Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

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course website
www-users.cse.umn.edu/~mcgehee/teaching/Math5490/

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Math 5490
Dynamical Systems

Bifurcation Theory

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Math 5490
Bifurcation Theory

Setup

$\dot{x} = f(x,\mu), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^m$

state variables ↗ ↖ parameters

rest point at $x = 0$ when $\mu = 0$: $f(0,0) = 0$

What happens when we change the parameters?

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Bifurcation Theory

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No Bifurcation (Poincaré Continuation)

The Jacobian matrix $D_1 f(0,0)$ is nonsingular, i.e., has no zero eigenvalues.

Conclusion

For small values of μ, there is a rest point $p(\mu)$ satisfying
$p(0) = 0, \quad f(p(\mu),\mu) = 0.$

The rest point "continues" for small parameter values.

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Bifurcation Theory

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Bifurcation Theory

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$p(0) = 0, \quad f(p(\mu),\mu) = 0.$

The rest point "continues" for small parameter values.

$D_1 f(0,0)$ is the Jacobian with respect to the first variable, holding the second variable constant.

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Bifurcation Theory

$$\dot{x} = f(x,\mu), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^m$$

rest point at $x = 0$ when $\mu = 0$: $f(0,0) = 0$

Poincaré Continuation

If Jacobian matrix $D_1 f(0,0)$ is nonsingular, then, for small values of μ, there is a rest point $p(\mu)$ satisfying

$$p(0) = 0, \quad f(p(\mu),\mu) = 0.$$

Idea of Proof

We can write

$$f(x,\mu) = Ax + B\mu + \mathrm{O}^2(x,\mu) = 0,$$

where $A = D_1 f(0,0)$ and B is an $n \times m$ matrix, and solve for x:

$$x = p(\mu) = A^{-1}B\mu + \mathrm{O}^2(\mu).$$

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$$\dot{x} = f(x,\mu), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^m$$

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higher order terms

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rest point at $x = 0$ when $\mu = 0$: $f(0,0) = 0$

Poincaré Continuation

If Jacobian matrix $D_1 f(0,0)$ is nonsingular, then, for small values of μ, there is a rest point $p(\mu)$ satisfying

$$p(0) = 0, \quad f(p(\mu),\mu) = 0.$$

There's more!

If f is continuously differentiable (C^1), then the Jacobian matrix $D_1 f(p(\mu),\mu)$ varies continuously with μ, as do the eigenvalues and eigenvectors.

If the rest point at $\mu = 0$ is hyperbolic (a saddle, or a stable node, or an unstable node, or a stable spiral, or an unstable spiral), then the rest point $p(\mu)$ inherits the property for small values of μ.

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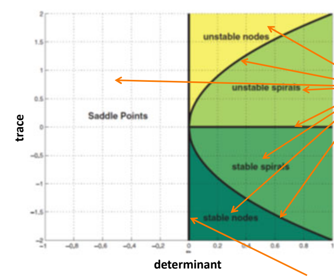
Bifurcation Theory

Classification



Poincare continuation

Poincare continuation fails when determinant = 0.

Kaper & Engler, 2013

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Bifurcation Theory

Example

$$\dot{x} = f(x,\mu) = \mu - 2x - x^2$$

$D_1 f(x,\mu) = -2 - 2x$, so $D_1 f(0,0) = -2 \neq 0$,

so there is a rest point $x = p(\mu)$ satisfying $p(0) = 0$.

$$x = p(\mu) = \frac{\mu}{2} + \mathrm{O}^2(\mu).$$

For each value of μ close to 0, there is a unique rest point near $x = 0$.

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Bifurcation Theory

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For each value of μ close to 0, there is a unique rest point near $x = 0$.

In this example, we can solve explicitly:

$$x^2 + 2x - \mu = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 4\mu}}{2} = -1 \pm \sqrt{1+\mu}.$$

Since $p(0) = 0$, we take the "+" sign:

$$x = p(\mu) = -1 + \sqrt{1+\mu}.$$

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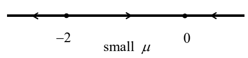
Bifurcation Theory

Example

$$\dot{x} = f(x,\mu) = \mu - 2x - x^2$$

$D_1 f(x,\mu) = -2 - 2x$, so $D_1 f(0,0) = -2$,
so the rest point $x = p(\mu)$ has an eigenvalue near -2 for small μ
and hence is asymptotically stable.

Note that there is another rest point at $x = -2$ for $\mu = 0$.
Its eigenvalue is $D_1 f(-2,0) = -2 - 2(-2) = 2$, so it is unstable.
Futhermore, for small values of μ, there is a unique rest point $\tilde{p}(\mu)$
near $x = -2$, and that rest point is unstable.



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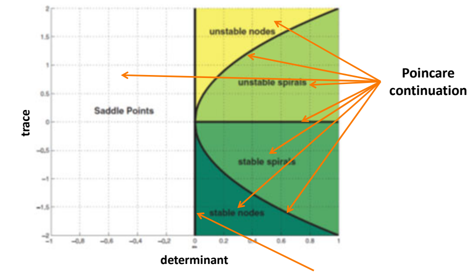

Poincare continuation fails when determinant = 0.

What happens when it fails?

