Permafrost Response to Climate Change via Budyko's Model*

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Mathematics of Climate Seminar
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*Based on a course project by Aileen Zebrowski.

Permafrost Response

What is permafrost?

http://alaska.usgs.gov/science/interdisciplinary_science/cae/arctic_coastal_plain.php

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Permafrost Response

What is permafrost?

https://www.nps.gov/gaar/learn/nature/permafrost.htm

Permafrost Response

Where is the permafrost?

Average latitude of permafrost boundary: 61°

(yellow circle)

(Aileen Zebrowski)

Permafrost Response

What is permafrost?

https://climatevulture.com/2016/08/28/satellite-remote-sensing-of-permafrost/

Permafrost Response

Potential Carbon Emissions

The National Snow and Ice Data Center estimates that there are ***1400*** Gigatonnes of carbon (GtC) stored in the permafrost.

By comparison, the atmosphere currently holds about ***850*** GtC.

https://nsidc.org/cryosphere/frozenground/methane.html

Permafrost Response

Paris climate conference (COP21)

http://www.cbc.ca/radio/thecurrent/a-special-edition-of-the-current-for-november-30-2-degrees-1.3343179

Permafrost Response

How much carbon would be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?

Permafrost Response

Paris climate conference (COP21)

http://www.npr.org/sections/thetwo-way/2015/12/12/459502597/2-degrees-100-billion-the-world-climate-agreement-by-the-numbers

Permafrost Response

Conservation of Energy

temperature change ~ energy in – energy out

energy in: short wave energy from the Sun

energy out: long wave energy from the Earth

Everything else is detail.

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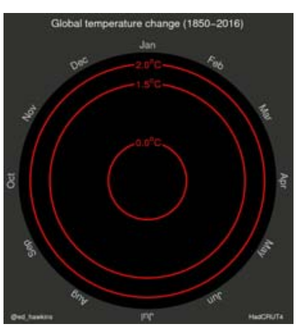

http://www.climate-lab-book.ac.uk/2016/spiralling-global-temperatures/

Permafrost Response

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\bar{T}-T)$$

$\bar{T} = \int_0^1 T(y)\,dy$

- T: surface temperature
- y: sin(latitude)
- R: heat capacity
- $Qs(y)$: insolation
- $\alpha(y)$: albedo
- $A+BT$: OLR
- $C(\bar{T}-T)$: heat transport

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 + s_2\left(3y^2 - 1\right), \quad \text{where} \quad s_2 = -0.241$$

Permafrost Response

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\bar{T} - T)$$

parameter	value	units
Q	343	Wm^{-2}
s_2	−2.41	dimensionless
A	202	Wm^{-2}
B	1.9	$Wm^{-2}K^{-1}$
C	3.04	$Wm^{-2}K^{-1}$
α_1	0.32	dimensionless
α_2	0.62	dimensionless
T_c	−10	$^\circ C$

K.K. Tung, *Topics in Mathematical Modeling*, Princeton University Press, 2007, Chapter 8.

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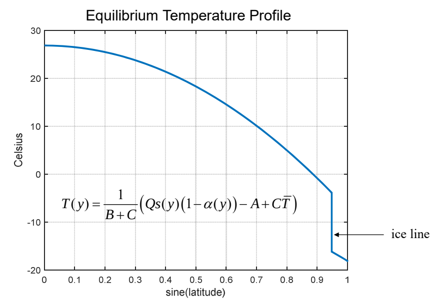

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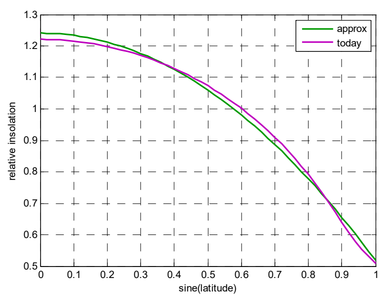
Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.4°



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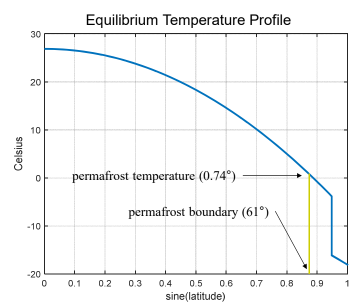


