## Permafrost Response to Climate Change via Budyko's Model*

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Mathematics of Climate Seminar
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*Based on a course project by Aileen Zebrowski.

## Permafrost Response

*What is permafrost?*

https://www.nps.gov/gaar/learn/nature/permafrost.htm



## Permafrost Response

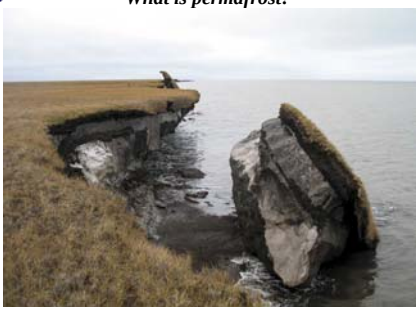
*What is permafrost?*

https://climatevulture.com/2016/08/28/satellite-remote-sensing-of-permafrost/



## Permafrost Response

*What is permafrost?*



http://alaska.usgs.gov/science/interdisciplinary_science/cae/arctic_coastal_plain.php



## Permafrost Response

*Where is the permafrost?*

Average latitude of permafrost boundary: 61°

(yellow circle)

(Aileen Zebrowski)




## Permafrost Response

**Potential Carbon Emissions**

The National Snow and Ice Data Center estimates that there are ***1400*** Gigatonnes of carbon (GtC) stored in the permafrost.

By comparison, the atmosphere currently holds about ***850*** GtC.

https://nsidc.org/cryosphere/frozenground/methane.html

## Permafrost Response

Paris climate conference (COP21)


http://www.cbc.ca/radio/thecurrent/a-special-edition-of-the-current-for-november-30-2-degrees-1.3343179



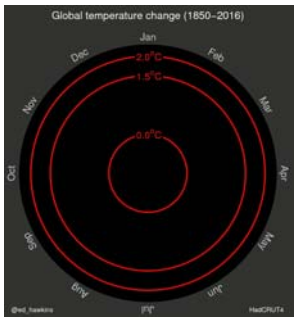
## Permafrost Response

Paris climate conference (COP21)


http://www.npr.org/sections/thetwo-way/2015/12/12/459502597/2-degrees-100-billion-the-world-climate-agreement-by-the-numbers



## Permafrost Response




http://www.climate-lab-book.ac.uk/2016/spiralling-global-temperatures/



## Permafrost Response

How much carbon would be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?



## Permafrost Response

**Conservation of Energy**

**temperature change ~ energy in – energy out**

short wave energy from the Sun

long wave energy from the Earth

***Everything else is detail.***



## Permafrost Response

**Budyko's Equation**

surface temperature

sin(latitude)

$\bar{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\bar{T}-T)$$

heat capacity

insolation

albedo

OLR

heat transport

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Chylek and Coakley's quadratic approximation:

$s(y) \approx 1 + s_2(3y^2 - 1)$, where $s_2 = -0.241$

## Permafrost Response

**Budyko's Equation**

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

| parameter | value | units |
|---|---|---|
| $Q$ | 343 | $\mathrm{Wm}^{-2}$ |
| $s_2$ | −2.41 | dimensionless |
| $A$ | 202 | $\mathrm{Wm}^{-2}$ |
| $B$ | 1.9 | $\mathrm{Wm}^{-2}\mathrm{K}^{-1}$ |
| $C$ | 3.04 | $\mathrm{Wm}^{-2}\mathrm{K}^{-1}$ |
| $\alpha_1$ | 0.32 | dimensionless |
| $\alpha_2$ | 0.62 | dimensionless |
| $T_c$ | −10 | °C |

K.K. Tung, *Topics in Mathematical Modeling*, Princeton University Press, 2007, Chapter 8.

