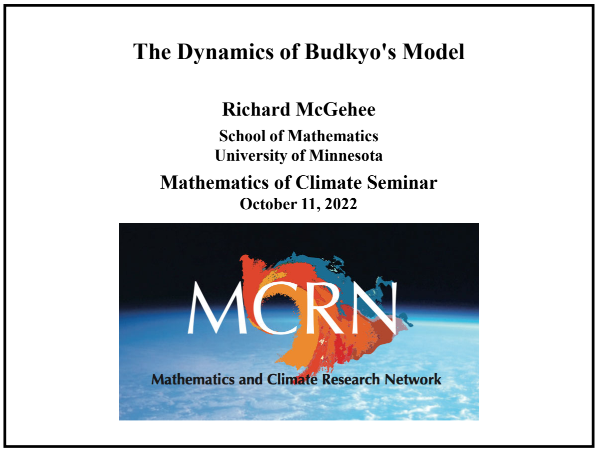

The Dynamics of Budkyo's Model

Richard McGehee
School of Mathematics
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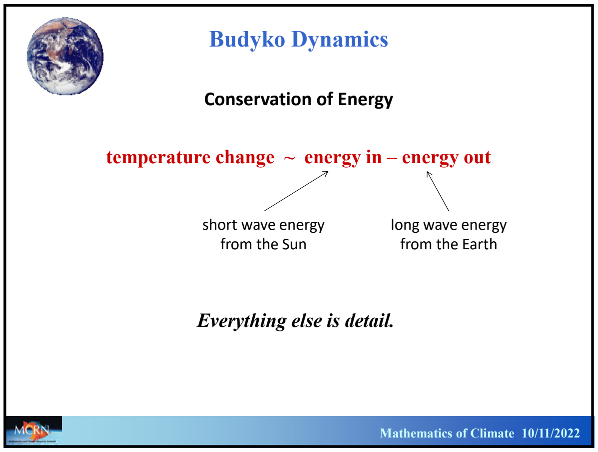
Mathematics of Climate Seminar
October 11, 2022



Budyko Dynamics

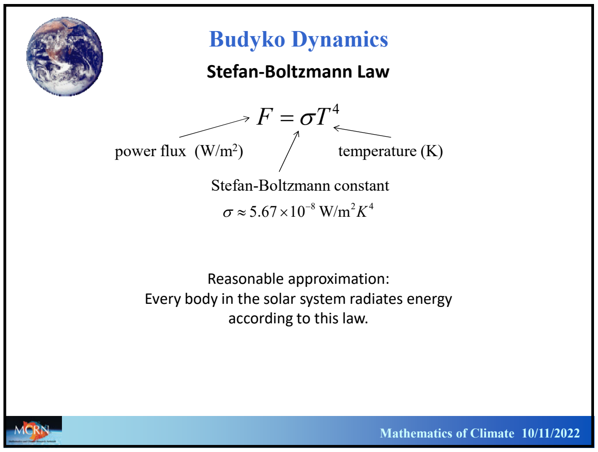
Conservation of Energy

temperature change ~ energy in – energy out

short wave energy
from the Sun

long wave energy
from the Earth

Everything else is detail.



Budyko Dynamics

Stefan-Boltzmann Law

$$F = \sigma T^4$$

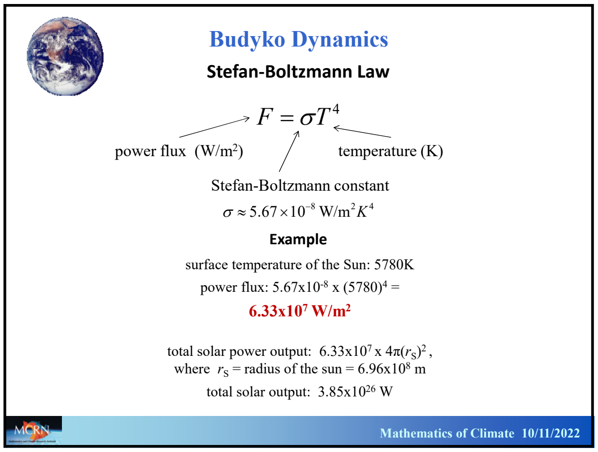
power flux (W/m^2)

temperature (K)

Stefan-Boltzmann constant

$\sigma \approx 5.67 \times 10^{-8}$ W/m$^2K^4$

Reasonable approximation:
Every body in the solar system radiates energy
according to this law.



Budyko Dynamics

Stefan-Boltzmann Law

$$F = \sigma T^4$$

power flux (W/m^2)

temperature (K)

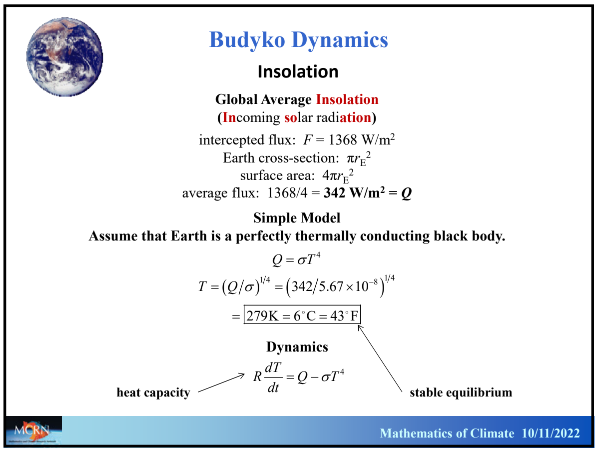
Stefan-Boltzmann constant

$\sigma \approx 5.67 \times 10^{-8}$ W/m$^2K^4$

Example

surface temperature of the Sun: 5780K
power flux: 5.67x10^{-8} x $(5780)^4$ =
6.33x10^7 W/m^2

total solar power output: 6.33x10^7 x $4\pi(r_S)^2$,
where r_S = radius of the sun = 6.96x10^8 m
total solar output: 3.85x10^{26} W



