Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

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course website
www-users.cse.umn.edu/~mcgehee/teaching/Math5490/

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Dynamical Systems

Can We Predict the Future?

If we know the state of a system now, do we know its state in the future?

For models based on differential equations, the answer is 'yes".

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad x = x_0 \text{ when } t = 0$$

If f is sufficiently smooth (e.g., continuously differentiable) then there is a unique solution of the differential equations satisfying the initial condition.

Interpretation:

If we know the state of the system now, we can compute its state in the future.

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Example of a late 19th century mathematical question:

Is the solar system stable?

Example of a late 20th century mathematical question:

Can we predict the weather?

What happened?

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Example of a late 19th century mathematical question:

Is the solar system stable?

Example of a late 20th century mathematical question:

Can we predict the weather?

What happened?

Quantum Mechanics
Elementary particles are probability distributions.
Heisenberg Uncertainty

Gödel's Incompleteness Theorem
There are mathematical statements which can be neither proved nor disproved.

Chaos Theory
Deterministic systems can exhibit random behavior.

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Coin Flips

How do we model a sequence of coin flips?

Flip a fair coin N times.
Record H for heads and T for tails to obtain a sequences of Hs and Ts.

$$H \; T \; H \; H \; H \; T \; T \; T \; H \; T$$

To reveal the sequence of flip results, just march down the line.

```
        H  T  H  H  H  T  T  T  H  T
flip 1  ↑
        H  T  H  H  H  T  T  T  H  T
flip 2     ↑
        H  T  H  H  H  T  T  T  H  T
flip 3        ↑
```

etc.

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Coin Flips

Alternatively:

Keep the pointer fixed, and shift the sequence to the left.

```
           H  T  H  H  H  T  T  T  H  T
flip 1     ↑
        H  T  H  H  H  T  T  T  H  T
flip 2     ↑
     H  T  H  H  H  T  T  T  H  T
flip 3     ↑
  H  T  H  H  H  T  T  T  H  T
flip 4     ↑
```

etc.

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Coin Flips

Replace H with 1 and T with 0. Throw away stuff to the left of the arrow.

flip 1	1 0 1 1 1 0 0 0 1 0 ↑
flip 2	0 1 1 1 0 0 0 1 0 ↑
flip 3	1 1 1 0 0 0 1 0 ↑
flip 4	1 1 0 0 0 1 0 ↑

etc.

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Coin Flips

	Replace H with 1 and T with 0. Throw away stuff to the left of the arrow.	Replace the arrow with a point.
flip 1	1 0 1 1 1 0 0 0 1 0 ↑	.1011100010
flip 2	0 1 1 1 0 0 0 1 0 ↑	.011100010
flip 3	1 1 1 0 0 0 1 0 ↑	.11100010
flip 4	1 1 0 0 0 1 0 ↑	.1100010

etc.

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Coin Flips

	Replace H with 1 and T with 0. Throw away stuff to the left of the arrow.	Replace the arrow with a point.	Interpret as a binary number.
flip 1	1 0 1 1 1 0 0 0 1 0 ↑	.1011100010	$\frac{1}{2}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{512}$ $=0.72073125$
flip 2	0 1 1 1 0 0 0 1 0 ↑	.011100010	$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{256}$ $=0.44140625$
flip 3	1 1 1 0 0 0 1 0 ↑	.11100010	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{128}$ $=0.8828125$
flip 4	1 1 0 0 0 1 0 ↑	.1100010	$\frac{1}{2}+\frac{1}{4}+\frac{1}{64}$ $=0.765625$

etc.

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Coin Flips

Interpret as a map.

$.1011100010 \;=\; 0.72073125 \;=\; x$

Shift the sequence to the right, leaving the binary point fixed.
Equivalent to multiplication by 2.

$1.011100010 \;=\; 1.44140625 \;=\; 2x$

Throw away the digit to the left of the binary point.
Equivalent to mod 1.

