An Introduction to Budyko's Energy Balance Model

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Mathematics and Climate Research Network

Budyko's Model

Conservation of Energy

temperature change ~ energy in – energy out

energy in: short wave energy from the Sun

energy out: long wave energy from the Earth

Everything else is detail.

Budyko's Model

Dynamical Models

	Model	Equilibrium
Perfectly Thermally Conducting Black Body	$R\frac{dT}{dt} = Q - \sigma T^4$	$T = (Q/\sigma)^{1/4}$
Plus Albedo	$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$	$T = ((1-\alpha)Q/\sigma)^{1/4}$
Switch to Surface Temperature	$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$	$T = ((1-\alpha)Q - A)/B$
Dependence on Latitude	$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A+BT(y,t))$	$T(y) = ((1-\alpha)Qs(y) - A)/B$

Budyko's Model

Dynamical Models

Add Heat Transport $\quad R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T} - T)$

global mean temperature $\quad \bar{T}(t) = \int_0^1 T(y,t)\,dy$

Second Law of Thermodynamics:
Energy travels from hot places to cold places.

Equilibrium temperature profile?

Budyko's Model

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\bar{T} - T)$$

- R: heat capacity
- ∂T: surface temperature
- $Qs(y)$: insolation
- y: sin(latitude)
- $\alpha(y)$: albedo
- $(A+BT)$: OLR
- $C(\bar{T}-T)$: heat transport, with $\bar{T} = \int_0^1 T(y)\,dy$

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241(3y^2 - 1)$$

Budyko's Model

Budyko's Equilibrium

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\bar{T} - T)$$

albedo depends on latitude

equilibrium solution: $T = T^*(y)$

$$Qs(y)(1-\alpha(y)) - (A + BT^*(y)) + C(\bar{T}^* - T^*(y)) = 0$$

Integrate:

$$\int_0^1 \left(Qs(y)(1-\alpha(y)) - (A + BT^*(y)) + C(\bar{T}^* - T^*(y))\right)dy = 0$$

$$Q\underbrace{\int_0^1 s(y)\,dy}_{1} - Q\underbrace{\int_0^1 s(y)\alpha(y)\,dy}_{\bar{\alpha}} - A\underbrace{\int_0^1 dy}_{1} - B\underbrace{\int_0^1 T^*(y)\,dy}_{\bar{T}^*} + C\left(\underbrace{\int_0^1 \bar{T}^*\,dy}_{\bar{T}^*} - \underbrace{\int_0^1 T^*(y)\,dy}_{\bar{T}^*}\right) = 0$$

$$Q(1-\bar{\alpha}) - (A + B\bar{T}^*) = 0$$

equilibrium global mean temperature $\quad \boxed{\bar{T}^* = \frac{1}{B}\left(Q(1-\bar{\alpha}) - A\right)}$

Budyko's Model

Budyko's Equilibrium

$$Qs(y)(1-\alpha(y))-\left(A+BT^*(y)\right)+C\left(\bar{T}^*-T^*(y)\right)=0$$

Global mean temperature at equilibrium:

$$\bar{T}^*=\frac{1}{B}\left(Q(1-\bar{\alpha})-A\right) \qquad \left(\bar{\alpha}=\int_0^1\alpha(y)s(y)dy\right)$$

Solve for $T^*(y)$.

$$Qs(y)(1-\alpha(y))-A+C\bar{T}^*=BT^*(y)+CT^*(y)=(B+C)T^*(y)$$

$$T^*(y)=\frac{1}{B+C}\left(Qs(y)(1-\alpha(y))-A+C\bar{T}^*\right)$$

Equilibrium temperature profile:

$$\boxed{T^*(y)=\frac{1}{B+C}\left(Qs(y)(1-\alpha(y))-A+C\bar{T}^*\right)}$$

where $\bar{T}^*=\frac{1}{B}\left(Q(1-\bar{\alpha})-A\right)$ and $\bar{\alpha}=\int_0^1\alpha(y)s(y)dy$