Budyko Dynamics

Insolation

Global Average Insolation
(Incoming solar radiation)

intercepted flux: F = 1368 W/m^2
Earth cross-section: $\pi r_E{}^2$
surface area: $4\pi r_E{}^2$
average flux: 1368/4 = **342 W/m^2 = Q**

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

$$Q = \sigma T^4$$

$$T = (Q/\sigma)^{1/4} = \left(342/5.67 \times 10^{-8}\right)^{1/4}$$

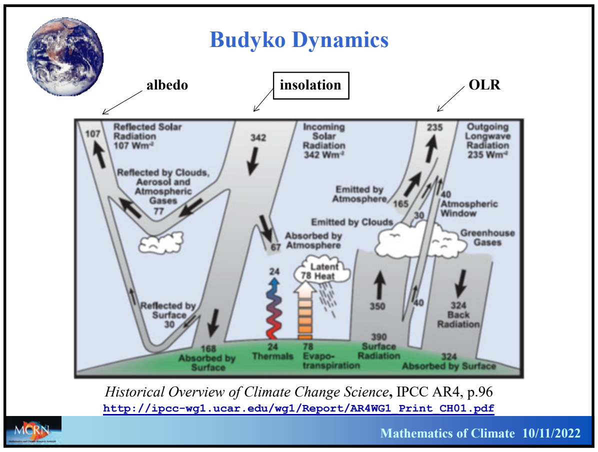
$$= \boxed{279\text{K} = 6^\circ\text{C} = 43^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q - \sigma T^4$$

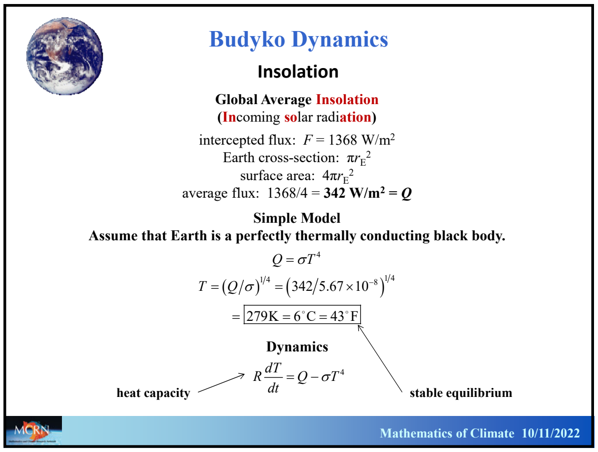
heat capacity

stable equilibrium



Budyko Dynamics

Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf



Budyko Dynamics

Insolation

Global Average Insolation
(Incoming solar radiation)

intercepted flux: $F = 1368$ W/m^2
Earth cross-section: πr_E^2
surface area: $4\pi r_E^2$
average flux: 1368/4 = **342 W/m² = *Q***

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

$$Q = \sigma T^4$$

$$T = (Q/\sigma)^{1/4} = \left(342/5.67\times10^{-8}\right)^{1/4}$$

$$= \boxed{279\text{K} = 6^\circ\text{C} = 43^\circ\text{F}}$$

Dynamics

heat capacity

$$R\frac{dT}{dt} = Q - \sigma T^4$$

stable equilibrium

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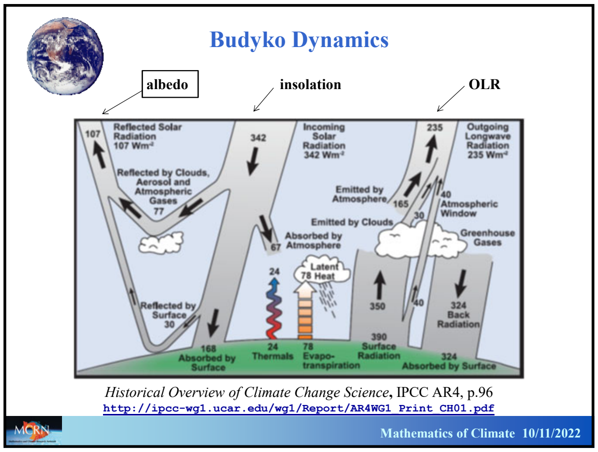

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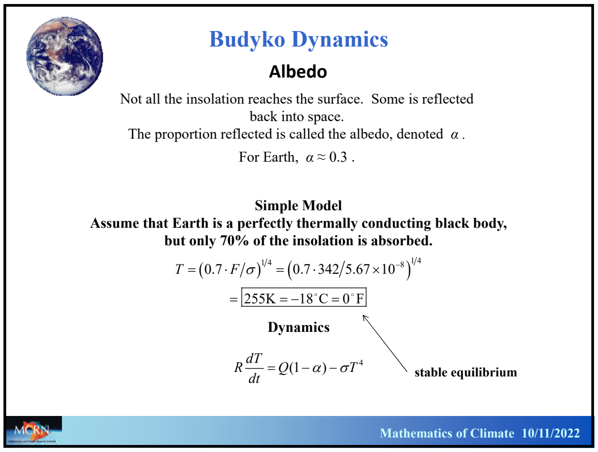


Budyko Dynamics

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α .

For Earth, $\alpha \approx 0.3$.