$.011100010 \;=\; .44140625 \;=\; 2x \bmod 1$

Altogether:

$$x \mapsto \varphi(x) = 2x \bmod 1$$

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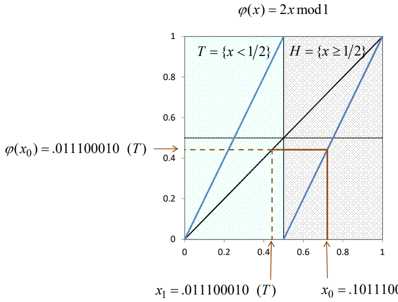
Coin Flips

$\varphi(x) = 2x \bmod 1$



$x_0 = .1011100010\ (H)$
$x_1 = .011100010\ (T)$

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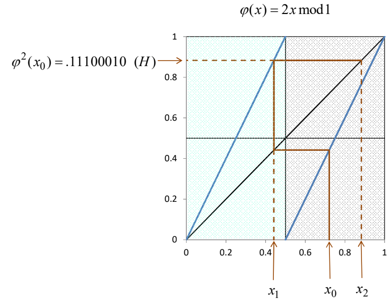
Coin Flips

$\varphi(x) = 2x \bmod 1$



$x_0 = .1011100010\ (H)$
$x_1 = .011100010\ (T)$
$x_2 = .11100010\ (H)$

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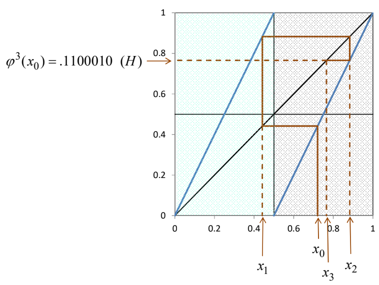
Coin Flips
$\varphi(x) = 2x \bmod 1$



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$x_2 = .11100010\ (H)$
$x_3 = .1100010\ (H)$

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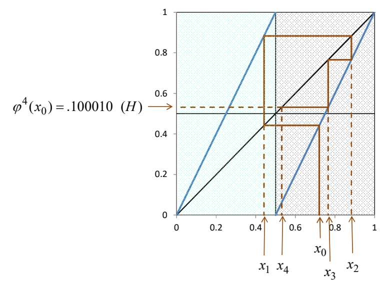
Coin Flips
$\varphi(x) = 2x \bmod 1$



$x_0 = .1011100010\ (H)$
$x_1 = .011100010\ (T)$
$x_2 = .11100010\ (H)$
$x_3 = .1100010\ (H)$
$x_4 = .100010\ (H)$

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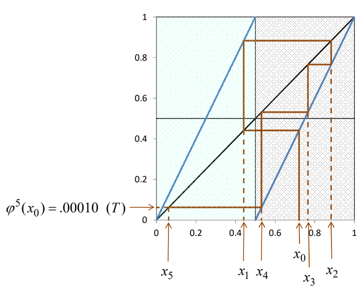
Coin Flips
$\varphi(x) = 2x \bmod 1$



$x_0 = .1011100010\ (H)$
$x_1 = .011100010\ (T)$
$x_2 = .11100010\ (H)$
$x_3 = .1100010\ (H)$
$x_4 = .100010\ (H)$
$x_5 = .00010\ (T)$

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Coin Flips
No reason to stop.

Flip a fair coin forever.
Record H for heads and T for tails to obtain a sequences of Hs and Ts.

$H\ T\ H\ H\ H\ T\ T\ T\ H\ T \ldots$

Switch to 0s and 1s

$1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0 \ldots$

to binary

$.1011100010$

to decimal

$\frac{1}{2}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{512}+\cdots = 0.72073125\ldots$

same map:

$x \mapsto \varphi(x) = 2x \bmod 1$

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Coin Flips

Interpretation

For every infinite sequence of coin flips there is a binary number between 0 and 1. Numbers greater than ½ correspond to heads, while numbers less than ½ correspond to tails. Observing the events of sequential flips corresponds to shifting the binary point and discarding the integer part, which corresponds to iterating the map

$x \mapsto \varphi(x) = 2x \bmod 1$

and observing whether the iterate is to the left or to the right of ½.