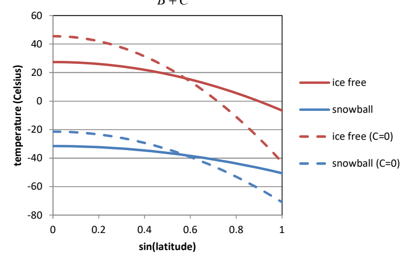
Budyko's Model

Budyko's Equilibrium

$$Qs(y)(1-\alpha(y))-\left(A+BT^*(y)\right)+C\left(\bar{T}^*-T^*(y)\right)=0$$

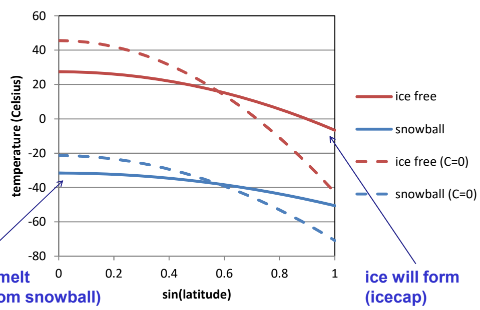
Equilibrium temperature profile: $T^*(y)=\frac{1}{B+C}\left(Qs(y)(1-\alpha(y))-A+C\bar{T}^*\right)$

$C = 3.04$
$\alpha(y) = 0.32$: ice free
$\alpha(y) = 0.62$: snowball
(constant albedo)



Budyko's Model

Budyko's Equilibrium



ice won't melt (no exit from snowball)

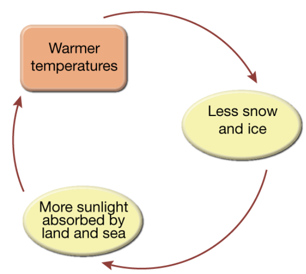
ice will form (icecap)

Budyko's Model

Ice Albedo Feedback

temperature warms
ice melts
albedo decreases
more sunlight absorbed
temperature warms
REPEAT

Why would it stop?



http://www.i-fink.com/melting-polar-ice/

Budyko's Model

Ice Albedo Feedback

temperature warms
ice melts
albedo decreases
more sunlight absorbed
temperature warms
REPEAT

Why would it stop?

M. I. Budyko, "The effect of solar radiation variations on the climate of the Earth," **Tellus XXI**, 611-619 , 1969.

http://www.inenco.org/index_principals.html

Budyko's Model

Ice Albedo Feedback

Why would it stop?

Budyko's Equation

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y))-(A+BT)+C(\bar{T}-T) \qquad \bar{T}=\int_0^1 T(y)dy$$

surface temperature; sin(latitude); heat capacity; insolation; albedo; OLR; heat transport

Budyko's Model

Ice Albedo Feedback

What if the albedo is not constant?

Ice Line Assumption: There is a single ice line at $y=\eta$ between the equator and the pole. The albedo is α_1 below the ice line and α_2 above it.

$$\alpha(y,\eta)=\begin{cases}\alpha_1 & y<\eta\\ \alpha_2 & y>\eta\end{cases}$$

Equilibrium condition:

$$Qs(y)\big(1-\alpha(y,\eta)\big)-\big(A+BT_\eta^*(y)\big)+C\big(\bar{T}^*-T_\eta^*(y)\big)=0$$

Equilibrium solution:

$$T_\eta^*(y)=\frac{1}{B+C}\big(Qs(y)\big(1-\alpha(y,\eta)\big)-A+C\bar{T}_\eta^*\big)$$

where $\bar{T}_\eta^*=\frac{1}{B}\big(Q\big(1-\bar{\alpha}(\eta)\big)-A\big)$ $\quad\left(\bar{\alpha}(\eta)=\int_0^1\alpha(y,\eta)s(y)dy\right)$

Budyko's Model

Ice Albedo Feedback

equilibrium temperature profile:

$$T_\eta^*(y)=\frac{1}{B+C}\big(Qs(y)\big(1-\alpha(y,\eta)\big)-A+C\bar{T}_\eta^*\big),\ \text{where}\ \bar{T}_\eta^*=\frac{1}{B}\big(Q\big(1-\bar{\alpha}(\eta)\big)-A\big)$$

albedo: $\alpha(y,\eta)=\begin{cases}\alpha_1=0.32, & y<\eta\\ \alpha_2=0.62, & y>\eta\end{cases}$

global albedo: $\bar{\alpha}(\eta)=\int_0^1\alpha(y,\eta)s(y)dy=\int_0^\eta\alpha_1 s(y)dy+\int_\eta^1\alpha_2 s(y)dy$

let: $S(\eta)=\int_0^\eta s(y)dy,\quad 1-S(\eta)=\int_\eta^1 s(y)dy,\quad \text{since}\quad 1=\int_0^1 s(y)dy$

then: $\bar{\alpha}(\eta)=\alpha_1 S(\eta)+\alpha_2(1-S(\eta))=\alpha_2-(\alpha_2-\alpha_1)S(\eta)=0.62-0.3S(\eta)$

$$S(\eta)=\int_0^\eta s(y)dy=\int_0^\eta\big(1-0.241\big(3y^2-1\big)\big)dy=\eta-0.241\big(\eta^3-\eta\big)$$