Permafrost Response

Equilibrium Temperature Profile

$$T(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha(y)) - A + C\bar{T}\right)$$

where $\alpha(y) = \begin{cases} \alpha_1 = 0.32, & y < \eta, \\ \alpha_2 = 0.62, & y > \eta, \end{cases}$ ← current ice boundary

global mean temperature → $\bar{T} = \frac{1}{B}\left(Q(1-\bar{\alpha}) - A\right)$, and $\bar{\alpha} = \int_0^1 \alpha(y)s(y)dy.$ ← average albedo

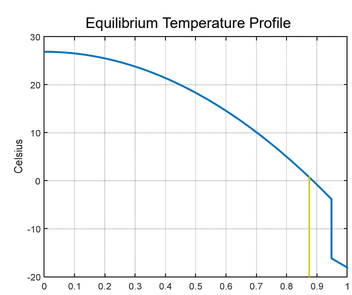
$$T(y) = \begin{cases} \frac{1}{B+C}\left(Qs(y)(1-\alpha_1) - A + C\bar{T}\right), & y < \eta, \\ \frac{1}{B+C}\left(Qs(y)(1-\alpha_2) - A + C\bar{T}\right), & y > \eta. \end{cases}$$

piecewise quadratic

Permafrost Response

If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

First approximation: Raise the temperature profile by 2°, and use linear approximation to determine the latitude where the new temperature is the permafrost temperature.



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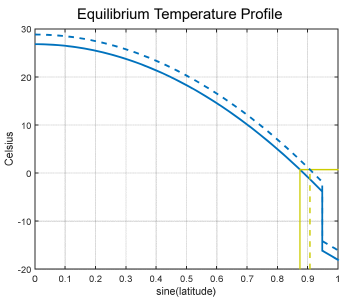

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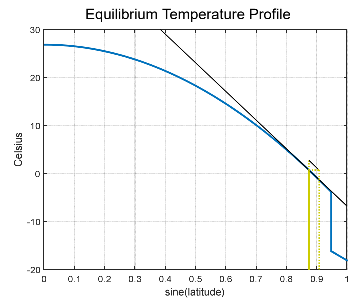

Permafrost Response

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$$T(y)=\begin{cases}\dfrac{1}{B+C}\left(Qs(y)(1-\alpha_1)-A+C\bar{T}\right), & y<\eta,\\ \dfrac{1}{B+C}\left(Qs(y)(1-\alpha_2)-A+C\bar{T}\right), & y>\eta,\end{cases}$$
$$=\begin{cases}26.85-34.13y^2, & y<0.948,\\ 1.00-19.07y^2, & y>0.948.\end{cases}$$

permafrost boundary: $y_p=\sin(61^\circ)\approx 0.875$

$$T'(y_p)=-68.26y_p=-59.70$$

$$\hat{T}(y)=T(y)+2$$

$$T(y_p)=\hat{T}(y_p+\Delta y)\approx\hat{T}(y_p)+\hat{T}'(y_p)\Delta y=T(y_p)+2+T'(y_p)\Delta y$$

$$\Delta y\approx\frac{-2}{T'(y_p)}=\frac{-2}{-59.70}=0.0335$$

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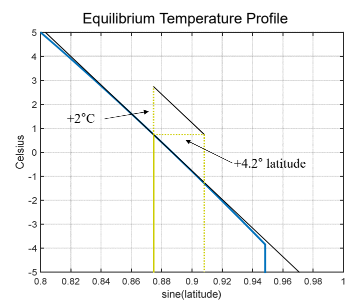

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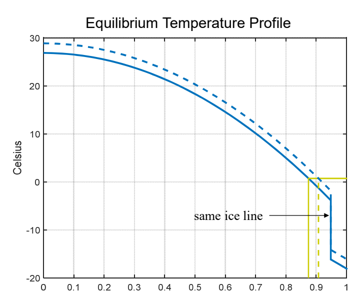
new permafrost boundary:
$\hat{y}_p\approx 0.875+0.0335\approx 0.908$

new permafrost boundary in degrees latitude:
$\sin^{-1}(\hat{y}_p)\approx 65.2^\circ$ latitude

Permafrost Response

If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

We have not taken into account that the ice line might move.



