Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

Richard McGehee, Instructor
458 Vincent Hall
mcgehee@umn.edu
www-users.cse.umn.edu/~mcgehee/

course website
www-users.cse.umn.edu/~mcgehee/teaching/Math5490/

Math 5490
Energy Balance

What determines the Earth's surface temperature?

Math 5490
Energy Balance

What determines the Earth's surface temperature?

temperature change ~ energy in – energy out

energy in from the Sun energy out from the Earth

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

energy in from the Sun $Q = 342\ \mathrm{W/m^2}$

energy out from the Earth $\sigma T^4\ \mathrm{W/m^2}$

$$T = (342/\sigma)^{1/4} = (342/5.67\times10^{-8})^{1/4} =$$
$$279\mathrm{K}\ =\ 6^\circ\mathrm{C}\ =\ 43^\circ\mathrm{F}$$

Earth's global mean temperature: 57°F ***Not bad!***

Math 5490
Energy Balance

What determines the Earth's surface temperature?

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

energy in from the Sun $Q = 342\ \mathrm{W/m^2}$

energy out from the Earth $\sigma T^4\ \mathrm{W/m^2}$

$$T = (342/\sigma)^{1/4} = (342/5.67\times10^{-8})^{1/4} =$$
$$279\mathrm{K}\ =\ 6^\circ\mathrm{C}\ =\ 43^\circ\mathrm{F}$$

Note that there is a differential equation lurking here.

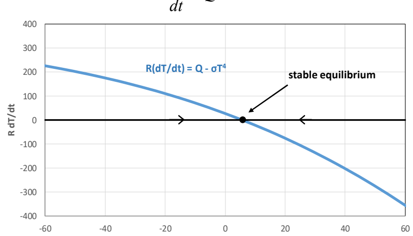
temperature change ~ energy in – energy out

heat capacity → $R\dfrac{dT}{dt} = Q - \sigma T^4$

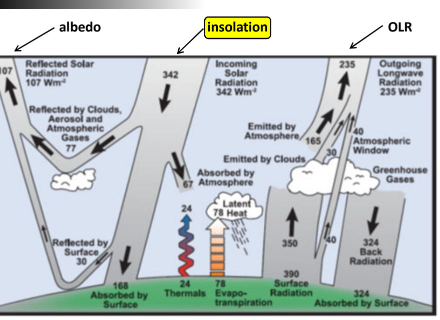
Math 5490
Energy Balance

temperature change ~ energy in – energy out

heat capacity → $R\dfrac{dT}{dt} = Q - \sigma T^4$

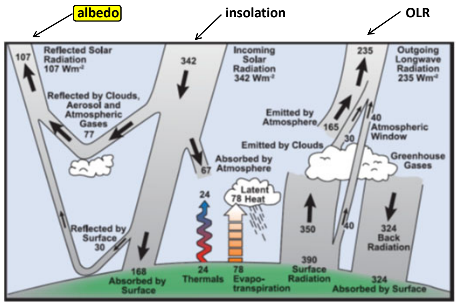


Math 5490
Energy Balance



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Math 5490
Energy Balance



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Math 5490 9/12/2023

Math 5490
Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the ***albedo***, denoted α.

What is Earth's albedo?

Math 5490 9/12/2023

Math 5490
Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α.

What is Earth's albedo?

The average albedo of the Earth from the upper atmosphere, its *planetary albedo*, is 30–35% ... .

https://en.wikipedia.org/wiki/Albedo

For now, let's round to 30%.

$\boldsymbol{\alpha = 0.3}$

Math 5490 9/12/2023

Math 5490
Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α.

For Earth, $\alpha \approx 0.3$.

Math 5490 9/12/2023

Math 5490
Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α.

For Earth, $\alpha \approx 0.3$.

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = (0.7 \cdot Q/\sigma)^{1/4} = (0.7 \cdot 342/5.67\times10^{-8})^{1/4}$$

$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium

Math 5490 9/12/2023

Math 5490
Energy Balance

Albedo

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = (0.7 \cdot Q/\sigma)^{1/4} = (0.7 \cdot 342/5.67\times10^{-8})^{1/4}$$

$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$ ***??? Really cold!***

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

Math 5490 9/12/2023

Math 5490
Energy Balance

Albedo

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = (0.7 \cdot Q/\sigma)^{1/4} = (0.7 \cdot 342/5.67 \times 10^{-8})^{1/4}$$
$$= \boxed{255\text{K} = -18^{\circ}\text{C} = 0^{\circ}\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

Wait a minute! Way too cold!
What just happened???

What happened to 57°F?