Simple Model
Assume that Earth is a perfectly thermally conducting black body,
but only 70% of the insolation is absorbed.

$$T = (0.7\cdot F/\sigma)^{1/4} = \left(0.7\cdot 342/5.67\times10^{-8}\right)^{1/4}$$

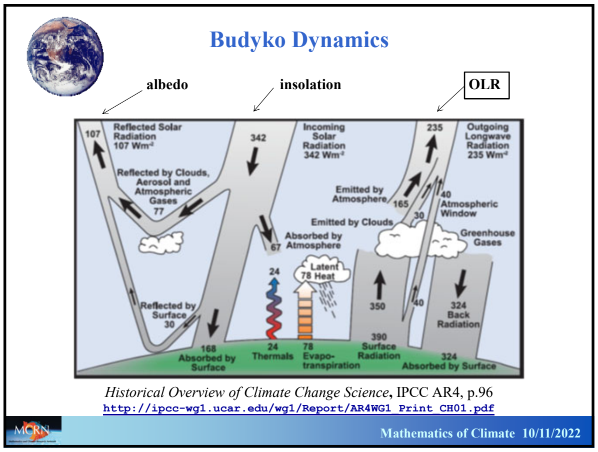
$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium

Mathematics of Climate 10/11/2022



Budyko Dynamics

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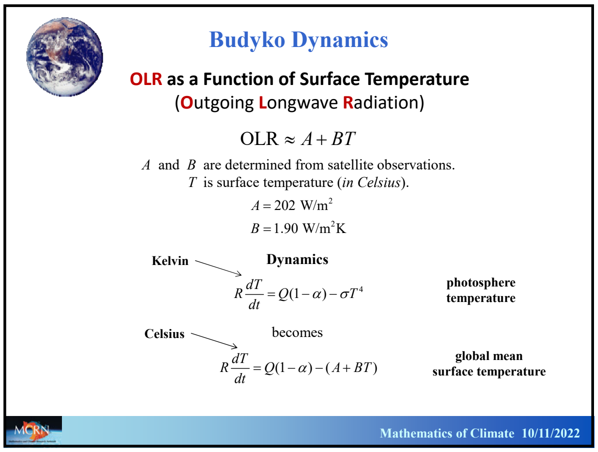

Budyko Dynamics

OLR as a Function of Surface Temperature
(Outgoing Longwave Radiation)

$$\text{OLR} \approx A + BT$$

A and B are determined from satellite observations.
T is surface temperature (*in Celsius*).

$$A = 202\ \text{W/m}^2$$

$$B = 1.90\ \text{W/m}^2\text{K}$$

Kelvin

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

photosphere temperature

Celsius

becomes

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature

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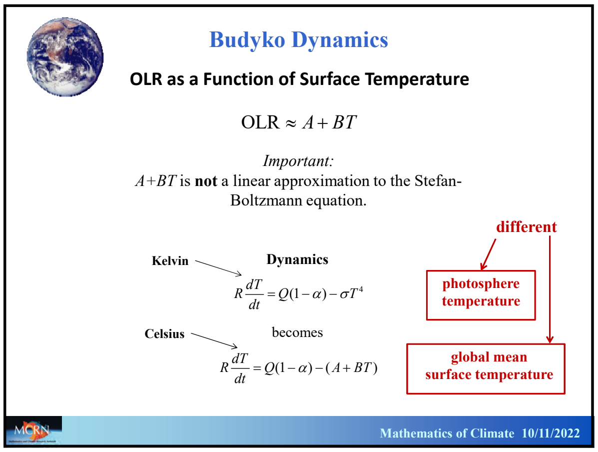

Budyko Dynamics

OLR as a Function of Surface Temperature

$$\text{OLR} \approx A + BT$$

Important:
$A+BT$ is **not** a linear approximation to the Stefan-Boltzmann equation.

different

Kelvin

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

photosphere temperature

Celsius

becomes

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature

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Budyko Dynamics

Homogeneous Earth

$$R\frac{dT}{dt}=Q(1-\alpha)-(A+BT)$$

Equilibrium Temperature: $Q(1-\alpha)-A-BT_{eq}=0$

$$T_{eq}=\frac{Q(1-\alpha)-A}{B}$$ Stable, since $B>0$.

Ice-free planet: $\alpha=0.32$, $T_{\text{eq}}=16$ °C
Snowball planet: $\alpha=0.62$, $T_{\text{eq}}=-38$ °C

No glacier would form on an ice-free Earth.

No glacier would melt on a snowball Earth.

Easy question:
Why do we have ice caps?

Hard question:
If Earth was ever a snowball, how did we get out?

Budyko Dynamics

Latitude Dependence

Make T depend on $y=\sin(\text{latitude})$

$$R\frac{\partial T(y,t)}{\partial t}=Qs(y)(1-\alpha)-\left(A+BT(y,t)\right)$$

insolation distribution

$Q=$ global annual average insolation $=342\text{W/m}^2$

$s(y)=$ distribution across latitudes $\left(\int_0^1 s(y)dy=1\right)$

One can show that

$$s(y)=\frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta-y\cos\beta\right)^2}\,d\theta$$

β = obliquity = 23.4°

Chylek and Coakley's quadratic approximation:

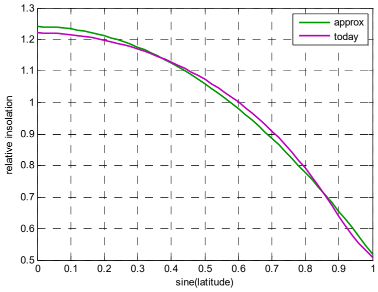
$$s(y)\approx 1-0.241\left(3y^2-1\right)$$

Budyko Dynamics

Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.4°



Budyko Dynamics

Latitude Dependence

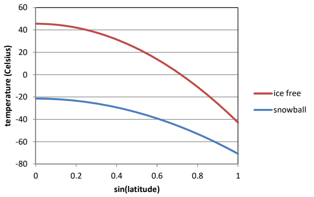
$$R\frac{\partial T(y,t)}{\partial t}=Qs(y)(1-\alpha)-\left(A+BT(y,t)\right)$$

Note that y is just a parameter.