Kaper & Engler, 2013

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Bifurcation Theory

Example

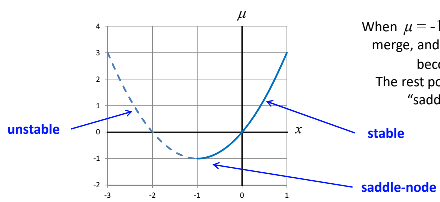
$$\dot{x} = f(x,\mu) = \mu - 2x - x^2$$

Two rest points for $\mu > -1$: $\mu - 2x - x^2 = 0$, $x = -1 \pm \sqrt{1+\mu}$

$D_1 f(x,\mu) = -2 - 2x$

rest point: $p_1 = -1 - \sqrt{1+\mu}$ eigenvalue: $2\sqrt{1+\mu}$

rest point: $p_2 = -1 + \sqrt{1+\mu}$ eigenvalue: $-2\sqrt{1+\mu}$



When $\mu = -1$, the rest points merge, and the eigenvalue becomes 0.
The rest point becomes a "saddle-node".

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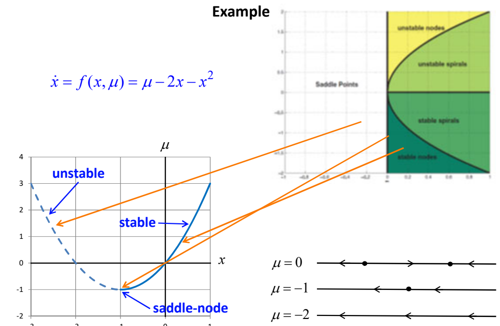
Example

$$\dot{x} = f(x,\mu) = \mu - 2x - x^2$$



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Example

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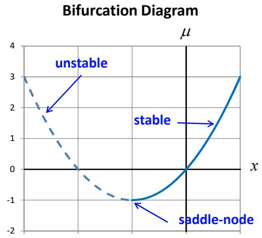
Bifurcation Diagram



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Dynamical Systems

Discussion

Sketch the bifurcation diagram for

$$\frac{dx}{dt} = f(x,\mu) = x^2 - \mu$$

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Dynamical Systems

Discussion

Sketch the bifurcation diagram for

$$\frac{dx}{dt} = f(x,\mu) = x^2 - \mu$$

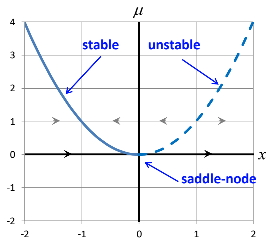
$$D_1 f(x,\mu) = 2x$$

restpoint: $x = \sqrt{\mu}$

$D_1 f(x,\mu) = 2\sqrt{\mu} > 0$ unstable

restpoint: $x = -\sqrt{\mu}$

$D_1 f(x,\mu) = -2\sqrt{\mu} > 0$ stable



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Bifurcation Theory

Two Variable Example

vector field $\dot{x} = -\mu + x^2$, $\dot{y} = -2y$

Rest Points:

$\mu > 0$: two rest points: $(x,y) = (\pm\sqrt{\mu}, 0)$

$\mu = 0$: one rest point: $(x,y) = (0,0)$

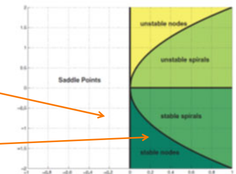
$\mu < 0$: no rest point

Jacobian $D_1 f((x,y),\mu) = \begin{bmatrix} 2x & 0 \\ 0 & -2 \end{bmatrix}$

$\mu > 0$

$D_1 f((+\sqrt{\mu},0),\mu) = \begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -2 \end{bmatrix}$ saddle

$D_1 f((-\sqrt{\mu},0),\mu) = \begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -2 \end{bmatrix}$ stable node



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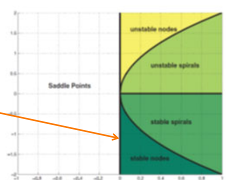
Jacobian $D_1 f((x,y),\mu) = \begin{bmatrix} 2x & 0 \\ 0 & -2 \end{bmatrix}$

$\mu = 0$

$D_1 f((0,0),\mu) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$ determinant = 0, trace < 0

The local structure is not determined by the linearized equations.



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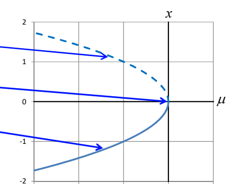

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Two Variable Example

$\dot{x} = -\mu + x^2$

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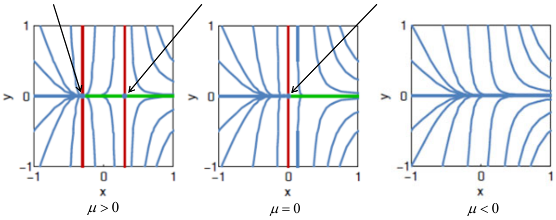

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Dynamical Systems

Discussion

Sketch the bifurcation diagram for

$$\frac{dx}{dt} = \mu - x^2$$

$$\frac{dy}{dt} = -y$$

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Dynamical Systems

Discussion

Sketch the bifurcation diagram for

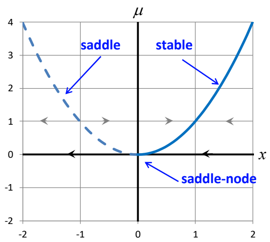
$$\frac{dx}{dt} = \mu - x^2 \qquad D_1 f(x,\mu) = \begin{bmatrix} -2x & 0 \\ 0 & -1 \end{bmatrix}$$
$$\frac{dy}{dt} = -y$$

restpoint: $x = \left(\sqrt{\mu}, 0\right)$

$$D_1 f(x,\mu) = \begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix} \quad \text{stable}$$

restpoint: $x = \left(-\sqrt{\mu}, 0\right)$

$$D_1 f(x,\mu) = \begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix} \quad \text{saddle}$$