Math 5490
Energy Balance

Albedo

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = (0.7 \cdot Q/\sigma)^{1/4} = (0.7 \cdot 342/5.67 \times 10^{-8})^{1/4}$$
$$= \boxed{255\text{K} = -18^{\circ}\text{C} = 0^{\circ}\text{F}}$$ ← ***Way too cold!***

Maybe we got the albedo wrong.
Wikipedia said 30-35%. Let's try 35%, so only 65% of the insolation is absorbed.

$$T = (0.65 \cdot Q/\sigma)^{1/4} = (0.65 \cdot 342/5.67 \times 10^{-8})^{1/4}$$
$$= \boxed{250\text{K} = -23^{\circ}\text{C} = -9^{\circ}\text{F}}$$ ← ***Even colder!***

Oops! Wrong direction.

Math 5490
Energy Balance

Albedo

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = (0.7 \cdot Q/\sigma)^{1/4} = (0.7 \cdot 342/5.67 \times 10^{-8})^{1/4}$$
$$= \boxed{255\text{K} = -18^{\circ}\text{C} = 0^{\circ}\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

Wait a minute! Way too cold!
What just happened???

255 K is actually the *photosphere* temperature.

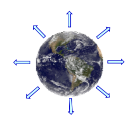
Math 5490
Energy Balance

Photosphere

A photosphere is the deepest region of a luminous object, usually a star, that is transparent to photons of certain wavelengths.

`https://en.wikipedia.org/wiki/Photosphere`

For the Earth, the photosphere is where the long wave photons escape into space. It is high in the atmosphere where the temperature is 255 K.



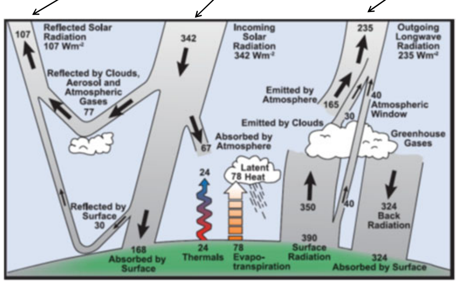
$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

T = photosphere temperature.

What about the surface temperature?

Math 5490
Energy Balance

albedo insolation OLR



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Math 5490
Energy Balance

OLR as a Function of Surface Temperature
(Outgoing Longwave Radiation)

$$\text{OLR} \approx A + BT$$

A and B are determined from satellite observations.
T is surface temperature (*in Celsius*).

$$A = 202 \text{ W/m}^2$$
$$B = 1.90 \text{ W/m}^2\text{K}$$

Dynamics

Kelvin

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

photosphere temperature

becomes

Celsius

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature

Math 5490
Energy Balance

OLR as a Function of Surface Temperature

$$\text{OLR} \approx A + BT$$

Important:
$A+BT$ is **not** a linear approximation to the Stefan-Boltzmann equation.

different

Kelvin → **Dynamics**

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

photosphere temperature

Celsius → becomes

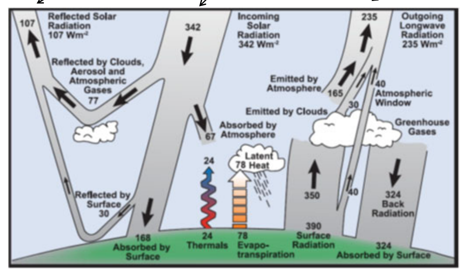
$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

global mean surface temperature

Math 5490 9/12/2023

Math 5490
Energy Balance

albedo insolation OLR



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Math 5490 9/12/2023

Math 5490
Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha) - A}{B}$$

Recall: Q = 342 W/m², A = 202 W/m², B = 1.9 W/m²K

Wikipedia: $0.30 \le \alpha \le 0.35$

$\alpha = 0.30$, $T^* = 19.7°\text{C} = 67°\text{F}$
$\alpha = 0.32$, $T^* = 16.1°\text{C} = 61°\text{F}$
$\alpha = 0.33$, $T^* = 14.3°\text{C} = 58°\text{F}$ ← ***Not bad!***
$\alpha = 0.35$, $T^* = 10.7°\text{C} = 51°\text{F}$

Earth's global mean temperature: 57°F

Math 5490 9/12/2023

Math 5490
Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha) - A}{B}$$

Is it stable?

Math 5490 9/12/2023

Math 5490
Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha) - A}{B}$$

Is it stable?

$$R\frac{dT}{dt} = \left(Q(1-\alpha) - A\right) - BT$$

Stable, since $B > 0$.

Math 5490 9/12/2023

Math 5490
Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha) - A}{B}$$

Is it stable?

$$R\frac{dT}{dt} = \left(Q(1-\alpha) - A\right) - BT$$

Stable, since $B > 0$.

What if Earth had more ice or less ice?
That would change the albedo α.