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Global Mean Temperature

$$\bar{T}(\eta)=\frac{1}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right),\ \text{where}\ \bar{\alpha}(\eta)=\int_0^1\alpha(y,\eta)s(y)dy,$$

$$\text{where}\ \alpha(y)=\begin{cases}\alpha_1=0.32, & y<\eta,\\ \alpha_2=0.62, & y>\eta,\end{cases}\quad \text{ice line}$$

The ice line is determined by the assumption that the average temperature across the ice line is T_c, usually take to be $-10°\text{C}$. This condition reduces to*

$$\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)\right)-A+C\bar{T}(\eta)\Big)=T_c,\ \text{where}\ \alpha_0=\frac{1}{2}(\alpha_1+\alpha_2)$$

outgoing long wave radiation varies with greenhouse gases.

$$h(\eta,A)=\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)\right)-A+\frac{C}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right)\Big)-T_c=0$$

e.g.*, McGehee & Widiasih 2014, *SIAM J. Applied Dynamical Systems* **13, pp 518-536.

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How to Proceed?

1. Determine how the ice line varies with the parameter A.
2. Determine the change in A giving an increase of 2 degrees Celsius in the global mean temperature.
3. Determine the change in the location of the permafrost boundary given the change in A.

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Step 1

Solve for η as a function of A:

$$h(\eta,A)=\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)\right)-A+\frac{C}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right)\Big)-T_c=0,$$

where

$$\bar{\alpha}(\eta)=\int_0^\eta\alpha_1 s(y)dy+\int_\eta^1\alpha_2 s(y)dy$$

$$=\alpha_1\int_0^\eta s(y)dy+\alpha_2\left(1-\int_0^\eta s(y)dy\right)=\alpha_2-(\alpha_2-\alpha_1)\int_0^\eta s(y)dy$$

Numerically,

$$h(\eta,A)=h_0(\eta)-0.5236A,\ \text{where}\ h_0(\eta)=-8.0309\eta^3-26.6024\eta^2+41.3542\eta+97.8714$$

$$h_0'(\eta)\frac{d\eta}{dA}-0.5236=0$$

Evaluate at $\eta=0.9483$: $\frac{d\eta}{dA}=\frac{0.5236}{-30.7672}=-0.0171$ $\boxed{\frac{d\eta}{dA}=-0.0171}$

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Step 2

Compute $\frac{d\bar{T}}{dA}$:

$$\bar{T}(\eta,A)=\frac{1}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right),\ \text{where}\ \bar{\alpha}(\eta)=\alpha_2-(\alpha_2-\alpha_1)\int_0^\eta s(y)dy$$

$$\frac{d\bar{T}}{dA}=\frac{\partial\bar{T}}{\partial\eta}\frac{d\eta}{dA}+\frac{\partial\bar{T}}{\partial A}=-\frac{Q}{B}\bar{\alpha}'(\eta)\frac{d\eta}{dA}-\frac{1}{B}=\frac{Q}{B}(\alpha_2-\alpha_1)s(\eta)\frac{d\eta}{dA}-\frac{1}{B}$$

Evaluate at $\eta=0.9483$: $\frac{d\bar{T}}{dA}=-1.09172$

Change in A to increase T by 2 degrees:

$$\Delta A\approx\frac{\Delta T}{-1.09172}=\frac{2}{-1.09172}=-1.832$$

$\boxed{\Delta A\approx-1.832}$

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Step 3

Compute the change in y_p:

current temperature profile

$$T(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right)-A+C\bar{T}\right),\quad y<\eta$$
$$=26.85-34.13y^2$$

$\Delta\bar{T}=2$

$\Delta A\approx-1.832$

new temperature profile

$$\hat{T}(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right)-(A+\Delta A)+C(\bar{T}+\Delta\bar{T}\right)$$
$$=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right)-A+C\bar{T}\right)+\frac{C\Delta\bar{T}-A}{B+C}$$
$$=T(y)+1.60$$

permafrost boundary

$$y_p=\sin(61^\circ)\approx0.875$$

$$\Delta y\approx\frac{-1.60}{T'(y_p)}=\frac{-1.60}{-59.70}=0.027$$

as before, but with 1.6 instead of 2

new permafrost boundary $\hat{y}_p=y_p+\Delta y=0.902$, corresponding to $\boxed{64.4^\circ\ \text{latitude}}$

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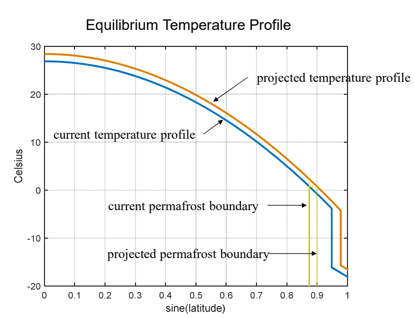

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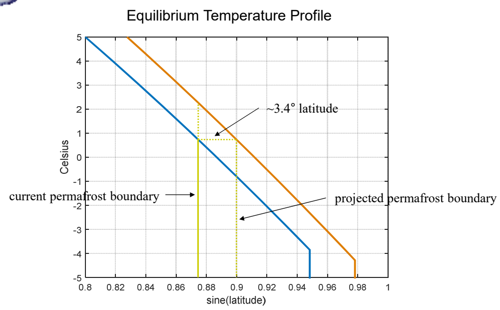
Permafrost Response



Permafrost Response

Future Projects

Budyko's model includes ice-albedo feedback, but not carbon feedback.

