline{\alpha}_{2n}(\eta) = (4n+1)\int_0^1 s(y)\alpha(y,\eta)p_{2n}(y)dy$
- $\frac{\partial}{\partial y}(1-y^2)\frac{\partial T}{\partial y} = -\sum_{n=0}^{N} 2n(2n+1)T_{2n}p_{2n}(y)$

Plug in, equate coefficients of $p_{2n}(y)$ ...

Budyko--Widiasih model: Diffusive heat transport

$$\boxed{T(y,t) = \sum_{n=0}^{N} T_{2n}(t)p_{2n}(y)}$$

$$\dot{T}_0 = -\frac{B}{R}(T_0 - f_0(\eta)),$$

$$\dot{T}_{2n} = -\frac{(B+2n(2n+1)D)}{R}(T_{2n} - f_{2n}(\eta)),$$

$$\dot{\eta} = \epsilon\left(\sum_{n=0}^{N} T_{2n}p_{2n}(\eta) - T_c\right)$$

$$f_0(\eta) = \frac{1}{B}(Q(s_0 - \overline{\alpha}_0(\eta)) - A),$$

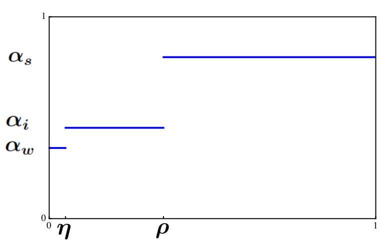
$$f_{2n}(\eta) = \frac{1}{(B+2n(2n+1)D)}(Q(s_{2n} - \overline{\alpha}_{2n}(\eta)),$$

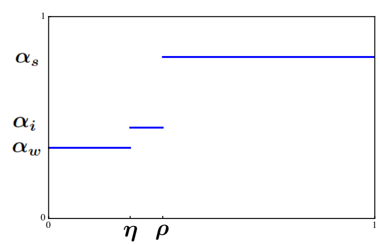
$$n = 1, \ldots, N$$

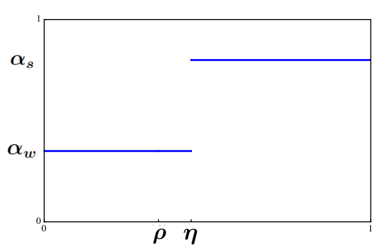
Idea: • $\epsilon = 0 \ \Rightarrow \ \exists$ globally attracting curve of rest points
• GSP theory $\Rightarrow$ for small ϵ, system behavior is well-approximated by

$$\dot{\eta} = \epsilon\left(\sum_{n=0}^{N} f_{2n}(\eta)p_{2n}(\eta) - T_c\right) = \epsilon h(\eta)$$

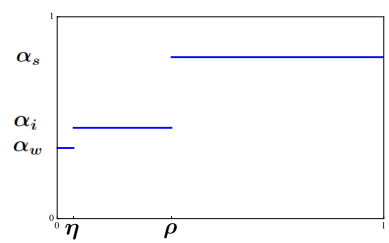
A simple Jorumungand albedo function

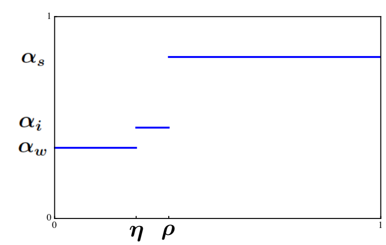


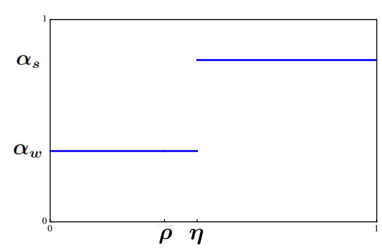




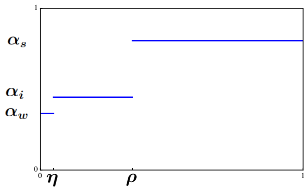
A simple Jorumungand albedo fucntion

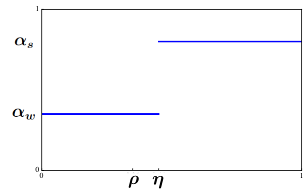






Now compute $\overline{\alpha}_{2n}(\eta)$!