Equilibrium Temperature Profile

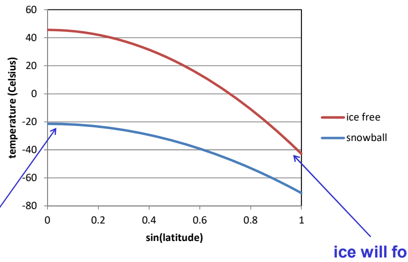
$$T_{eq}(y)=\frac{Qs(y)(1-\alpha)-A}{B}$$

α = 0.32: ice free
α = 0.62: snowball



Budyko Dynamics

Latitude Dependence



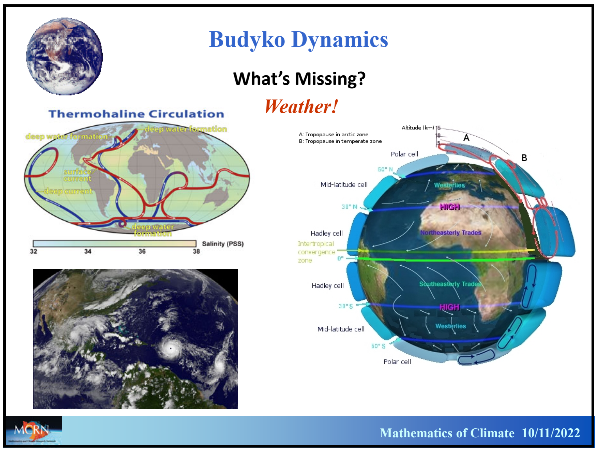
ice won't melt
(no exit from snowball)

ice will form
(icecap)

Budyko Dynamics

What's Missing?

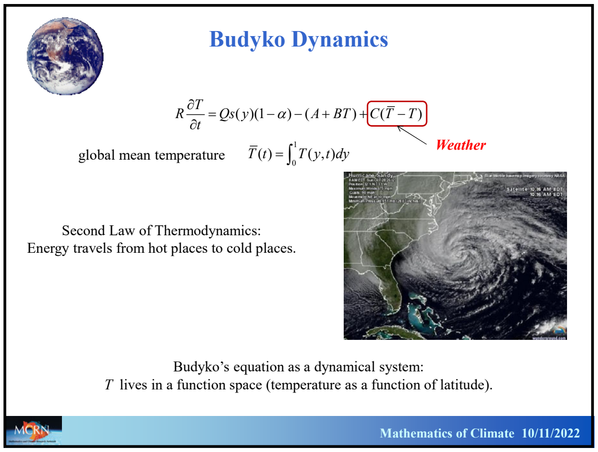
$$R\frac{\partial T(y,t)}{\partial t}=Qs(y)(1-\alpha)-\left(A+BT(y,t)\right)$$



Budyko Dynamics

What's Missing?

Weather!



Budyko Dynamics

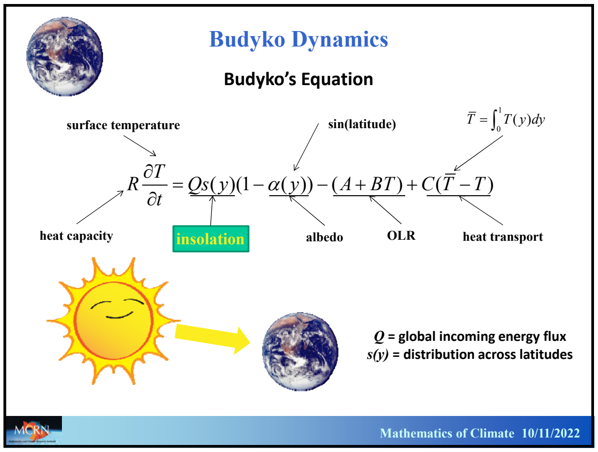
$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + \boxed{C(\overline{T}-T)}$$

Weather

global mean temperature $\overline{T}(t) = \int_0^1 T(y,t)dy$

Second Law of Thermodynamics:
Energy travels from hot places to cold places.

Budyko's equation as a dynamical system:
T lives in a function space (temperature as a function of latitude).



Budyko Dynamics

Budyko's Equation

surface temperature

sin(latitude)

$\overline{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = \underline{Qs(y)}(1-\underline{\alpha(y)}) - \underline{(A+BT)} + \underline{C(\overline{T}-T)}$$