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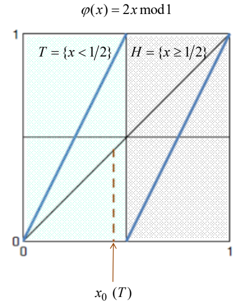
Coin Flips
$\varphi(x) = 2x \bmod 1$



$x_0 = 0.440301\ldots\ (T)$

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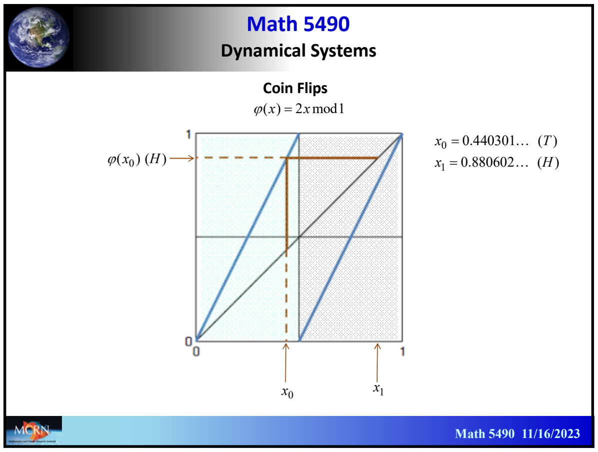
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Coin Flips
$\varphi(x) = 2x \bmod 1$
$\varphi(x_0)$ (H)
$x_0 = 0.440301\ldots$ (T)
$x_1 = 0.880602\ldots$ (H)
x_0
x_1
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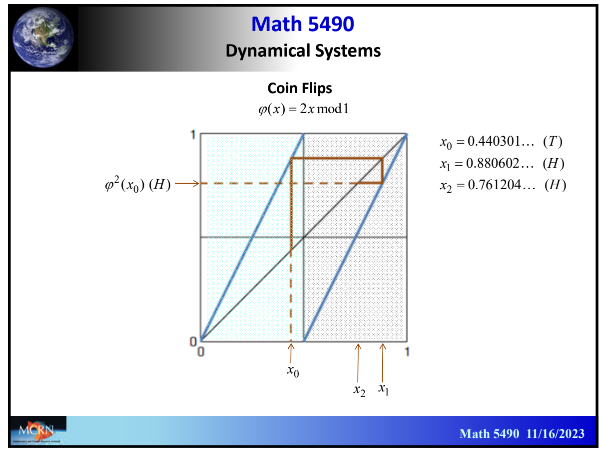
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Dynamical Systems
Coin Flips
$\varphi(x) = 2x \bmod 1$
$\varphi^2(x_0)$ (H)
$x_0 = 0.440301\ldots$ (T)
$x_1 = 0.880602\ldots$ (H)
$x_2 = 0.761204\ldots$ (H)
x_0
x_2 x_1
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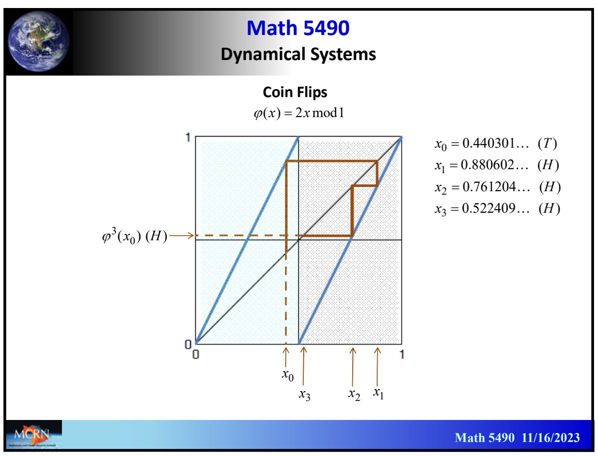
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Dynamical Systems
Coin Flips
$\varphi(x) = 2x \bmod 1$
$\varphi^3(x_0)$ (H)
$x_0 = 0.440301\ldots$ (T)
$x_1 = 0.880602\ldots$ (H)
$x_2 = 0.761204\ldots$ (H)
$x_3 = 0.522409\ldots$ (H)
x_0 x_3 x_2 x_1
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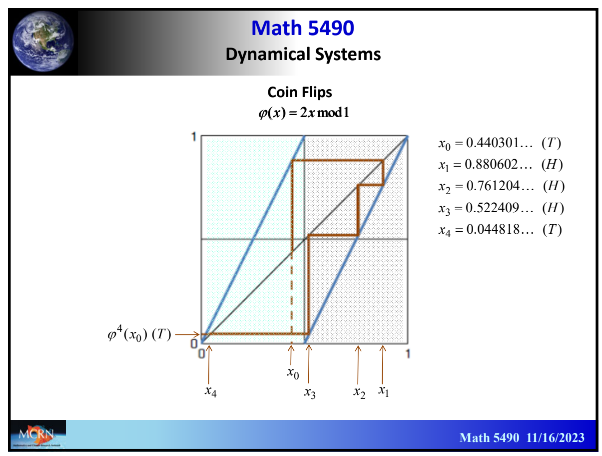
