


**Math 5490**  
**Topics in Applied Mathematics**  
**Introduction to the Mathematics of Climate**


Fall 2023  
**1:25 - 3:20 Tuesdays and Thursdays**  
**Amundson Hall 162**

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course website  
 www-users.cse.umn.edu/~mcgehee/teaching/Math5490/




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**Math 5490**  
**Energy Balance**

What determines the Earth's surface temperature?




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**Math 5490**  
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What determines the Earth's surface temperature?

**Conservation of Energy**  
 Heat is a form of energy.  
 Temperature measures heat.



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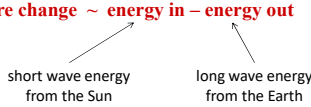



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**temperature change ~ energy in – energy out**





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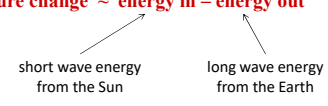


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*Everything else is detail.*



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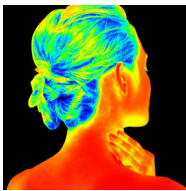


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**Energy Balance**

**Black-Body Radiation**

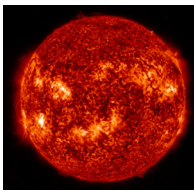
Every object emits electromagnetic radiation according to its temperature. The intensity of the radiation (power flux) is approximated by a theoretical object called a *perfect black body*.

**human**




<https://letstalkscience.ca/educational-resources/backgrounders/thermal-imaging>

**sun**



<https://solarsystem.nasa.gov/solar-system/sun/overview/>



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Energy Balance



*Black-Body Radiation*  
**Stefan-Boltzmann Law**

$F = \sigma T^4$   
 power flux (W/m<sup>2</sup>) ← temperature (K)

watts per square meter  
 watt = joule per second = "power"

kelvin  
 0 K = -273°C = "absolute zero"

Stefan-Boltzmann constant  
 $\sigma \approx 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$

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Reasonable approximation:  
Every body in the solar system radiates energy according to this law.

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*Lets try the Sun.*






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**Math 5490**  
Energy Balance

*What is the surface temperature of the Sun?*

<https://www.nasa.gov/sun>



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**10000°F = 5810K**




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surface temperature of the Sun: 5810K  
 power flux:  $5.67 \times 10^{-8} \times (5810)^4 =$   
 $6.46 \times 10^7 \text{ W/m}^2$

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Wikipedia: Prairie Island Nuclear Power Plant capacity =  
522 + 519 = 1041 MW  $\approx 10^9$  watts  
 $10^9 / (6.46 \times 10^7) \approx 15.5 \text{ m}^2$   
Sixteen square meters on the Sun's surface produces more  
power than a 1 gigawatt nuclear power plant.

[https://en.wikipedia.org/wiki/Prairie\\_Island\\_Nuclear\\_Power\\_Plant](https://en.wikipedia.org/wiki/Prairie_Island_Nuclear_Power_Plant)



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*What is the total energy output of the Sun?*



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*How many square meters on the surface of the Sun?*

*What is the total energy output of the Sun?*



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*What is the total energy output of the Sun?*

radius of the sun =  $6.957 \times 10^8$  meters  
 $\approx 7 \times 10^8$  meters

[https://en.wikipedia.org/wiki/Solar\\_radius](https://en.wikipedia.org/wiki/Solar_radius)



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total solar power output:  $6.46 \times 10^7 \times 4\pi(r_s)^2$ ,  
where  $r_s$  = radius of the sun =  $7 \times 10^8$  m  
total solar output: **3.98 × 10<sup>26</sup> W**



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How much is that?



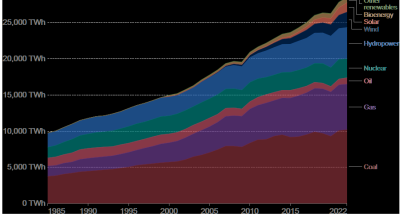
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### Math 5490 Energy Balance

#### Annual Electricity Production

2022 production: about 28,000 TWh

Electricity production by source, World



28,000 TWh =  $28 \times 10^{15} \text{ Wh}$   
 1 Wh =  $3.6 \times 10^3 \text{ J}$   
 28,000 TWh =  $28 \times 3.6 \times 10^{18} \text{ J} = 10^{20} \text{ J}$   
 Solar Output =  $3.98 \times 10^{26} \text{ J/s}$   
 seconds of solar output =  $10^{20} / 3.98 \times 10^{26} = 0.25 \times 10^{-6} = 250 \times 10^{-9} = 250 \text{ nanoseconds}$

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250 nanoseconds = time it takes for the Sun to produce the equivalent of the annual global electricity production.