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Bifurcation Theory

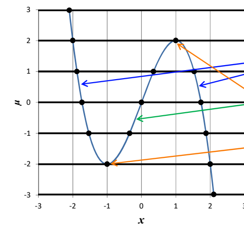
Example

$$\dot{x} = 3x - x^3 - \mu$$

Rest Points

$3x - x^3 - \mu = 0, \qquad \mu = 3x - x^3$

$D_1 f(x,\mu) = 3 - 3x^2$



stable: $D_1 f(x,\mu) < 0, \quad \text{if } |x| > 1$

unstable: $D_1 f(x,\mu) > 0, \quad \text{if } |x| < 1$

saddle-node $D_1 f(x,\mu) = 0, \quad \text{if } x = \pm 1$

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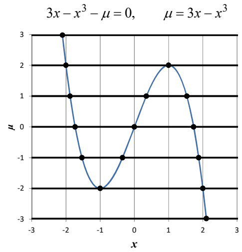

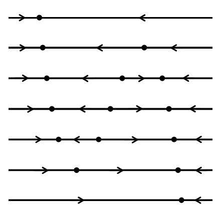

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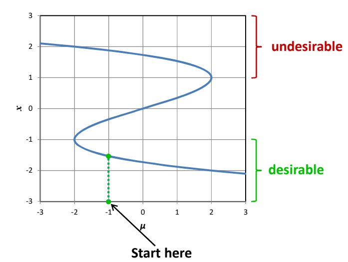

Hysteresis

The system has a memory of where it has been.

Returning parameters to the previous state might not return the system to the previous state

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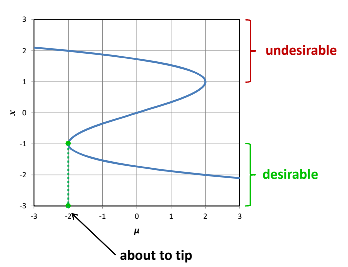
Example

$$\dot{x} = 3x - x^3 - \mu$$



Hysteresis

Decrease the parameter to -2 (the tipping point).

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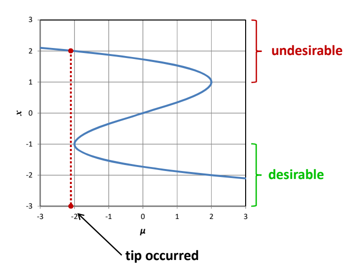
Example

$$\dot{x} = 3x - x^3 - \mu$$



Hysteresis

Decrease to below the tipping point. The system flips to the other stable state.

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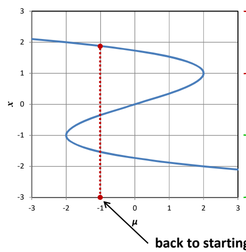
Example

$$\dot{x} = 3x - x^3 - \mu$$



Hysteresis

We now increase the parameter back to its starting value, but the system stays in the new state.

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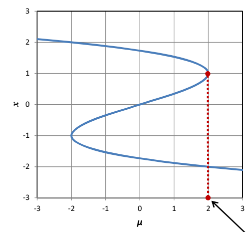
Example

$$\dot{x} = 3x - x^3 - \mu$$



Hysteresis

We must increase the parameter back to the other tipping point ($\mu = 2$) before we can return to the previous state.

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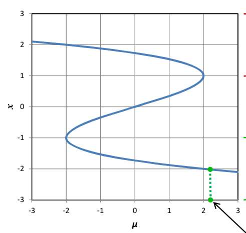
Example

$$\dot{x} = 3x - x^3 - \mu$$



Hysteresis

We must increase the parameter back to the other tipping point ($\mu = 2$) (*and beyond*) before we can return to the previous state.

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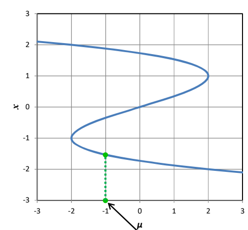
Example

$$\dot{x} = 3x - x^3 - \mu$$



Hysteresis

Now we can decrease the parameter back to its original value, returning the system to its original state.

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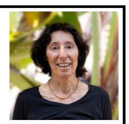
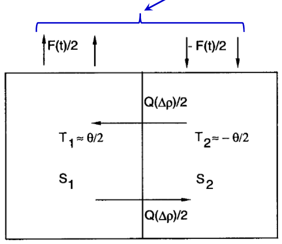


Remember Cessi's Model?

$$\frac{dy}{dt} = f(y,p) = -\left(1 + \mu^2 (y-1)^2\right) y + p$$



Recall that μ^2 is related to the diffusive and advective time scales, and we hold it at a constant 6.2.
We treat the influx p of fresh water as a parameter.