heat capacity

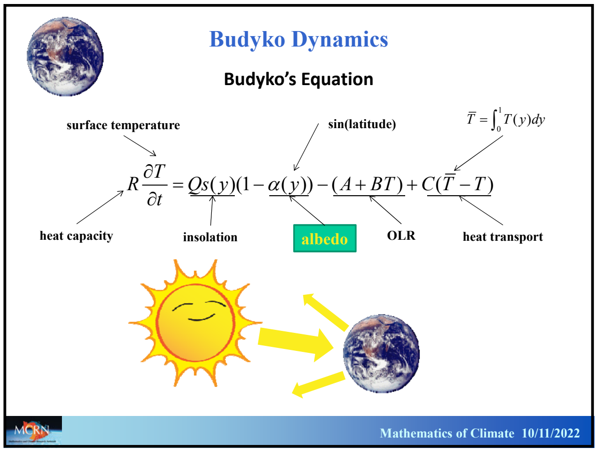
insolation

albedo

OLR

heat transport

Q = global incoming energy flux
s(y) = distribution across latitudes



Budyko Dynamics

Budyko's Equation

surface temperature

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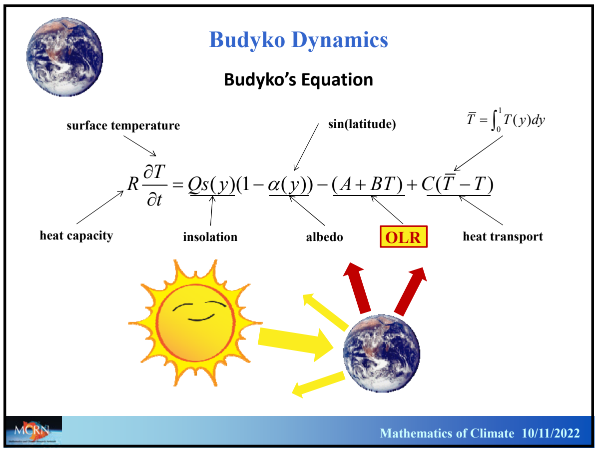
heat capacity

insolation

albedo

OLR

heat transport



Budyko Dynamics

Budyko's Equation

surface temperature

sin(latitude)

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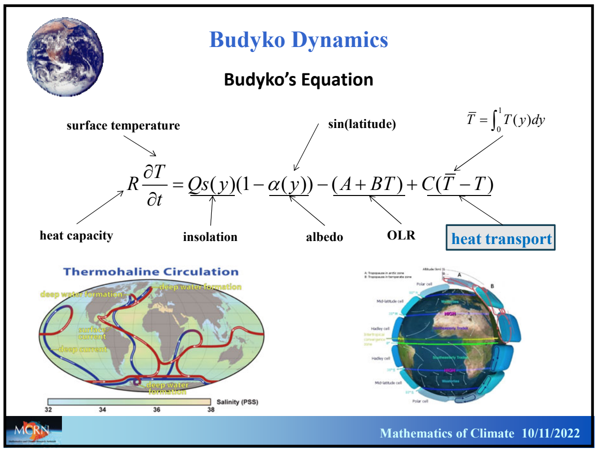
heat capacity

insolation

albedo

OLR

heat transport



Budyko Dynamics

Budyko's Equation

surface temperature

sin(latitude)

$\overline{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = \underline{Qs(y)}(1-\underline{\alpha(y)}) - \underline{(A+BT)} + \underline{C(\overline{T}-T)}$$

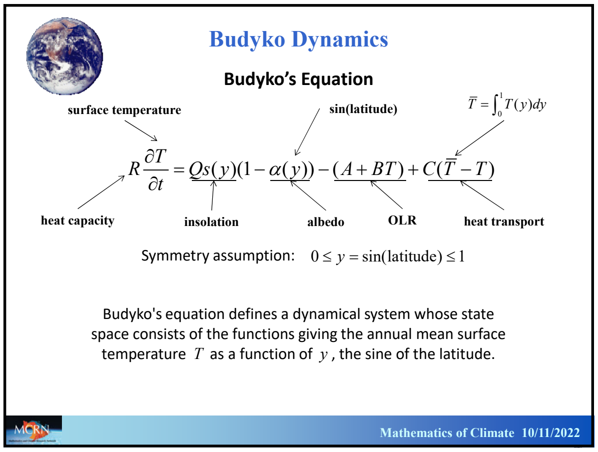
heat capacity

insolation

albedo

OLR

heat transport



Budyko Dynamics

Budyko's Equation

surface temperature

sin(latitude)

$\bar{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = \underline{Qs(y)}(1-\underline{\alpha(y)}) - \underline{(A+BT)} + \underline{C(\bar{T}-T)}$$

heat capacity

insolation

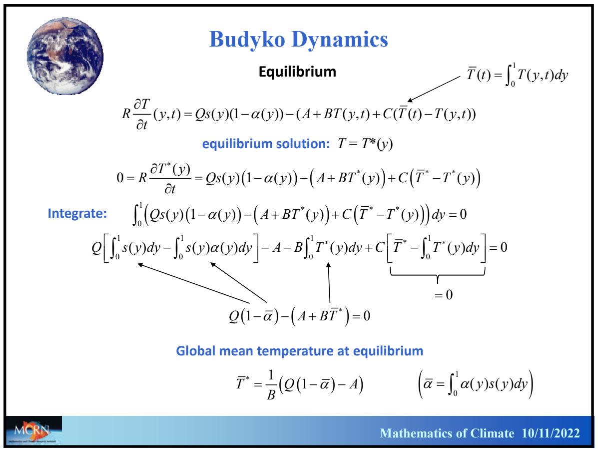
albedo

OLR

heat transport

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Budyko's equation defines a dynamical system whose state space consists of the functions giving the annual mean surface temperature T as a function of y, the sine of the latitude.



Budyko Dynamics

Equilibrium

$\bar{T}(t) = \int_0^1 T(y,t)dy$

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)(1-\alpha(y)) - (A+BT(y,t) + C(\bar{T}(t) - T(y,t))$$

equilibrium solution: $T = T^*(y)$

$$0 = R\frac{\partial T^*(y)}{\partial t} = Qs(y)\left(1-\alpha(y)\right) - \left(A+BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right)$$