Chylek & Coakley

Budyko's Model

Ice Albedo Feedback

equilibrium temperature profile:

$$\boxed{T_\eta^*(y)=\frac{1}{B+C}\big(Qs(y)\big(1-\alpha(y,\eta)\big)-A+C\bar{T}_\eta^*\big)},\ \text{where}\ \boxed{\bar{T}_\eta^*=\frac{1}{B}\big(Q\big(1-\bar{\alpha}(\eta)\big)-A\big)}$$

albedo: $\alpha(y,\eta)=\begin{cases}\alpha_1=0.32, & y<\eta\\ \alpha_2=0.62, & y>\eta\end{cases}$

global albedo: $\bar{\alpha}(\eta)=\int_0^1\alpha(y,\eta)s(y)dy=\int_0^\eta\alpha_1 s(y)dy+\int_\eta^1\alpha_2 s(y)dy$

let: $S(\eta)=\int_0^\eta s(y)dy,\quad 1-S(\eta)=\int_\eta^1 s(y)dy,\quad \text{since}\quad 1=\int_0^1 s(y)dy$

then: $\boxed{\bar{\alpha}(\eta)}=\alpha_1 S(\eta)+\alpha_2(1-S(\eta))=\alpha_2-(\alpha_2-\alpha_1)S(\eta)=\boxed{0.62-0.3S(\eta)}$

$$\boxed{S(\eta)}=\int_0^\eta s(y)dy=\int_0^\eta\big(1-0.241\big(3y^2-1\big)\big)dy=\boxed{\eta-0.241\big(\eta^3-\eta\big)}$$

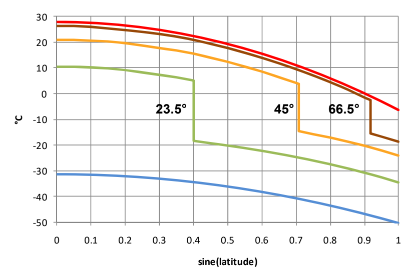
Chylek & Coakley

Budyko's Model

Ice Albedo Feedback

$$T_\eta^*(y)=\frac{1}{B+C}\big(Qs(y)\big(1-\alpha(y,\eta)\big)-A+C\bar{T}_\eta^*\big)$$

For each fixed η, there is an equilibrium solution for Budyko's equation



Budyko's Model

Dynamics

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y))-(A+BT)+C(\bar{T}-T)$$

Let X be the space of functions where T lives. (*e.g.* $L^1([0,1])$)
Let

$$L:X\to X:LT=C\bar{T}-(B+C)T,$$
$$f(y)=Qs(y)\big(1-\alpha(y)\big)-A$$

Budyko's equation can be written as a linear vector field on X.

$$R\frac{dT}{dt}=f+LT$$

The operator L has only point spectrum, with all eigenvalues negative.
Therefore, all solutions are stable.
True for any albedo function.

experts only

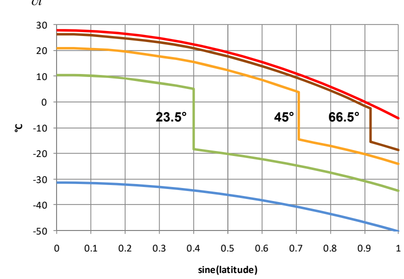
Budyko's Model

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\bar{T}-T)$$

For each fixed η, there is a **globally stable** equilibrium solution for Budyko's equation.

How to pick one?



Budyko's Model

Ice Albedo Feedback

Summary

If we artificially hold the ice line at a fixed latitude, then the surface temperature will come to an equilibrium.

However,
if the temperature is high, we would expect ice to melt and the ice line to retreat to higher latitudes. If the temperature is low, we would expect ice to form and the ice line to advance to lower latitudes.

How to model this expectation?