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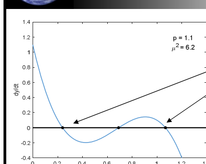
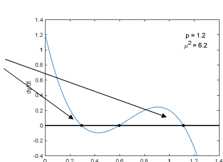
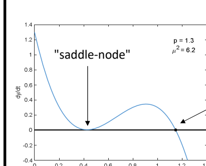
Cessi's Model



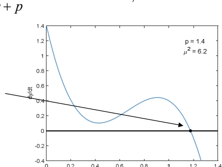


two stable rest points, with an unstable rest point in between

$$\frac{dy}{dt} = -\left(1 + \mu^2 (y-1)^2\right) y + p$$



one stable rest point



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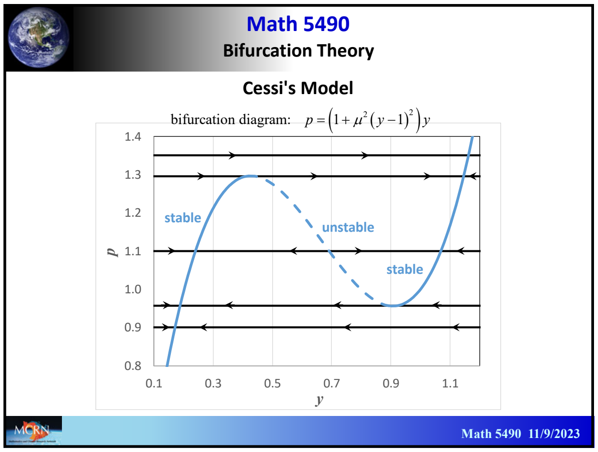
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Bifurcation Theory
Cessi's Model
bifurcation diagram: $p = \left(1 + \mu^2 (y-1)^2\right) y$
stable
unstable
stable
p
y
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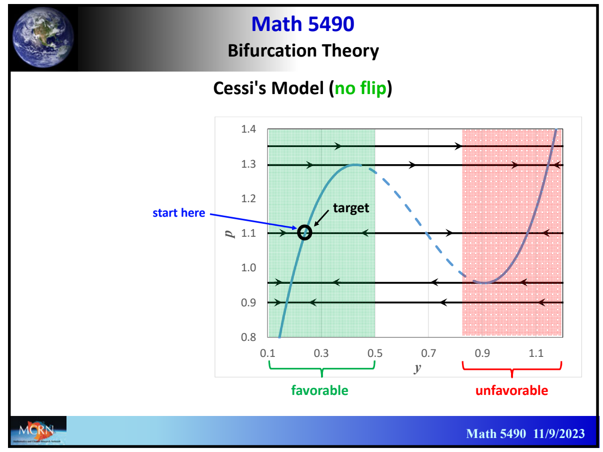
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Bifurcation Theory
Cessi's Model (no flip)
start here
target
favorable
unfavorable
p
y
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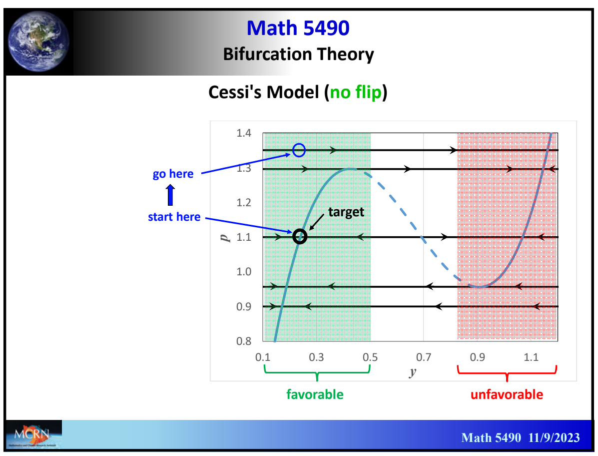
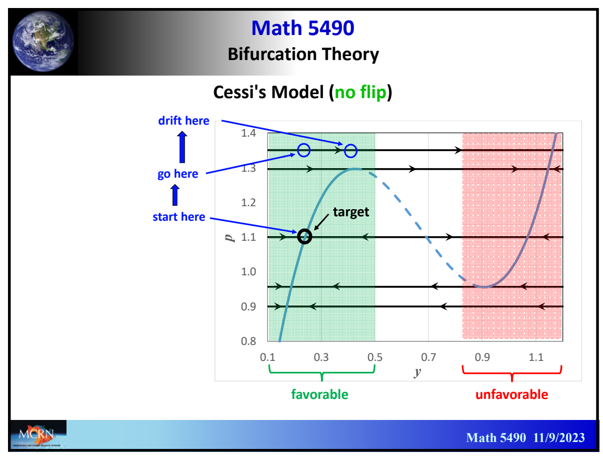
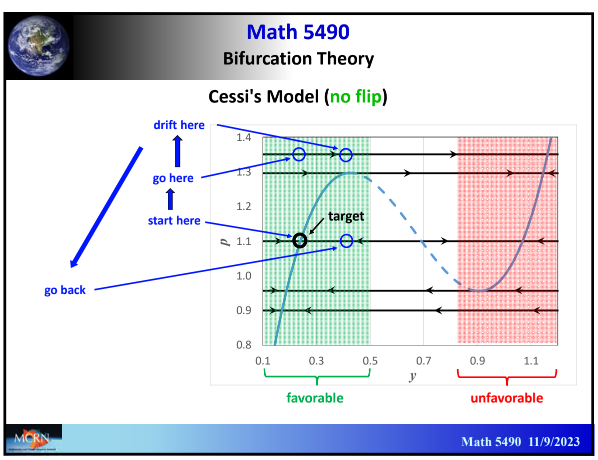
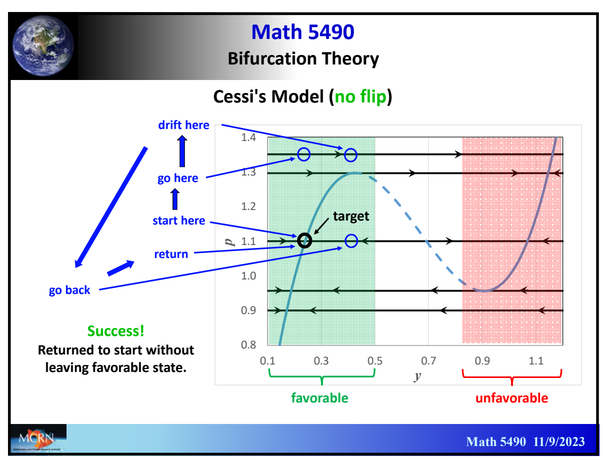
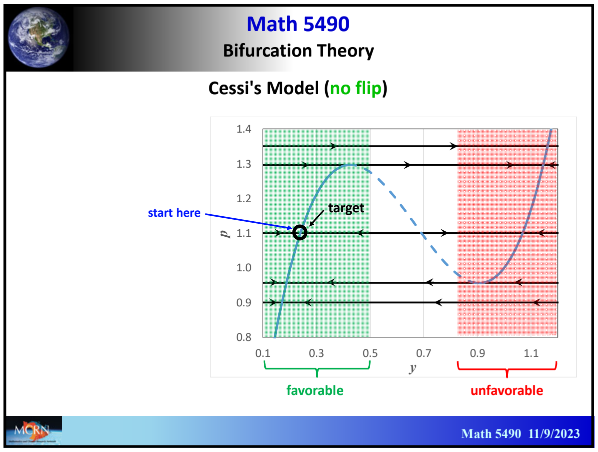
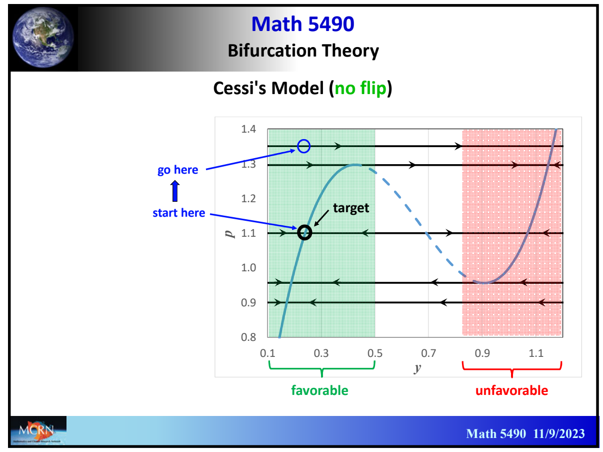