Math 5490 9/12/2023

Math 5490
Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha) - A}{B}$$

Current Earth: $\alpha = 0.33$, $T^* = 14.3°C = 58°F$

guesses →

Ice-free Earth: $\alpha = \mathbf{0.28}$, $T^* = 23.3°C = 74°F$

Snowball Earth: $\alpha = \mathbf{0.62}$, $T^* = -38\ °C = -36°F$

Why do we have ice caps?

If Earth was ever a snowball, how did we get out?

Math 5490 9/12/2023

Math 5490
Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

What's missing?

Earth is not homogeneous. For example, it is warmer at the equator and colder at the poles. The temperature should depend on latitude.

Make T depend on $y = \sin(\text{latitude})$

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation distribution

$s(y) = \text{distribution across latitudes}$ $\left(\int_0^1 s(y)dy = 1\right)$

Math 5490 9/12/2023

Math 5490
Energy Balance

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Why do we use y = sine(latitude) instead of just latitude?

Math 5490 9/12/2023

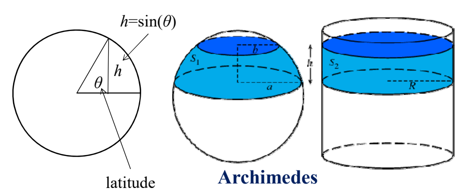
Math 5490
Energy Balance

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Why do we use y = sine(latitude) instead of just latitude?

Because y is directly proportional to surface area.

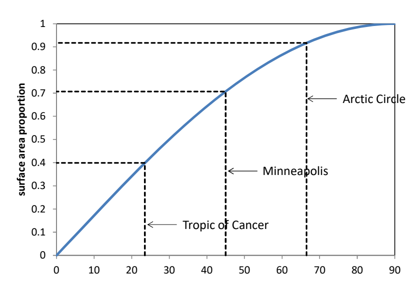


http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html

Math 5490 9/12/2023

Math 5490
Energy Balance

Why y ?



Math 5490 9/12/2023

Math 5490
Energy Balance

Latitiude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

$s(y) = \text{distribution across latitudes}$ $\left(\int_0^1 s(y)dy = 1\right)$

One can show that **β = obliquity = 23.4°**

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\,d\theta$$

McGehee & Lehman, *SIAM J. Applied Dynamical Systems* **11** (2) (2012), 684–707.

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241\left(3y^2 - 1\right)$$

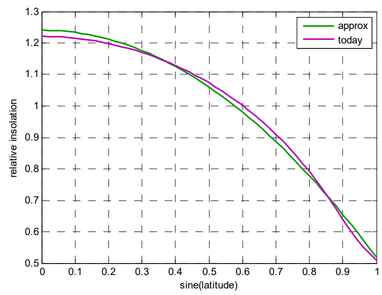
Chylek & Coakley, *J. Atmos. Sci.* **32** (1975), 675–679.

Math 5490 9/12/2023

Math 5490
Energy Balance
Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.5°



Math 5490
Energy Balance
Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Note that an equilibrium temperature T is a function of y = sine(latitude), denoted $T^*(y)$

$$Qs(y)(1-\alpha) - (A + BT^*(y)) = 0$$

$$T^*(y) = \frac{Qs(y)(1-\alpha) - A}{B}$$

Math 5490
Energy Balance
Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Equilibrium Solution

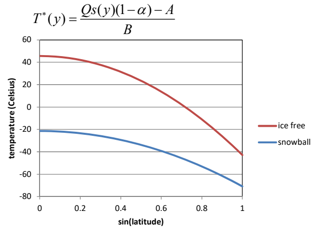
$$T^*(y) = \frac{Qs(y)(1-\alpha) - A}{B}$$

Tung*

α = 0.32: ice free
α = 0.62: snowball

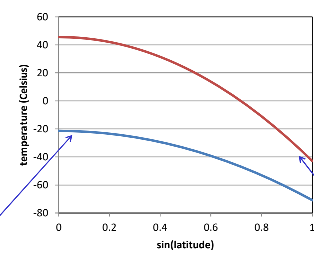
Note that α = 0.32 is in the range of current Earth.

* K.K. Tung, *Topics in Mathematical Modeling*, Princeton U. Press, 2007



Math 5490
Energy Balance
Latitude Dependence

α = 0.32: ice free
α = 0.62: snowball



Current Earth has ice caps, so we are in the ballpark.

ice won't melt (no exit from snowball)

ice will form (icecap)

Math 5490
Energy Balance

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

Math 5490
Energy Balance

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

Math 5490
Energy Balance

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

One simple statement of the law is that heat always moves from hotter objects to colder objects ...