$\boxed{\eta < \rho}$

$$\overline{\alpha^-}_{2n}(\eta) = (4n+1)\left(\alpha_w \int_0^\eta s(y)p_{2n}(y)dy + \alpha_i \int_\eta^\rho s(y)p_{2n}(y)dy + \alpha_s \int_\rho^1 s(y)p_{2n}(y)dy\right)$$

$= \text{polynomial in } \eta$

$\boxed{\eta > \rho}$

$$\overline{\alpha^+}_{2n}(\eta) = (4n+1)\int_0^1 \alpha(y,\eta)s(y)p_{2n}(y)dy$$

$$= (4n+1)\left(\alpha_w \int_0^\eta s(y)p_{2n}(y)dy + \alpha_s \int_\eta^1 s(y)p_{2n}(y)dy\right)$$

$= \text{polynomial in } \eta$

N.B. $\overline{\alpha^-}_{2n}(\rho) = \overline{\alpha^+}_{2n}(\rho)$

A piecewise-defined vector field $V : U \subset \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$

V^-: $\boxed{\eta < \rho}$

$$\begin{cases} \dot{T}_0 = -\dfrac{B}{R}(T_0 - f_0^-(\eta)), \\ \dot{T}_{2n} = -\dfrac{(B+2n(2n+1)D)}{R}(T_{2n} - f_{2n}^-(\eta)), \\ \dot{\eta} = \epsilon\left(\sum_{n=0}^{N} T_{2n}p_{2n}(\eta) - T_c\right), \end{cases}$$

where

$$f_0^-(\eta) = \frac{1}{B}(Q(s_0 - \overline{\alpha^-}_0(\eta)) - A),$$

$$f_{2n}^-(\eta) = \frac{Q(s_{2n} - \overline{\alpha^-}_{2n}(\eta))}{(B+2n(2n+1)D)}, \quad n = 1, \dots, N .$$

V^+: $\boxed{\eta > \rho}$

$$\begin{cases} \dot{T}_0 = -\dfrac{B}{R}(T_0 - f_0^+(\eta)), \\ \dot{T}_{2n} = -\dfrac{(B+2n(2n+1)D)}{R}(T_{2n} - f_{2n}^+(\eta)), \\ \dot{\eta} = \epsilon\left(\sum_{n=0}^{N} T_{2n}p_{2n}(\eta) - T_c\right), \end{cases}$$

where

$$f_0^+(\eta) = \frac{1}{B}(Q(s_0 - \overline{\alpha^+}_0(\eta)) - A),$$

$$f_{2n}^+(\eta) = \frac{Q(s_{2n} - \overline{\alpha^+}_{2n}(\eta))}{(B+2n(2n+1)D)}, \quad n = 1, \dots, N .$$

- V^-, V^+ are polynomial vector fields
- V^-, V^+ agree on the set Σ of all points for which $\eta = \rho$
- V is a continuous vector field

$\dot{x} = V(x)$, where $x = (T_0, \dots, T_{2n}, \eta)$: solutions exist

A piecewise-defined vector field $V : U \subset \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$

V^-: $\boxed{\eta < \rho}$

$$\begin{cases} \dot{T}_0 = -\dfrac{B}{R}(T_0 - f_0^-(\eta)), \\ \dot{T}_{2n} = -\dfrac{(B+2n(2n+1)D)}{R}(T_{2n} - f_{2n}^-(\eta)), \\ \dot{\eta} = \epsilon\left(\sum_{n=0}^{N} T_{2n}p_{2n}(\eta) - T_c\right), \end{cases}$$

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$$f_{2n}^+(\eta) = \frac{Q(s_{2n} - \overline{\alpha^+}_{2n}(\eta))}{(B+2n(2n+1)D)}, \quad n = 1, \dots, N .$$

Prop. V is not C^1 on Σ (i.e., when $\eta = \rho$)

A piecewise-defined vector field $V : U \subset \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$

V^-: $\boxed{\eta < \rho}$

$$\begin{cases} \dot{T}_0 = -\dfrac{B}{R}(T_0 - f_0^-(\eta)), \\ \dot{T}_{2n} = -\dfrac{(B+2n(2n+1)D)}{R}(T_{2n} - f_{2n}^-(\eta)), \\ \dot{\eta} = \epsilon\left(\sum_{n=0}^{N} T_{2n}p_{2n}(\eta) - T_c\right), \end{cases}$$