Math 5490
Bifurcation Theory
Cessi's Model (no flip)
go here
start here
target
favorable
unfavorable
p
y
Math 5490 11/9/2023

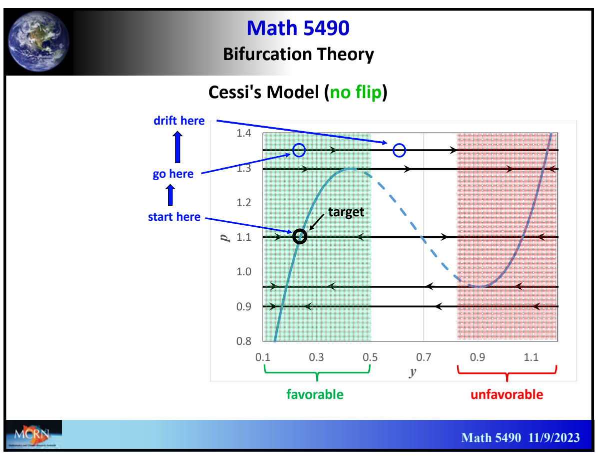

Math 5490
Bifurcation Theory
Cessi's Model (no flip)
drift here
go here
start here
target
favorable
unfavorable
p
y
Math 5490 11/9/2023

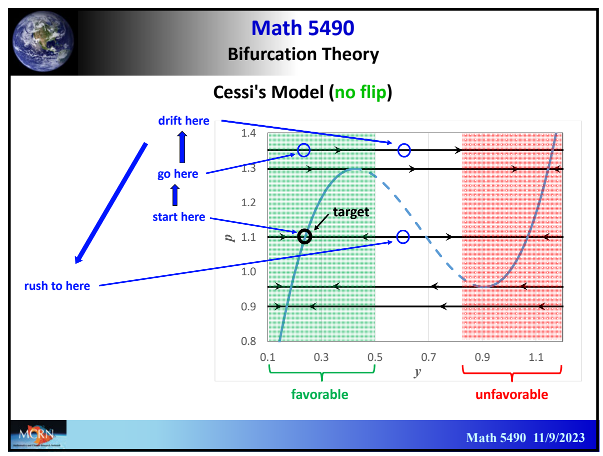

Math 5490
Bifurcation Theory
Cessi's Model (no flip)
drift here
go here
start here
go back
target
favorable
unfavorable
p
y
Math 5490 11/9/2023