- Can we modify the model to include the effects of the release of carbon to the atmosphere due to permafrost melt?

Permafrost Response

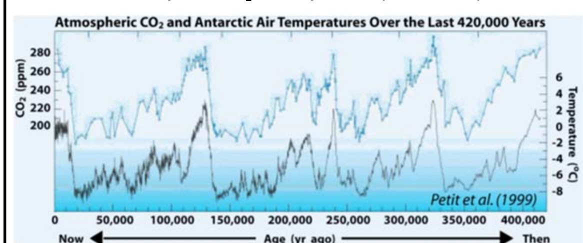
Where is the permafrost?

Average latitude of permafrost boundary: 61° (yellow circle)

Projected permafrost boundary (orange circle)

Permafrost Response

Atmospheric CO_2 & Temperature (Vostok data)



Pam Martin, University of Chicago, 2010

Permafrost Response

How much carbon would be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?

Recall that the surface area is proportional to y, the sine of the latitude.

Current permafrost boundary: $y_p = \sin(61^\circ) \approx 0.875$

Proportion of globe cover by permafrost: $1 - y_p = 0.125$

$\Delta y \approx 0.027$

Proportion of permafrost melted: $\frac{0.027}{0.125} = 0.216$

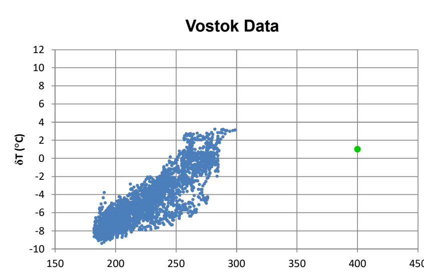
Amount of carbon released: $0.216 \times 1400 = \boxed{302 \text{ GtC}}$

Total fossil fuel emissions since 1751: 375 GtC

To hold the GMT at 2°C, we will have to withdraw 302 GtC from the atmosphere as the permafrost melts.

Permafrost Response

Current conditions are well outside the range recorded in the ice core data.



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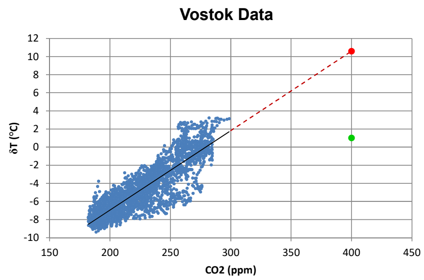
Extrapolate linear regression to 400 ppm CO2.



Permafrost Response

Future Projects

- Could we use the data we have about current permafrost to model the glacial retreats during the Pleistocene?
  - To what extent was the "dead ice" in the Holocene similar to today's permafrost?
    - Can the ^{13}C data be reconciled with the Budyko model?

Permafrost Response

Future Projects

Budyko's model includes ice-albedo feedback, but not carbon feedback.

- Can we modify the model to include the effects of the release of carbon to the atmosphere due to permafrost melt?
- Could we use the data we have about current permafrost to model the glacial retreats during the Pleistocene?*
- To what extent was the "dead ice" in the Holocene similar to today's permafrost?**

*e.g., J.A. Walsh, E. Widiasih, J. Hahn & R. McGehee, *Nonlinearity* **29**, 1843-1864 (2016).

H. Wright & I. Stefanova, *Acta Palaeobotanica* **44, 141–146 (2004).

Permafrost Response

http://www.rootsweb.ancestry.com/~ndpembin/html/lake_agassiz.htm

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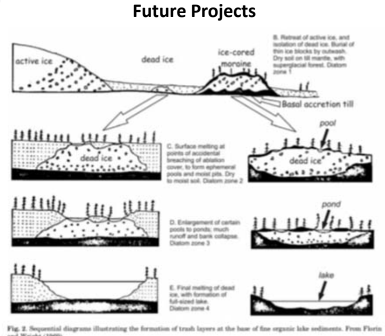
Future Projects



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Could this be a picture of the North shore of Lake Agassiz?