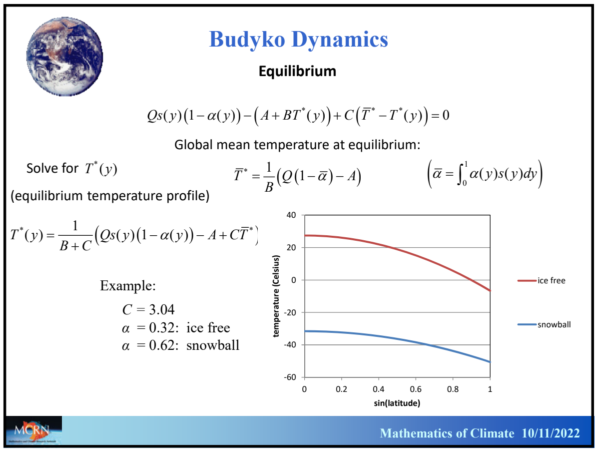
Integrate: $\int_0^1 \left(Qs(y)\left(1-\alpha(y)\right) - \left(A+BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right)\right)dy = 0$

$$Q\left[\int_0^1 s(y)dy - \int_0^1 s(y)\alpha(y)dy\right] - A - B\int_0^1 T^*(y)dy + C\underbrace{\left[\bar{T}^* - \int_0^1 T^*(y)dy\right]}_{=0} = 0$$

$$Q\left(1-\bar{\alpha}\right) - \left(A+B\bar{T}^*\right) = 0$$

Global mean temperature at equilibrium

$$\bar{T}^* = \frac{1}{B}\left(Q\left(1-\bar{\alpha}\right) - A\right) \qquad \left(\bar{\alpha} = \int_0^1 \alpha(y)s(y)dy\right)$$



Budyko Dynamics

Equilibrium

$$Qs(y)\left(1-\alpha(y)\right) - \left(A+BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right) = 0$$

Global mean temperature at equilibrium:

$$\bar{T}^* = \frac{1}{B}\left(Q\left(1-\bar{\alpha}\right) - A\right) \qquad \left(\bar{\alpha} = \int_0^1 \alpha(y)s(y)dy\right)$$

Solve for $T^*(y)$

(equilibrium temperature profile)

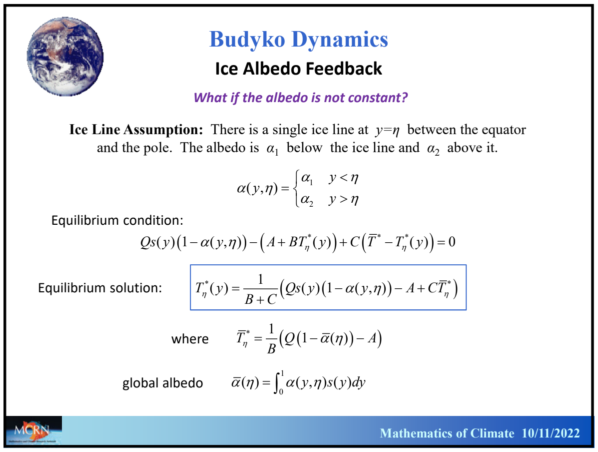
$$T^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y)\right) - A + C\bar{T}^*\right)$$

Example:

$C = 3.04$

$\alpha = 0.32$: ice free

$\alpha = 0.62$: snowball



Budyko Dynamics

Ice Albedo Feedback

What if the albedo is not constant?

Ice Line Assumption: There is a single ice line at $y=\eta$ between the equator and the pole. The albedo is α_1 below the ice line and α_2 above it.

$$\alpha(y,\eta) = \begin{cases} \alpha_1 & y < \eta \\ \alpha_2 & y > \eta \end{cases}$$

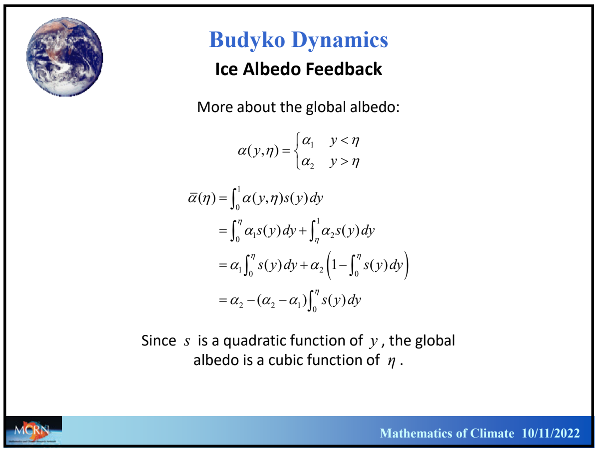
Equilibrium condition:

$$Qs(y)\left(1-\alpha(y,\eta)\right) - \left(A+BT_\eta^*(y)\right) + C\left(\bar{T}^* - T_\eta^*(y)\right) = 0$$

Equilibrium solution: $\boxed{T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\bar{T}_\eta^*\right)}$

where $\bar{T}_\eta^* = \frac{1}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right) - A\right)$

global albedo $\bar{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy$



Budyko Dynamics

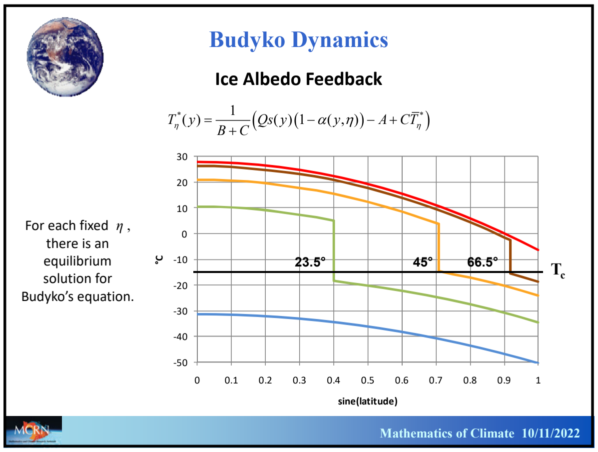
Ice Albedo Feedback

More about the global albedo:

$$\alpha(y,\eta) = \begin{cases} \alpha_1 & y < \eta \\ \alpha_2 & y > \eta \end{cases}$$

$$\begin{aligned} \bar{\alpha}(\eta) &= \int_0^1 \alpha(y,\eta)s(y)dy \\ &= \int_0^\eta \alpha_1 s(y)dy + \int_\eta^1 \alpha_2 s(y)dy \\ &= \alpha_1\int_0^\eta s(y)dy + \alpha_2\left(1 - \int_0^\eta s(y)dy\right) \\ &= \alpha_2 - (\alpha_2 - \alpha_1)\int_0^\eta s(y)dy \end{aligned}$$

Since s is a quadratic function of y, the global albedo is a cubic function of η.