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Budyko's Model

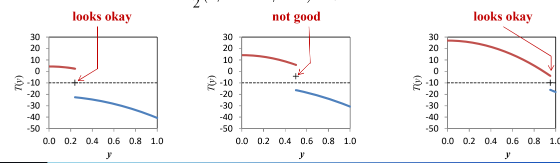
Ice Albedo Feedback

For each fixed η , there is a stable equilibrium solution for Budyko's equation.

Standard assumption: Permanent ice forms if the annual average temperature is below T_c=−10 °C and melts if the annual average temperature is above T_c.

Additional condition: The average temperature across the ice boundary is the critical temperature T_c .

$$\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=T_c=-10$$



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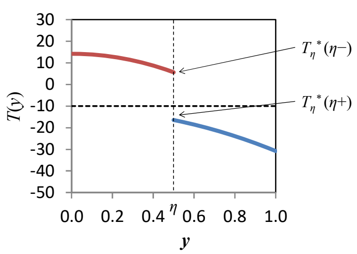
Budyko's Model

Ice Albedo Feedback

ice line condition: $\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=T_c=-10$



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Budyko's Model

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\bar{T}-T)$$

Equilibrium: $T_\eta^*(y)=\frac{1}{B+C}\left(Qs(y)(1-\alpha(y,\eta))-A+C\bar{T}_\eta^*\right)$

Ice line condition: $\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=T_c=-10$

Albedo: $\alpha(y,\eta)=\begin{cases}\alpha_1=0.32, & y<\eta\\ \alpha_2=0.62, & y>\eta\end{cases}$ $\quad\alpha(\eta-,\eta)=\alpha_1,\quad \alpha(\eta+,\eta)=\alpha_2$

$$T_\eta^*(\eta+)=\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_2)-A+C\bar{T}_\eta^*\right)\qquad T_\eta^*(\eta-)=\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_1)-A+C\bar{T}_\eta^*\right)$$

Ice line condition:

$$\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0)-A+C\bar{T}_\eta^*\right)=T_c=-10$$

where: $\alpha_0=\frac{\alpha_1+\alpha_2}{2}=0.47$

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Budyko's Model

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\bar{T}-T)$$

Ice line condition: $\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0)-A+C\bar{T}_\eta^*\right)=T_c=-10$

Rewrite: $h(\eta)=\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0)-A+C\bar{T}_\eta^*\right)-T_c=0$

Recall equilibrium GMT: $\bar{T}_\eta^*=\frac{1}{B}\left(Q(1-\bar{\alpha}(\eta))-A\right)$

Recall average albedo: $\bar{\alpha}(\eta)=\int_0^1\alpha(y,\eta)s(y)dy=\alpha_2-(\alpha_2-\alpha_1)S(\eta)=0.62-0.3S(\eta)$

where: $S(\eta)=\int_0^\eta s(y)dy=\eta-0.241\left(\eta^3-\eta\right)$

$$h(\eta)\equiv\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0)+\frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)S(\eta)\right)\right)-\frac{A}{B}-T_c=0$$

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Budyko's Model

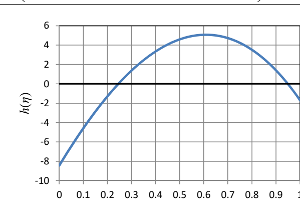
Ice Albedo Feedback

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\bar{T}-T)$$

The additional condition: $\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=T_c=-10$

can be written: $h(\eta)\equiv\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0)+\frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)S(\eta)\right)\right)-\frac{A}{B}-T_c=0$

Two equilibria (zeros of h) satisfy the additional condition.



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Budyko's Model

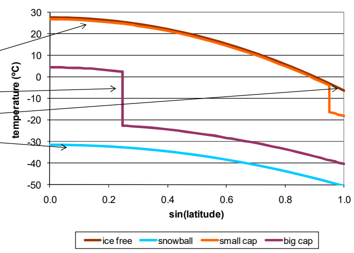
Ice Albedo Feedback

Equilibrium temperature profiles

$$T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha(y,\eta)) - A + C\overline{T}_\eta^*\right)$$

Interesting Solutions:

- small cap
- large cap
- ice free
- snowball



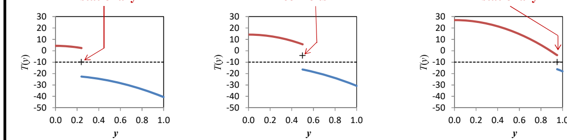
Budyko's Model

Dynamics of the Ice Line

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T} - T)$$

Idea:

If the average temperature across the ice line is above the critical temperature, some ice will melt, moving the ice line toward the pole. If it is below the critical temperature, the ice will advance toward the equator.