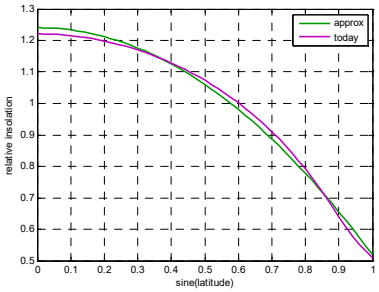


## Permafrost Response

**Insolation Distribution**

**green = quadratic approximation (Chylek & Coakley)**

**fuchsia = formula using obliquity of 23.4°**






## Permafrost Response

**Equilibrium Temperature Profile**

$$T(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha(y)) - A + C\overline{T}\right)$$

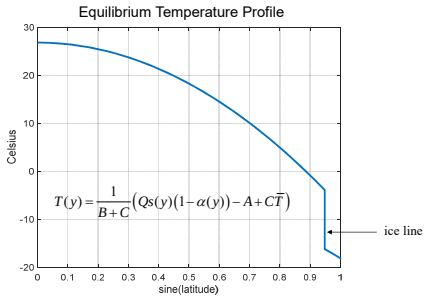
where $\alpha(y) = \begin{cases} \alpha_1 = 0.32, & y < \eta, \\ \alpha_2 = 0.62, & y > \eta, \end{cases}$ ← current ice boundary

global mean temperature → $\overline{T} = \frac{1}{B}\left(Q(1-\overline{\alpha}) - A\right)$, and $\overline{\alpha} = \int_0^1 \alpha(y)s(y)dy.$ ← average albedo

$$T(y) = \begin{cases} \frac{1}{B+C}\left(Qs(y)(1-\alpha_1) - A + C\overline{T}\right), & y < \eta, \\ \frac{1}{B+C}\left(Qs(y)(1-\alpha_2) - A + C\overline{T}\right), & y > \eta. \end{cases}$$ piecewise quadratic



## Permafrost Response



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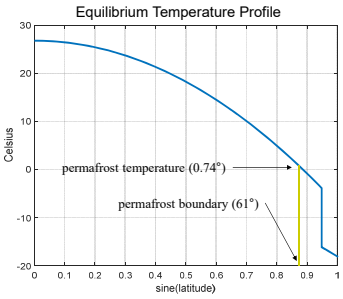
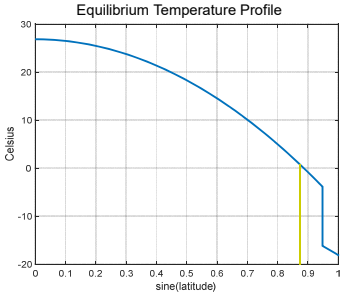





## Permafrost Response

If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

First approximation: Raise the temperature profile by 2°, and use linear approximation to determine the latitude where the new temperature is the permafrost temperature.



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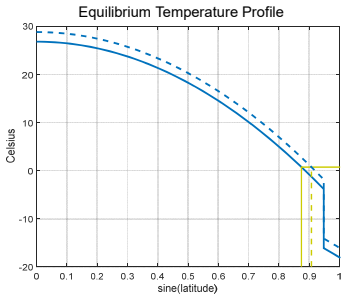




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If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

First approximation: Raise the temperature profile by 2°, and *use linear approximation* to determine the latitude where the new temperature is the permafrost temperature.

$$T(y)=\begin{cases}\frac{1}{B+C}\left(Qs(y)(1-\alpha_1)-A+C\bar{T}\right), & y<\eta,\\ \frac{1}{B+C}\left(Qs(y)(1-\alpha_2)-A+C\bar{T}\right), & y>\eta,\end{cases}$$

$$=\begin{cases}26.85-34.13y^2, & y<0.948,\\ 1.00-19.07y^2, & y>0.948.\end{cases}$$

permafrost boundary: $y_p=\sin(61^\circ)\approx 0.875$

$$T'(y_p)=-68.26y_p=-59.70$$

$$\hat{T}(y)=T(y)+2$$

$$T(y_p)=\hat{T}(y_p+\Delta y)\approx\hat{T}(y_p)+\hat{T}'(y_p)\Delta y=T(y_p)+2+T'(y_p)\Delta y$$

$$\Delta y\approx\frac{-2}{T'(y_p)}=\frac{-2}{-59.70}=0.0335$$



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new permafrost boundary:

$$\hat{y}_p\approx 0.875+0.0335\approx 0.908$$

new permafrost boundary in degrees latitude:

$$\sin^{-1}(\hat{y}_p)\approx 65.2^\circ \text{ latitude}$$



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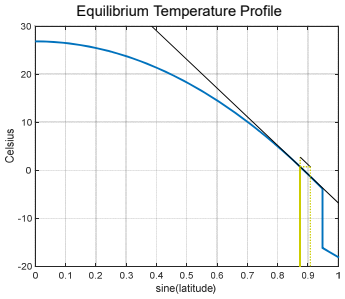