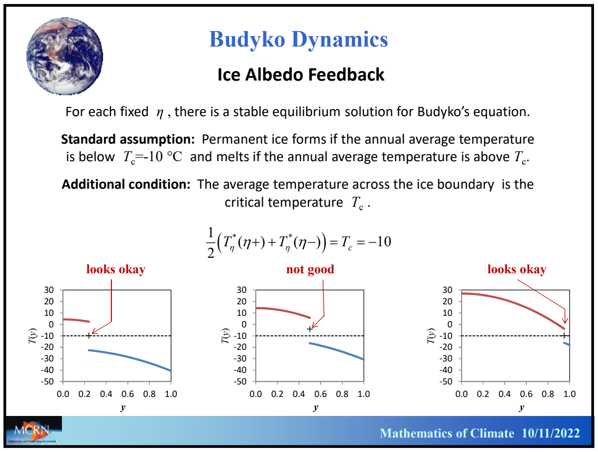


Budyko Dynamics

Ice Albedo Feedback

$$T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\bar{T}_\eta^*\right)$$

For each fixed η, there is an equilibrium solution for Budyko's equation.



Budyko Dynamics

Ice Albedo Feedback

For each fixed η , there is a stable equilibrium solution for Budyko's equation.

Standard assumption: Permanent ice forms if the annual average temperature is below T_c=-10 °C and melts if the annual average temperature is above T_c.

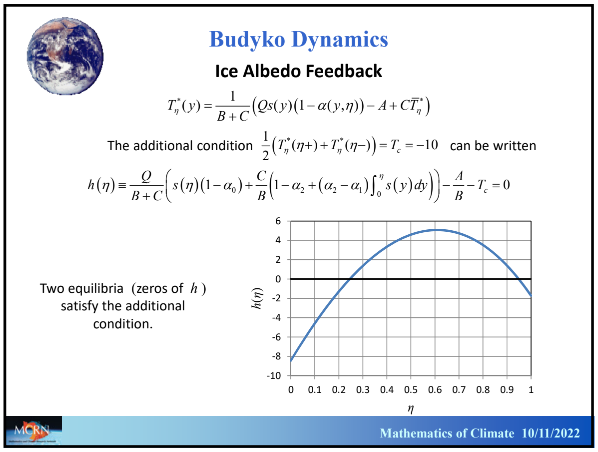
Additional condition: The average temperature across the ice boundary is the critical temperature T_c .

$$\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=T_c=-10$$

looks okay

not good

looks okay



Budyko Dynamics

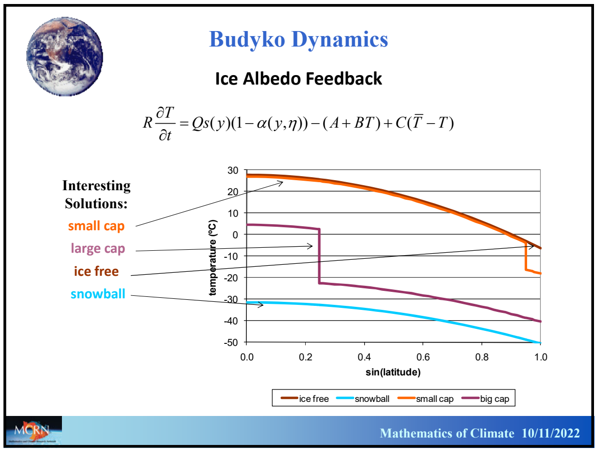
Ice Albedo Feedback

$$T_\eta^*(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right)-A+C\overline{T}_\eta^*\right)$$

The additional condition $\frac{1}{2}\left(T_\eta^*(\eta+)+T_\eta^*(\eta-)\right)=T_c=-10$ can be written

$$h(\eta)\equiv\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0)+\frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\int_0^\eta s(y)dy\right)\right)-\frac{A}{B}-T_c=0$$

Two equilibria (zeros of h) satisfy the additional condition.



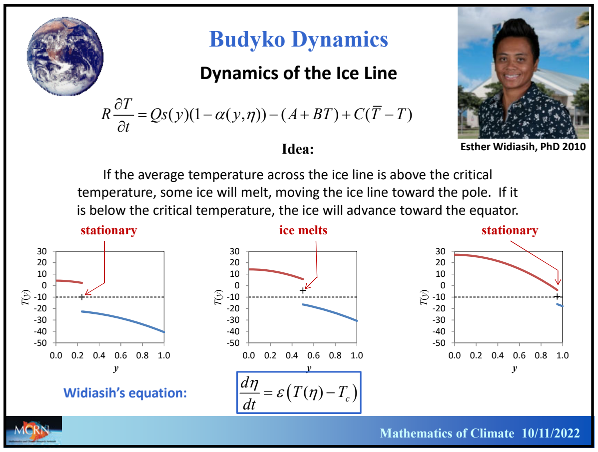
Budyko Dynamics

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\overline{T}-T)$$

Interesting Solutions:

- small cap
- large cap
- ice free
- snowball



Budyko Dynamics

Dynamics of the Ice Line

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\overline{T}-T)$$

Esther Widiasih, PhD 2010

Idea:

If the average temperature across the ice line is above the critical temperature, some ice will melt, moving the ice line toward the pole. If it is below the critical temperature, the ice will advance toward the equator.