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V^+: $\boxed{\eta > \rho}$

$$\begin{cases} \dot{T}_0 = -\dfrac{B}{R}(T_0 - f_0^+(\eta)), \\ \dot{T}_{2n} = -\dfrac{(B+2n(2n+1)D)}{R}(T_{2n} - f_{2n}^+(\eta)), \\ \dot{\eta} = \epsilon\left(\sum_{n=0}^{N} T_{2n}p_{2n}(\eta) - T_c\right), \end{cases}$$

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$$f_{2n}^+(\eta) = \frac{Q(s_{2n} - \overline{\alpha^+}_{2n}(\eta))}{(B+2n(2n+1)D)}, \quad n = 1, \dots, N .$$

Prop. V is not C^1 on Σ (i.e., when $\eta = \rho$)

Prop. V is locally Lipschitz

$\dot{x} = V(x)$, where $x = (T_0, \dots, T_{2n}, \eta)$: unique solutions exist

Apply GSP theory (twice!)

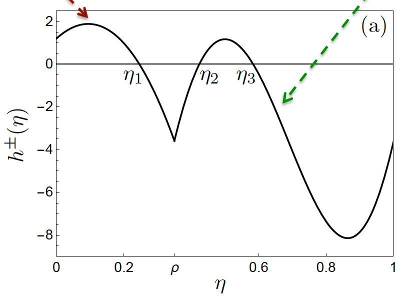
V^-: $\eta < \rho$

$\dot{\eta} = \epsilon h^-(\eta)$

V^+: $\eta > \rho$

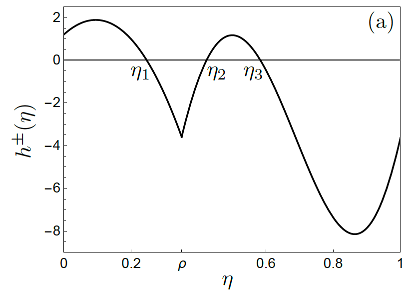
$\dot{\eta} = \epsilon h^+(\eta)$

$$T(y,t) = \sum_{n=0}^{1} T_{2n}(t) p_{2n}(y)$$

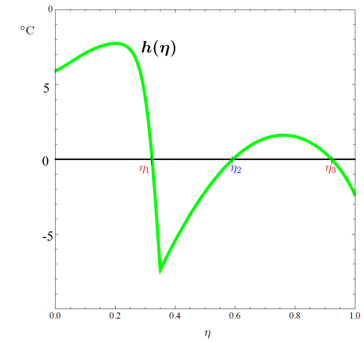


Comparison with infinite dimensional model

$$T(y,t) = \sum_{n=0}^{1} T_{2n}(t) p_{2n}(y)$$

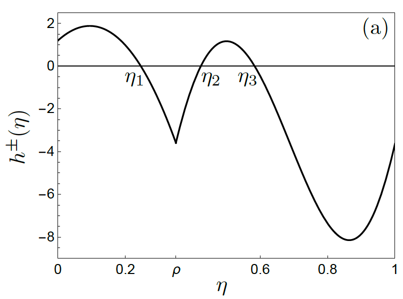


For sufficiently small ϵ, $\exists$ a locally attracting, invariant 1-D manifold on which the dynamics are described by the ODE $\dot{\eta} = \epsilon h(\eta)$

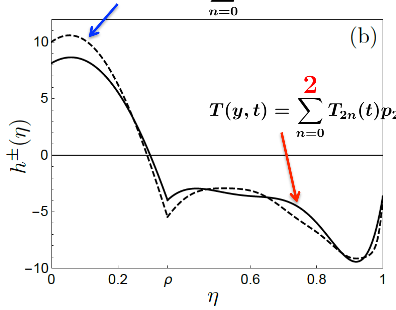


Including higher order modes

$$T(y,t) = \sum_{n=0}^{1} T_{2n}(t) p_{2n}(y)$$



$$T(y,t) = \sum_{n=0}^{5} T_{2n}(t) p_{2n}(y)$$



$$T(y,t) = \sum_{n=0}^{2} T_{2n}(t) p_{2n}(y)$$