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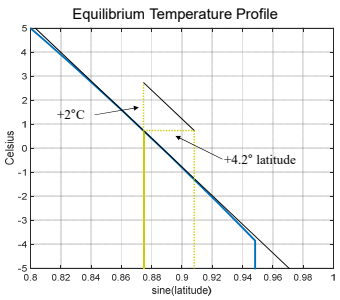
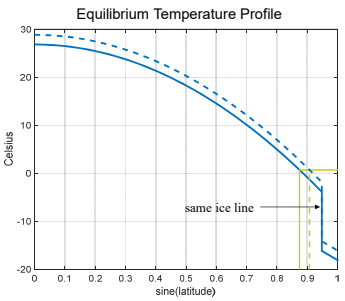





## Permafrost Response

If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

We have not taken into account that the ice line might move.



## Permafrost Response

Global Mean Temperature

$$\bar{T}(\eta)=\frac{1}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right),\ \text{where}\ \bar{\alpha}(\eta)=\int_0^1\alpha(y,\eta)s(y)dy,$$

$$\text{where}\ \alpha(y)=\begin{cases}\alpha_1=0.32, & y<\eta,\\ \alpha_2=0.62, & y>\eta,\end{cases}\quad\text{ice line}$$

The ice line is determined by the assumption that the average temperature across the ice line is $T_c$, usually take to be $-10^\circ$C. This condition reduces to*

$$\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)\right)-A+C\bar{T}(\eta)\Big)=T_c,\ \text{where}\ \alpha_0=\frac{1}{2}(\alpha_1+\alpha_2)$$

outgoing long wave radiation varies with greenhouse gases.

$$h(\eta,A)=\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)\right)-A+\frac{C}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right)\Big)-T_c=0$$

*e.g., McGehee & Widiasih 2014, *SIAM J. Applied Dynamical Systems* **13**, pp 518-536.



## Permafrost Response

**How to Proceed?**

1. Determine how the ice line varies with the parameter $A$.
2. Determine the change in $A$ giving an increase of 2 degrees Celsius in the global mean temperature.
3. Determine the change in the location of the permafrost boundary given the change in $A$.



## Permafrost Response

**Step 1**

Solve for $\eta$ as a function of $A$:

$$h(\eta,A)=\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)\right)-A+\frac{C}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right)\Big)-T_c=0,$$

where

$$\bar{\alpha}(\eta)=\int_0^\eta\alpha_1 s(y)dy+\int_\eta^1\alpha_2 s(y)dy$$

$$=\alpha_1\int_0^\eta s(y)dy+\alpha_2\left(1-\int_0^\eta s(y)dy\right)=\alpha_2-(\alpha_2-\alpha_1)\int_0^\eta s(y)dy$$

Numerically,

$$h(\eta,A)=h_0(\eta)-0.5236A,\ \text{where}\ h_0(\eta)=-8.0309\eta^3-26.6024\eta^2+41.3542\eta+97.8714$$

$$h_0'(\eta)\frac{d\eta}{dA}-0.5236=0$$

Evaluate at $\eta=0.9483$: $\dfrac{d\eta}{dA}=\dfrac{0.5236}{-30.7672}=-0.0171$ $\boxed{\dfrac{d\eta}{dA}=-0.0171}$



## Permafrost Response

**Step 2**

Compute $\dfrac{d\bar{T}}{dA}$:

$$\bar{T}(\eta,A)=\frac{1}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right),\ \text{where}\ \bar{\alpha}(\eta)=\alpha_2-(\alpha_2-\alpha_1)\int_0^\eta s(y)dy$$

$$\frac{d\bar{T}}{dA}=\frac{\partial\bar{T}}{\partial\eta}\frac{d\eta}{dA}+\frac{\partial\bar{T}}{\partial A}=-\frac{Q}{B}\bar{\alpha}'(\eta)\frac{d\eta}{dA}-\frac{1}{B}=\frac{Q}{B}(\alpha_2-\alpha_1)s(\eta)\frac{d\eta}{dA}-\frac{1}{B}$$