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Dynamical Systems
Coin Flips
$\varphi(x) = 2x \bmod 1$
$\varphi^4(x_0)$ (T)
$x_0 = 0.440301\ldots$ (T)
$x_1 = 0.880602\ldots$ (H)
$x_2 = 0.761204\ldots$ (H)
$x_3 = 0.522409\ldots$ (H)
$x_4 = 0.044818\ldots$ (T)
x_4 x_0 x_3 x_2 x_1
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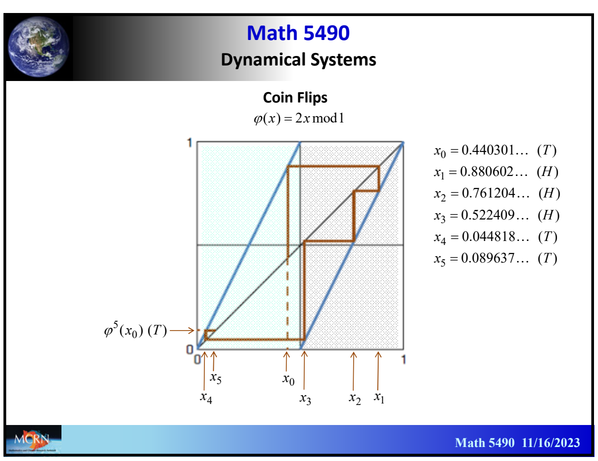
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Coin Flips
$\varphi(x) = 2x \bmod 1$
$\varphi^5(x_0)$ (T)
$x_0 = 0.440301\ldots$ (T)
$x_1 = 0.880602\ldots$ (H)
$x_2 = 0.761204\ldots$ (H)
$x_3 = 0.522409\ldots$ (H)
$x_4 = 0.044818\ldots$ (T)
$x_5 = 0.089637\ldots$ (T)
x_5 x_4 x_0 x_3 x_2 x_1
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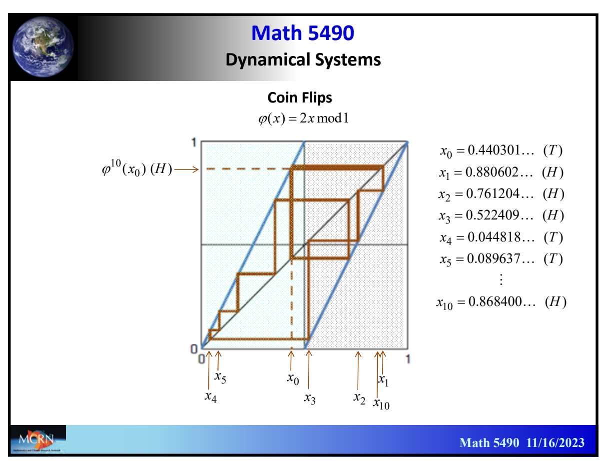
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Coin Flips
$\varphi(x) = 2x \bmod 1$
$\varphi^{10}(x_0)$ (H)
$x_0 = 0.440301\ldots$ (T)
$x_1 = 0.880602\ldots$ (H)
$x_2 = 0.761204\ldots$ (H)
$x_3 = 0.522409\ldots$ (H)
$x_4 = 0.044818\ldots$ (T)
$x_5 = 0.089637\ldots$ (T)
$\vdots$
$x_{10} = 0.868400\ldots$ (H)
x_5 x_4 x_0 x_3 x_2 x_1 x_{10}
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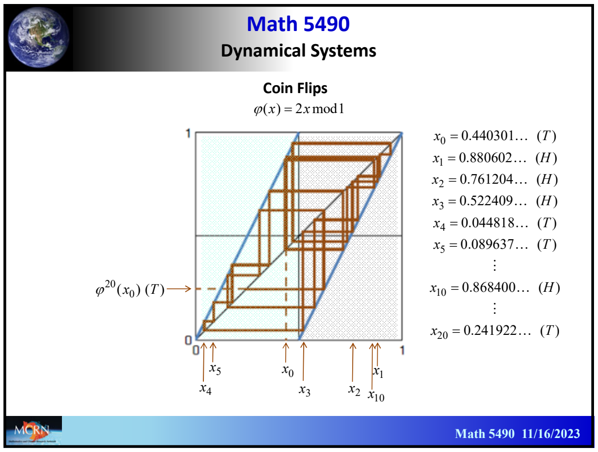