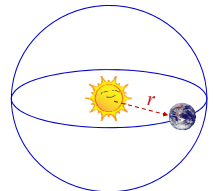


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### Math 5490 Insolation

#### Incoming solar Radiation

How much energy from the Sun is hitting the Earth?



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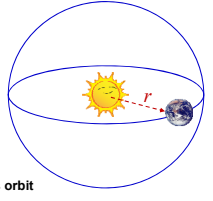
Solar flux at a distance  $r$  from the sun:

$$F = \frac{6.46 \times 10^7 \cdot 4\pi r_s^2}{4\pi r^2} = 6.46 \times 10^7 \left(\frac{r_s}{r}\right)^2 \text{ W/m}^2$$

$r_s = 7 \times 10^8 \text{ m}$   
 $r = 1.5 \times 10^{11} \text{ m}$

**$F = 1407 \text{ W/m}^2$**

solar flux at Earth's orbit



<https://solarsystem.nasa.gov/planets/earth/overview/>

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solar flux at Earth's orbit

Earth presents a disk to the Sun

area of disk:  $F \times \pi r_E^2$  W

power intercepted by the Earth:  $F \times \pi r_E^2$  W

Earth's surface area:  $4\pi r_E^2 \text{ m}^2$

area of surface

Average surface flux:  $\frac{F \times \pi r_E^2}{4\pi r_E^2} = \frac{F}{4} = 352 \text{ W/m}^2$

note the 4

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How much energy from the Sun is hitting the Earth?

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solar flux at Earth's orbit

**Note:**  
The IPCC uses  $F = 1368$  and  $F/4 = 342$ , which are the values we will use.

Average surface flux:  $\frac{F \times \pi r_e^2}{4\pi r_e^2} = \frac{F}{4} = 352 \text{ W/m}^2$

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**Incoming solar Radiation**

How much energy from the Sun is hitting the Earth?

Solar flux at a distance  $r$  from the sun:  $F = 1368 \text{ W/m}^2$

Power intercepted by Earth:  $F \times \pi r_e^2 \text{ W}$

$r_e = \text{radius of Earth} = 6.37 \times 10^6 \text{ m}$   
Earth surface area

[https://en.wikipedia.org/wiki/Earth\\_radius](https://en.wikipedia.org/wiki/Earth_radius)

Power intercepted by Earth:  
 $F \times \pi r_e^2 = 1368 \times \pi \times 6.37^2 \times 10^{12} = 1.74 \times 10^{17} \text{ W}$

How much is that?

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**Incoming solar Radiation**

How much energy from the Sun is hitting the Earth?

Solar power intercepted by Earth:  $1.74 \times 10^{17} \text{ W}$

**Biologically Stored Energy**  
total coal reserves:  $10^{15} \text{ kg}$   
energy content:  $3 \times 10^7 \text{ J/kg}$   
total energy in coal reserves:  $3 \times 10^{22} \text{ J}$   
How long:  $3 \times 10^{22} / 1.74 \times 10^{17}$   
 $\approx 1.7 \times 10^5 \text{ seconds}$   
 $\approx 47 \text{ hours}$   
 $\approx 2 \text{ days of insolation}$



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temperature change  $\sim$  energy in  $-$  energy out

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energy in from the Sun      energy out from the Earth

**Simple Model**  
Assume that Earth is a perfectly thermally conducting black body.

energy in from the Sun:  $342 \text{ W/m}^2$       energy out from the Earth:  $\sigma T^4 \text{ W/m}^2$

$$T = (342 / \sigma)^{1/4} = (342 / 5.67 \times 10^{-8})^{1/4} = 279 \text{ K} = 6^\circ \text{C} = 43^\circ \text{F}$$

Earth's global mean temperature:  $57^\circ \text{F}$  **Not bad!**

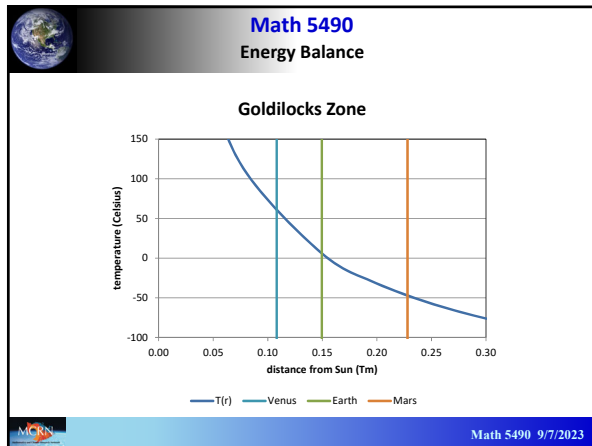


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**Energy Balance**

**Goldilocks Zone**

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**Group Questions**

Assume that Mars is a perfectly conducting black body.  
*What would be its surface temperature?*

*What about Venus?*

*What about Mercury?*

*What about Saturn?*

*What about Pluto?*

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