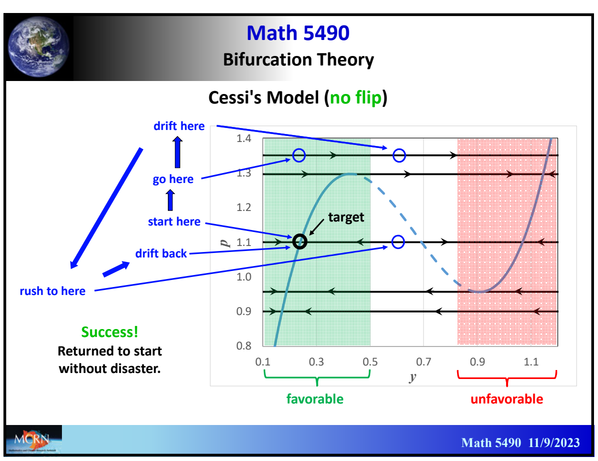

Math 5490
Bifurcation Theory
Cessi's Model (no flip)
drift here
go here
start here
return
go back
target
Success!
Returned to start without leaving favorable state.
favorable
unfavorable
p
y
Math 5490 11/9/2023

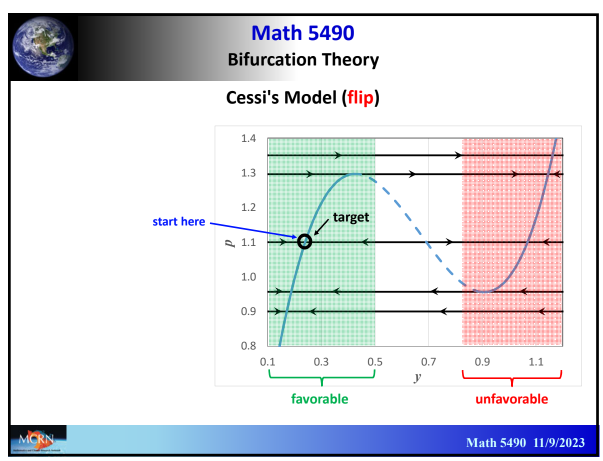

Math 5490
Bifurcation Theory
Cessi's Model (no flip)
start here
target
favorable
unfavorable
Math 5490 11/9/2023

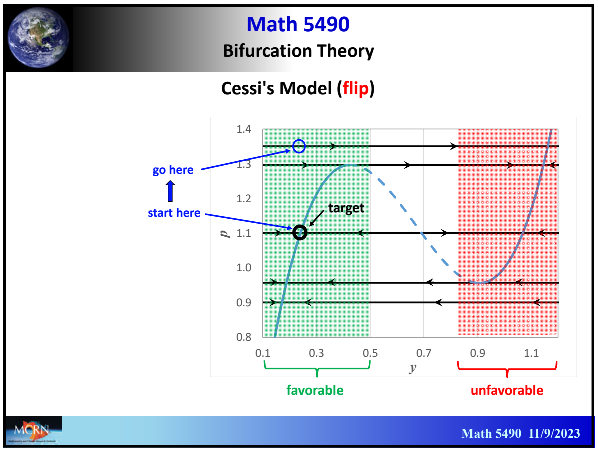

Math 5490
Bifurcation Theory
Cessi's Model (no flip)
go here
start here
target
favorable
unfavorable
Math 5490 11/9/2023

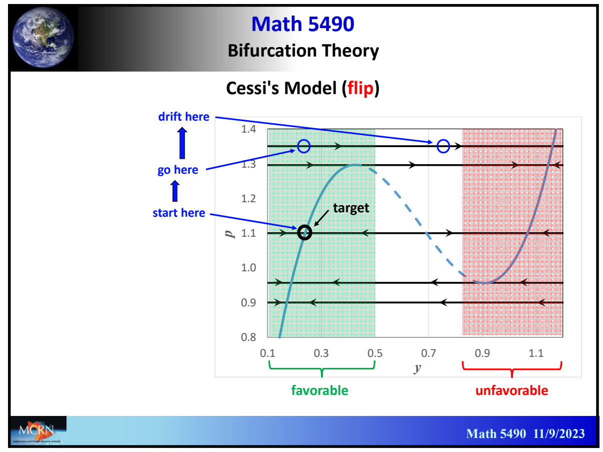

Math 5490
Bifurcation Theory
Cessi's Model (no flip)
drift here
go here
start here
target
favorable
unfavorable
Math 5490 11/9/2023

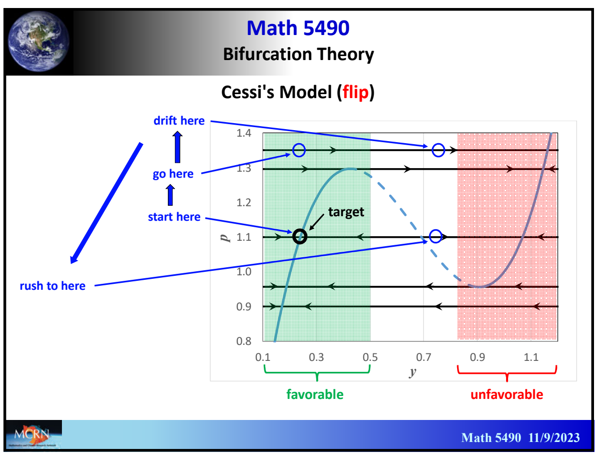

Math 5490
Bifurcation Theory
Cessi's Model (no flip)
drift here
go here
start here
target
rush to here
favorable
unfavorable
Math 5490 11/9/2023