Widiasih's equation: $\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$

Budyko's Model

Dynamics of the Ice Line

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T} - T)$$

State space: $[0,1]\times X$

Esther R. Widiasih, Dynamics of the Budyko Energy Balance Model, *SIAM J. Appl. Dyn. Syst.*, 12 (2013), 2068–2092.

Budyko's Model

Dynamics of the Ice Line

$$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$$

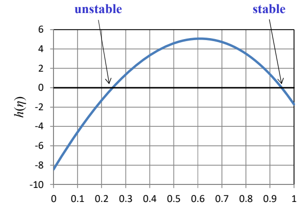
$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T} - T)$$

State space: $[0,1]\times X$

Widiasih's Theorem. For sufficiently small ε, the system has an attracting invariant curve given by the graph of a function $\Phi_\varepsilon : [0,1] \to X$. On this curve, the dynamics are approximated by the equation

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

experts only

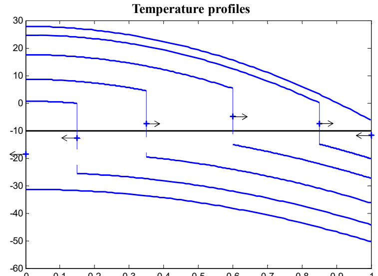


Esther R. Widiasih, Dynamics of the Budyko Energy Balance Model, *SIAM J. Appl. Dyn. Syst.*, 12 (2013), 2068–2092.

Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



Budyko's Model

Summary

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T} - T)$$

surface temperature; sin(latitude); $\overline{T} = \int_0^1 T(y)dy$; heat capacity; insolation; albedo; OLR; heat transport

reduces to

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

$$\frac{d\eta}{dt} = \varepsilon h(\eta) = \varepsilon\left(\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2 + (\alpha_2-\alpha_1)S(\eta)\right)\right) - \frac{A}{B} - T_c\right)$$

Budyko's Model

Budyko-Widiasih Model

surface temperature — sin(latitude) — $\overline{T}=\int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y))-(A+BT)+C(\overline{T}-T)$$

heat capacity — insolation — albedo — OLR — heat transport

What about the greenhouse effect?

$A+BT$ is the outgoing long wave radiation. This term decreases if the greenhouse gases increase.
We view A as a parameter.

$$\frac{d\eta}{dt}=\varepsilon h(\eta,A)$$

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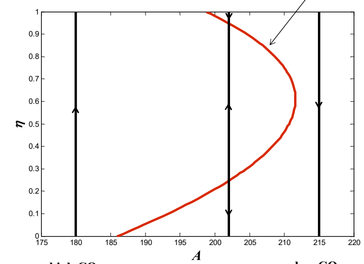
Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt}=\varepsilon h(\eta,A)$$

isocline $h(\eta,A)=0$



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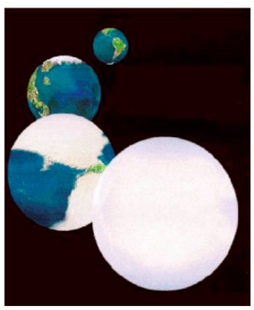
Snowball Earth

Is it possible for Earth to become completely covered in ice?
(**Snowball Earth**)

Did it ever happen?



http://www.astrobio.net/topic/solar-system/earth/new-information-about-snowball-earth-period/

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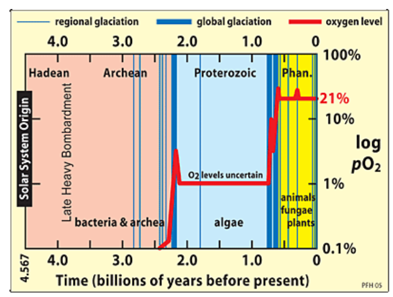
Snowball Earth

There is evidence that Snowball Earth has occurred, the last time about 600 million years ago.



http://www.snowballearth.org/when.html

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Snowball Earth

The continents were clustered near the equator.

Hoffman & Schrag, *Snowball Earth*, SCIENTIFIC AMERICAN, January 2000, 68-75

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Snowball Earth

"Ice-rafted debris" occurred in ocean sediments near the equator, indicating large equatorial glaciers calving icebergs.