`https://en.wikipedia.org/wiki/Second_law_of_thermodynamics`

Math 5490 9/12/2023

Math 5490
Energy Balance

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

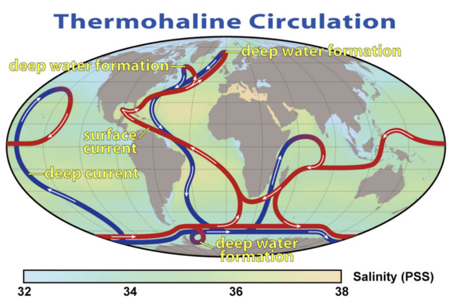
It's hotter at the equator than at the poles, so heat moves from the lower latitudes to the higher latitudes.

How?

Math 5490 9/12/2023

Math 5490
Energy Balance

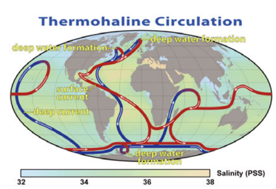
What's Missing?



Math 5490 9/12/2023

Math 5490
Energy Balance

What's Missing?



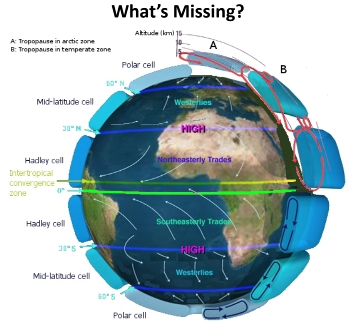
Example

The Gulf Stream carries warm salty surface water from the Gulf of Mexico to the North Atlantic, where it cools, becomes more dense, and sinks, flowing south in the deep ocean.

Math 5490 9/12/2023

Math 5490
Energy Balance

What's Missing?

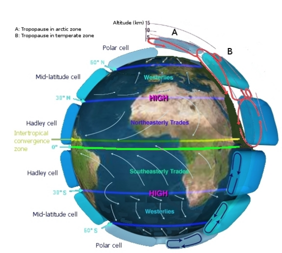


Math 5490 9/12/2023

Math 5490
Energy Balance

What's Missing?

Warm air from the surface along the equator rises and flows toward the poles in a series of cells moving heat from the equator to the poles.



Math 5490 9/12/2023

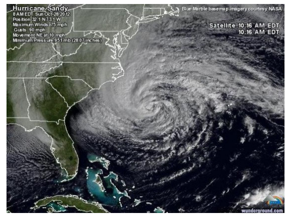
Math 5490
Energy Balance

What's Missing?

Math 5490 9/12/2023

Math 5490
Energy Balance

What's Missing?

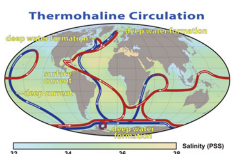


Atlantic hurricanes move heat from the equatorial Atlantic up the coast of North America.

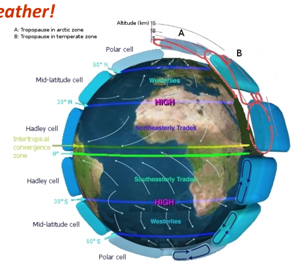
Math 5490 9/12/2023

Math 5490
Energy Balance

What's Missing?



Weather!



The second law of thermodynamics

Math 5490 9/12/2023

Math 5490
Energy Balance

Budyko's Equation

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right) + \boxed{C\left(\overline{T}(t) - T(y,t)\right)}$$

2nd law

global mean temperature $\overline{T}(t) = \int_0^1 T(y,t)dt$

$\left(\overline{T}(t) - T(y,t)\right)$ interpretation

Each point on Earth's surface is trying to assume the global mean temperature. If the temperature at a point is below the the global mean, then it heats up. If the temperature at that point is above the mean, then it cools off.

Math 5490 9/12/2023

Math 5490
Energy Balance

Budyko's Equation

M. I. Budyko, "The effect of solar radiation variations on the climate of the Earth," **Tellus XXI**, 611-619 , 1969.

$$\overline{T} = \int_0^1 T(y)dy$$

$$R\frac{\partial T}{\partial t} = \underline{Qs(y)}\underline{(1-\alpha)} - \underline{(A+BT)} + \underline{C(\overline{T}-T)}$$

surface temperature; sin(latitude); heat capacity; insolation; albedo; OLR; heat transport

Math 5490 9/12/2023

Math 5490
Energy Balance

Discussion

Back to slide on latitude dependence:

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Note that an equilbrium temperature T is a function of y = sine(latitude), denoted $T^*(y)$

$$Qs(y)(1-\alpha) - (A + BT_{eq}(y)) = 0$$

$$\boxed{T^*(y) = \frac{Qs(y)(1-\alpha) - A}{B}}$$

Is this equilibrium solution stable?

Math 5490 9/12/2023

Math 5490

Energy Balance

More Discussion

Why is $\overline{T} = \int_0^1 T(y)dy$ the global mean temperature?

How scary is Budyko's equation?

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right) + C\left(\overline{T}(t) - T(y,t)\right)$$

Is it a PDE?