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Dynamical Systems
Coin Flips
$\varphi(x) = 2x \bmod 1$
$x_0 = 0.440301\ldots\ (T)$
$x_1 = 0.880602\ldots\ (H)$
$x_2 = 0.761204\ldots\ (H)$
$x_3 = 0.522409\ldots\ (H)$
$x_4 = 0.044818\ldots\ (T)$
$x_5 = 0.089637\ldots\ (T)$
$x_{10} = 0.868400\ldots\ (H)$
$x_{20} = 0.241922\ldots\ (T)$
$\varphi^{20}(x_0)\ (T)$
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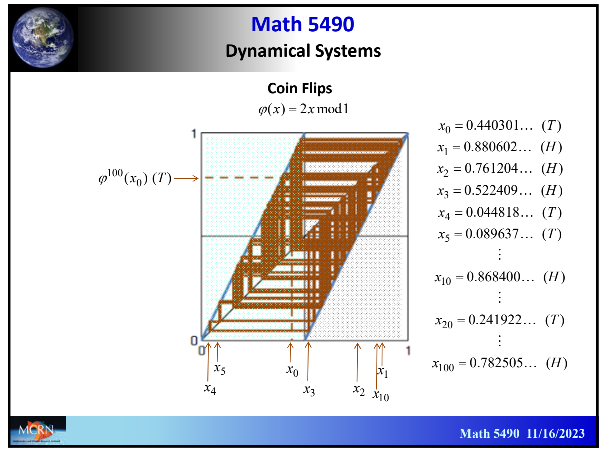
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Coin Flips
$\varphi(x) = 2x \bmod 1$
$x_0 = 0.440301\ldots\ (T)$
$x_1 = 0.880602\ldots\ (H)$
$x_2 = 0.761204\ldots\ (H)$
$x_3 = 0.522409\ldots\ (H)$
$x_4 = 0.044818\ldots\ (T)$
$x_5 = 0.089637\ldots\ (T)$
$x_{10} = 0.868400\ldots\ (H)$
$x_{20} = 0.241922\ldots\ (T)$
$x_{100} = 0.782505\ldots\ (H)$
$\varphi^{100}(x_0)\ (T)$
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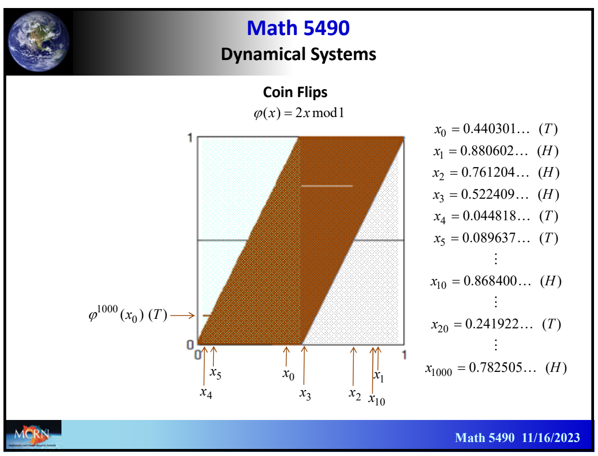
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Coin Flips
$\varphi(x) = 2x \bmod 1$
$x_0 = 0.440301\ldots\ (T)$
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$x_2 = 0.761204\ldots\ (H)$