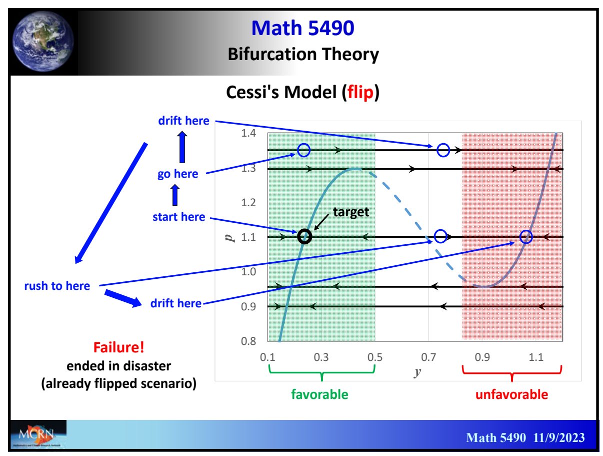

Math 5490
Bifurcation Theory
Cessi's Model (no flip)
drift here
go here
start here
target
drift back
rush to here
Success!
Returned to start
without disaster.
favorable
unfavorable
Math 5490 11/9/2023

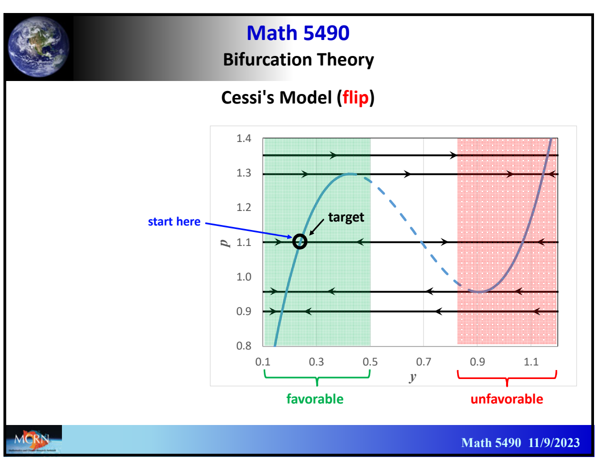

Math 5490
Bifurcation Theory
Cessi's Model (flip)
start here
target
favorable
unfavorable
Math 5490 11/9/2023

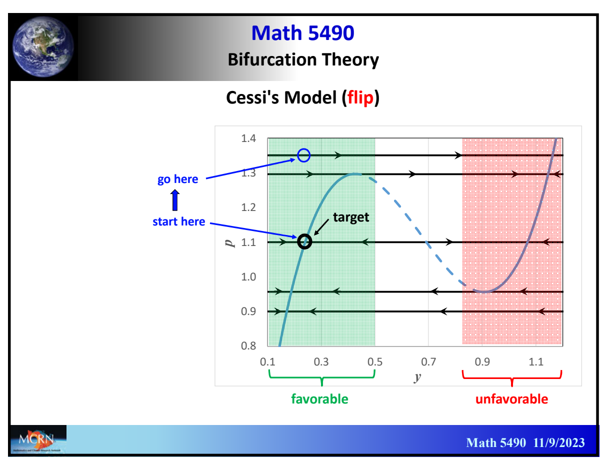

Math 5490
Bifurcation Theory
Cessi's Model (flip)
go here
start here
target
favorable
unfavorable
Math 5490 11/9/2023

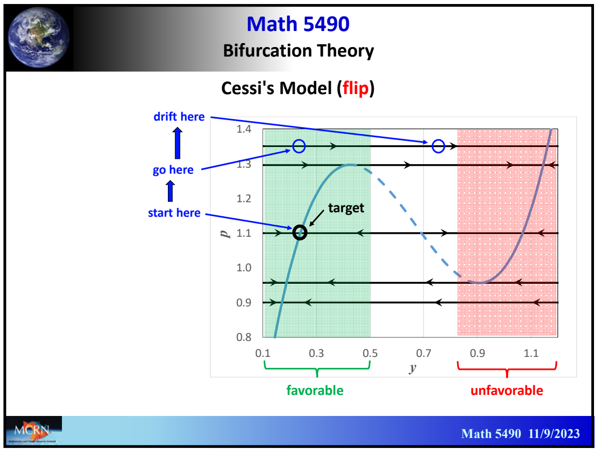

Math 5490
Bifurcation Theory
Cessi's Model (flip)
drift here
go here
start here
target
favorable
unfavorable
Math 5490 11/9/2023

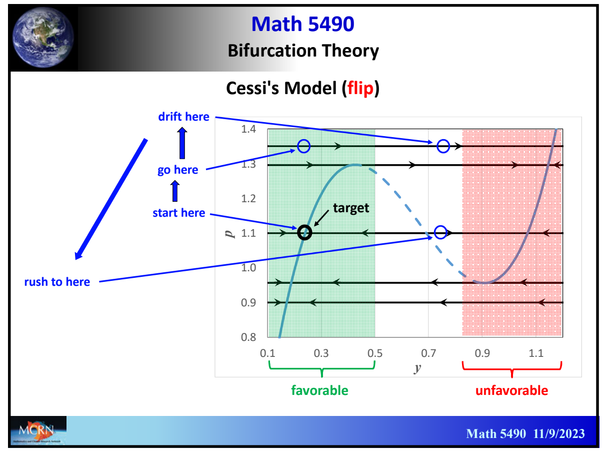

Math 5490
Bifurcation Theory
Cessi's Model (flip)
drift here
go here
start here
target
rush to here
favorable
unfavorable
Math 5490 11/9/2023