Hoffman & Schrag, *Snowball Earth*, SCIENTIFIC AMERICAN, January 2000, 68-75

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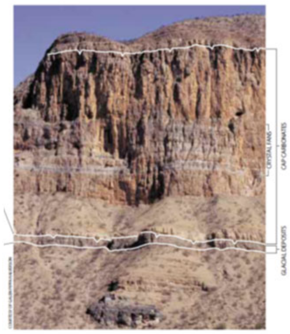
Snowball Earth

Large limestone deposits "cap carbonates" are found immediately above the glacial debris, indicating a rapid warming period following the snowball.



Hoffman & Schrag, *Snowball Earth*, Scientific American, January 2000, 68-75

Snowball Earth

Idea:

When the Earth is mostly ice-covered, silicate weathering slows down, but volcanic activity stays the same, allowing for a build-up of CO2 in the atmosphere.

When the Earth is mostly ice-free, silicate weathering speeds up, drawing down the CO2 in the atmosphere.

Budyko's Model

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

What if A is a dynamical variable?

Simple equation:

$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$

$$0 < \tilde{\varepsilon} \ll \varepsilon \ll 1$$

MCRN Paleocarbon equation (silicate weathering)

New system:

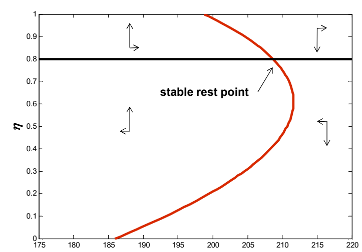
$$\frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$
$$\frac{d\eta}{dt} = \varepsilon h(\eta, A)$$

Anna M. Barry, Esther Widiasih, Richard McGehee, *Discrete & Continuous Dynamical Systems-B*, **22** (2017)
doi: 10.3934/dcdsb.2017125

Budyko's Model

Budyko-Widiasih-Paleocarbon Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A), \quad \frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$



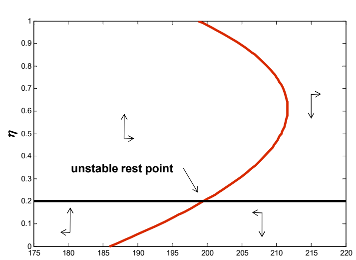
What if η_c were here?

η_c

Budyko's Model

Budyko-Widiasih-Paleocarbon Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta, A), \quad \frac{dA}{dt} = \tilde{\varepsilon}(\eta - \eta_c)$$



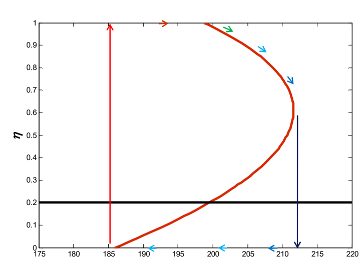
What if η_c were here?

η_c

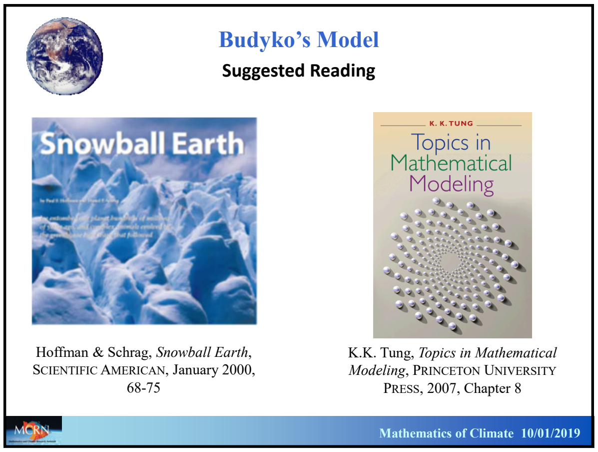
Budyko's Model

Budyko-Widiasih-Paleocarbon Model

Snowball – Hothouse Oscillations



η_c


Budyko's Model
Suggested Reading
Snowball Earth
K. K. TUNG
Topics in Mathematical Modeling
Hoffman & Schrag, Snowball Earth, SCIENTIFIC AMERICAN, January 2000, 68-75
K.K. Tung, Topics in Mathematical Modeling, PRINCETON UNIVERSITY PRESS, 2007, Chapter 8
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