$x_3 = 0.522409\ldots\ (H)$
$x_4 = 0.044818\ldots\ (T)$
$x_5 = 0.089637\ldots\ (T)$
$x_{10} = 0.868400\ldots\ (H)$
$x_{20} = 0.241922\ldots\ (T)$
$x_{1000} = 0.782505\ldots\ (H)$
$\varphi^{1000}(x_0)\ (T)$
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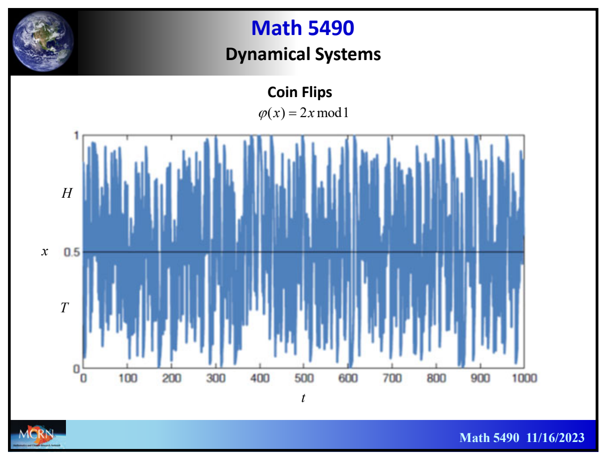
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$\varphi(x) = 2x \bmod 1$
H
T
x
t
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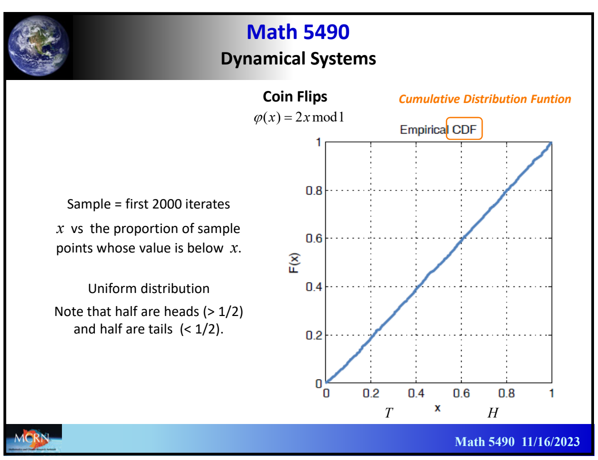
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Coin Flips
$\varphi(x) = 2x \bmod 1$
Cumulative Distribution Funtion
Empirical CDF
Sample = first 2000 iterates
x vs the proportion of sample points whose value is below x.
Uniform distribution
Note that half are heads (> 1/2) and half are tails (< 1/2).
F(x)
x
T
H
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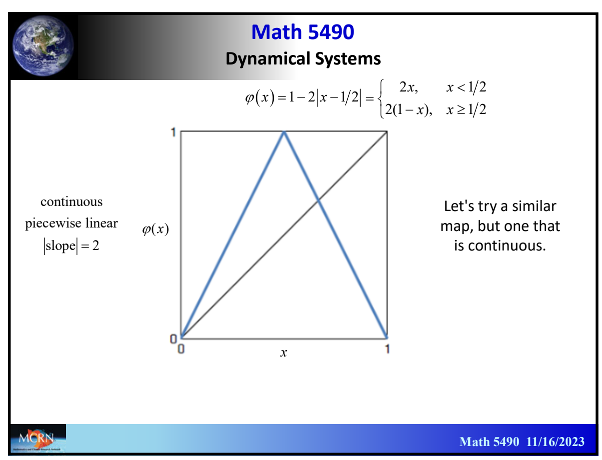
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$\varphi(x) = 1 - 2|x - 1/2| = \begin{cases} 2x, & x < 1/2 \\ 2(1-x), & x \ge 1/2 \end{cases}$
continuous
piecewise linear
$|\text{slope}| = 2$
$\varphi(x)$
x
Let's try a similar map, but one that is continuous.