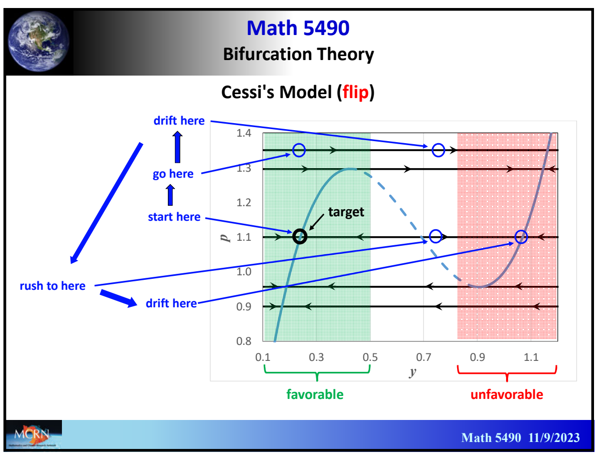

Math 5490
Bifurcation Theory
Cessi's Model (flip)
drift here
go here
start here
target
rush to here
drift here
Failure!
ended in disaster
(already flipped scenario)
favorable
unfavorable
Math 5490 11/9/2023

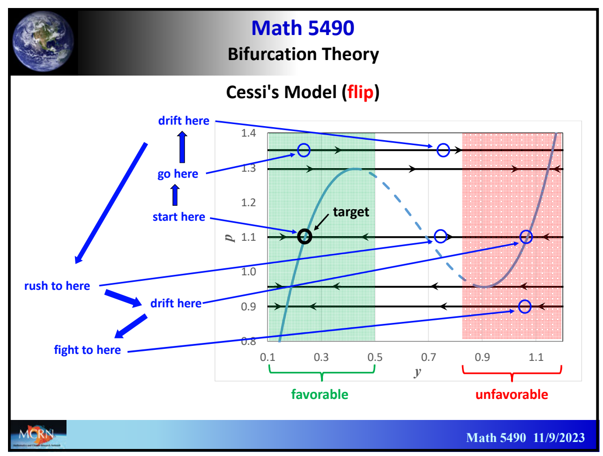

Math 5490
Bifurcation Theory
Cessi's Model (flip)
start here
target
favorable
unfavorable
Math 5490 11/9/2023

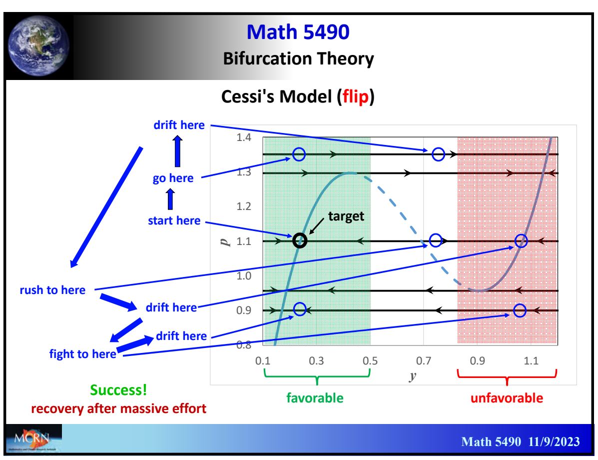

Math 5490
Bifurcation Theory
Cessi's Model (flip)
go here
start here
target
favorable
unfavorable
Math 5490 11/9/2023

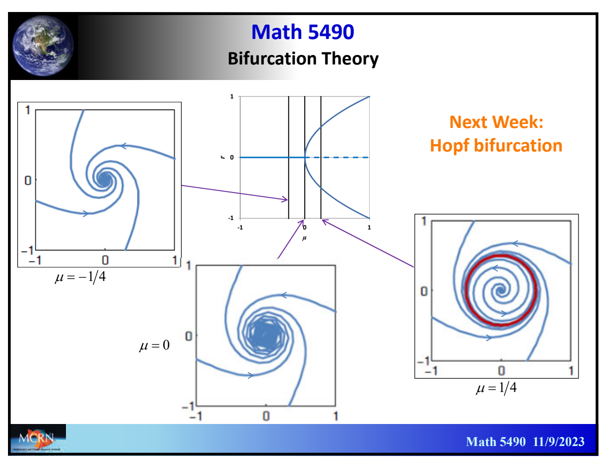

Math 5490
Bifurcation Theory
Cessi's Model (flip)
drift here
go here
start here
target
p
y
favorable
unfavorable
Math 5490 11/9/2023


Math 5490
Bifurcation Theory
Cessi's Model (flip)
drift here
go here
start here
target
rush to here
p
y
favorable
unfavorable
Math 5490 11/9/2023


Math 5490
Bifurcation Theory
Cessi's Model (flip)
drift here
go here
start here
target
rush to here
drift here
p
y
favorable
unfavorable
Math 5490 11/9/2023


Math 5490
Bifurcation Theory
Cessi's Model (flip)
drift here
go here
start here
target
rush to here
drift here
fight to here
p
y
favorable
unfavorable
Math 5490 11/9/2023


Math 5490
Bifurcation Theory
Cessi's Model (flip)
drift here
go here
start here
target
rush to here
drift here
drift here
fight to here
Success!
recovery after massive effort
p
y
favorable
unfavorable
Math 5490 11/9/2023


Math 5490
Bifurcation Theory
Next Week:
Hopf bifurcation
$\mu = -1/4$
$\mu = 0$
$\mu = 1/4$
Math 5490 11/9/2023