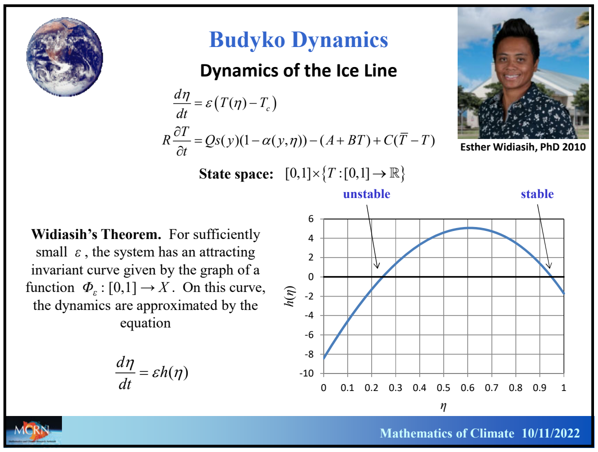
stationary

ice melts

stationary

Widiasih's equation:

$$\boxed{\frac{d\eta}{dt}=\varepsilon\left(T(\eta)-T_c\right)}$$



Budyko Dynamics

Dynamics of the Ice Line

$$\frac{d\eta}{dt}=\varepsilon\left(T(\eta)-T_c\right)$$

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta))-(A+BT)+C(\overline{T}-T)$$

Esther Widiasih, PhD 2010

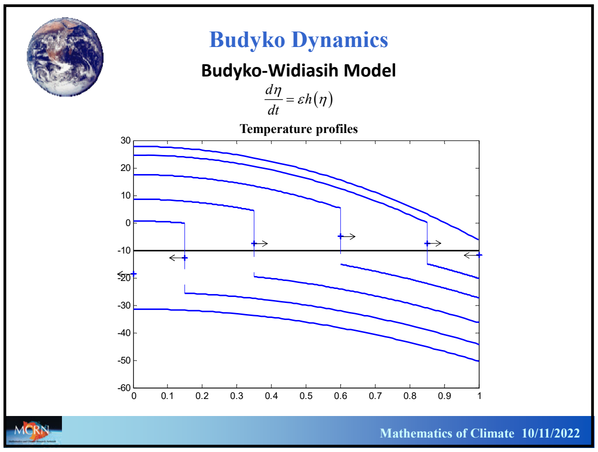
State space: $[0,1]\times\{T:[0,1]\to\mathbb{R}\}$

unstable

stable

Widiasih's Theorem. For sufficiently small ε , the system has an attracting invariant curve given by the graph of a function $\Phi_\varepsilon:[0,1]\to X$. On this curve, the dynamics are approximated by the equation

$$\frac{d\eta}{dt}=\varepsilon h(\eta)$$

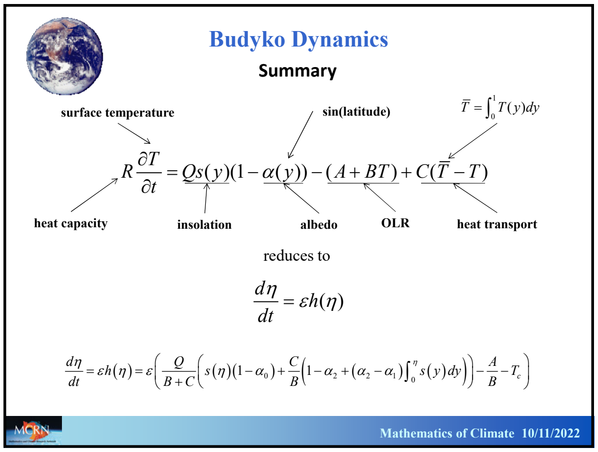


Budyko Dynamics

Budyko-Widiasih Model

$$\frac{d\eta}{dt}=\varepsilon h(\eta)$$

Temperature profiles



Budyko Dynamics

Summary

surface temperature

sin(latitude)

$\overline{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = \underline{Qs(y)}(1-\underline{\alpha(y)}) - \underline{(A+BT)} + \underline{C(\overline{T}-T)}$$

heat capacity

insolation

albedo

OLR

heat transport

reduces to

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

$$\frac{d\eta}{dt} = \varepsilon h(\eta) = \varepsilon\left(\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)\int_0^\eta s(y)dy\right)\right) - \frac{A}{B} - T_c\right)$$

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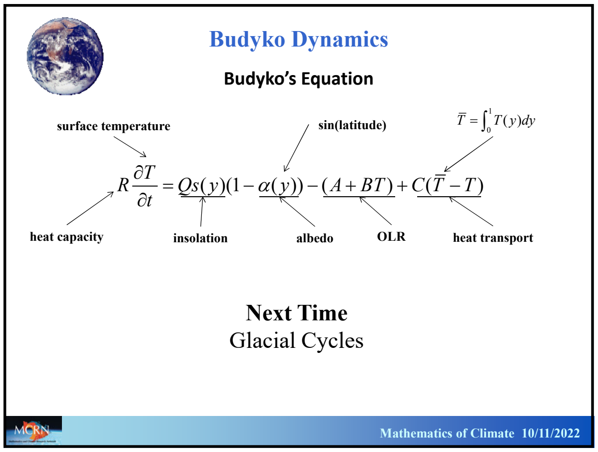

Budyko Dynamics

Budyko's Equation

surface temperature

sin(latitude)

$\overline{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = \underline{Qs(y)}(1-\underline{\alpha(y)}) - \underline{(A+BT)} + \underline{C(\overline{T}-T)}$$

heat capacity

insolation

albedo

OLR

heat transport

Next Time

Glacial Cycles

MCRN

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