Evaluate at $\eta=0.9483$: $\dfrac{d\bar{T}}{dA}=-1.09172$

Change in $A$ to increase $T$ by 2 degrees:

$$\Delta A\approx\frac{\Delta T}{-1.09172}=\frac{2}{-1.09172}=-1.832$$

$\boxed{\Delta A\approx-1.832}$



## Permafrost Response

**Step 3**

Compute the change in $y_p$:

current temperature profile

$$T(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right)-A+C\bar{T}\right),\quad y<\eta$$

$$=26.85-34.13y^2$$

$\Delta\bar{T}=2$

$\Delta A\approx-1.832$

new temperature profile

$$\hat{T}(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right)-(A+\Delta A)+C(\bar{T}+\Delta\bar{T})\right.$$

$$=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right)-A+C\bar{T}\right)+\frac{C\Delta\bar{T}-A}{B+C}$$

$$=T(y)+1.60$$

permafrost boundary

$$y_p=\sin(61^\circ)\approx0.875$$

as before, but with 1.6 instead of 2

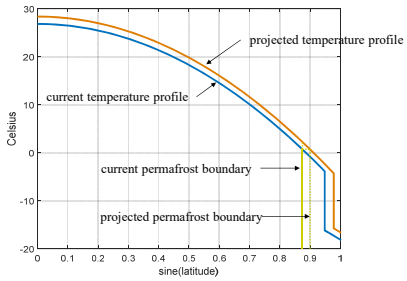
$$\Delta y\approx\frac{-1.60}{T'(y_p)}=\frac{-1.60}{-59.70}=0.027$$

new permafrost boundary $\hat{y}_p=y_p+\Delta y=0.902$, corresponding to $\boxed{64.4^\circ\ \text{latitude}}$



## Permafrost Response

Equilibrium Temperature Profile



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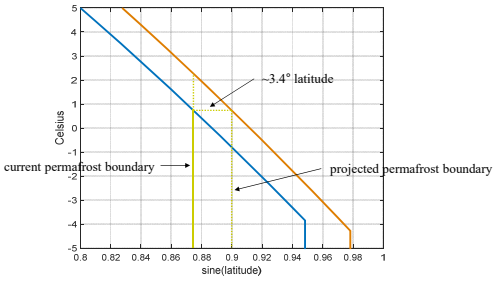





## Permafrost Response

***Where is the permafrost?***

Average latitude of permafrost boundary: 61° (yellow circle)

Projected permafrost boundary (orange circle)




## Permafrost Response

**How much carbon would be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?**

Recall that the surface area is proportional to $y$, the sine of the latitude.

Current permafrost boundary: $y_p = \sin(61^\circ) \approx 0.875$

Proportion of globe cover by permafrost: $1 - y_p = 0.125$

$\Delta y \approx 0.027$

Proportion of permafrost melted: $\dfrac{0.027}{0.125} = 0.216$

Amount of carbon released: $0.216 \times 1400 = \boxed{302 \text{ GtC}}$

Total fossil fuel emissions since 1751: 375 GtC

**To hold the GMT at 2°C, we will have to withdraw 302 GtC from the atmosphere as the permafrost melts.**



## Permafrost Response

**Future Projects**

Budyko's model includes ice-albedo feedback, but not carbon feedback.

- Can we modify the model to include the effects of permafrost melt?
- Could we use the data we have about current permafrost to model the glacial retreats during the Pleistocene?*
- To what extent was the "dead ice" in the Holocene similar to today's permafrost?**

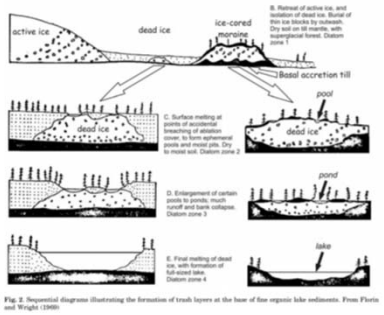
*e.g., J.A. Walsh, E. Widiasih, J. Hahn & R. McGehee, *Nonlinearity* **29**, 1843-1864 (2016).

**H. Wright & I. Stefanova, *Acta Palaeobotanica* **44**, 141–146 (2004).



## Permafrost Response

**Future Projects**






## Permafrost Response

**Future Projects**

- Could we use the data we have about current permafrost to model the glacial retreats during the Pleistocene?
- To what extent was the "dead ice" in the Holocene similar to today's permafrost?
- Can the $^{13}C$ data be reconciled with the Budyko model?

## Permafrost Response


http://www.rootsweb.ancestry.com/~ndpembin/html/lake_agassiz.htm



## Permafrost Response

Could this be a picture of the North shore of Lake Agassiz?