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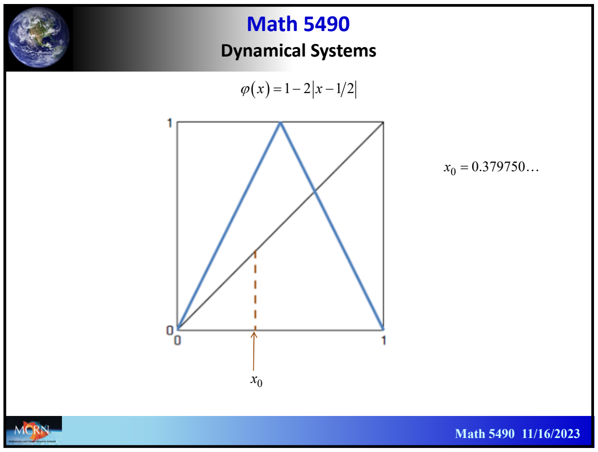
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φ(x) = 1 − 2|x − 1/2|
x0 = 0.379750…
x0
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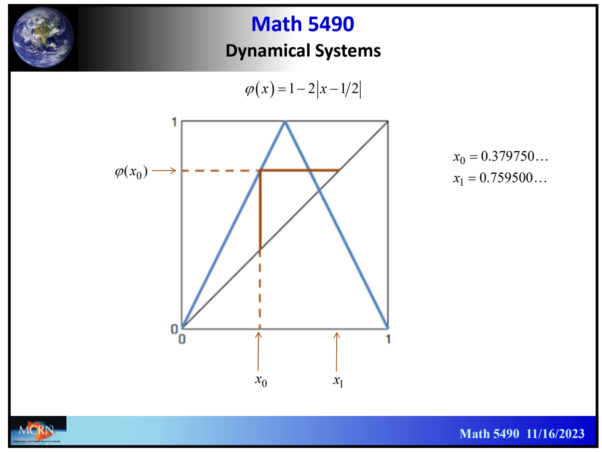
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φ(x) = 1 − 2|x − 1/2|
φ(x0)
x0 = 0.379750…
x1 = 0.759500…
x0
x1
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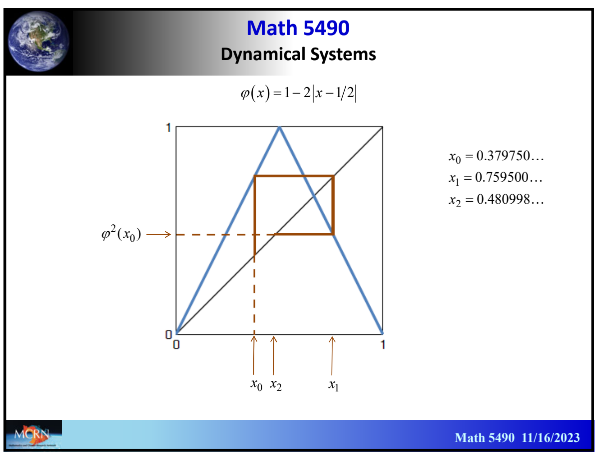
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φ(x) = 1 − 2|x − 1/2|
φ2(x0)
x0 = 0.379750…
x1 = 0.759500…
x2 = 0.480998…
x0 x2 x1
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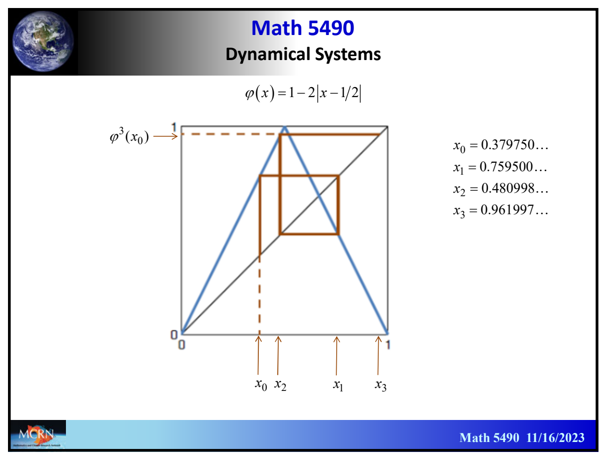
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φ(x) = 1 − 2|x − 1/2|
φ3(x0)
x0 = 0.379750…
x1 = 0.759500…
x2 = 0.480998…
x3 = 0.961997…
x0 x2 x1 x3
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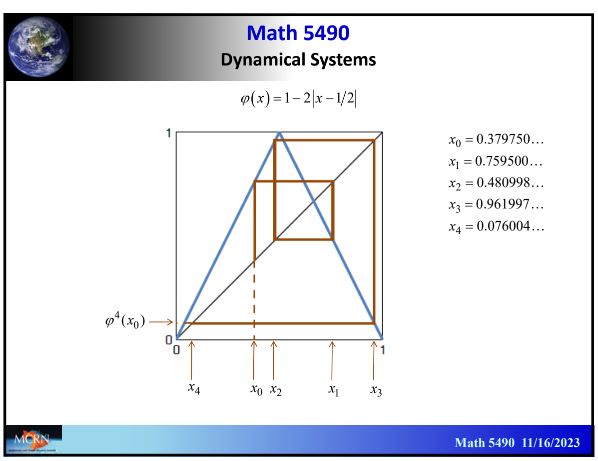
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φ(x) = 1 − 2|x − 1/2|
φ4(x0)
x0 = 0.379750…
x1 = 0.759500…
x2 = 0.480998…
x3 = 0.961997…
x4 = 0.076004…
x4 x0 x2 x1 x3
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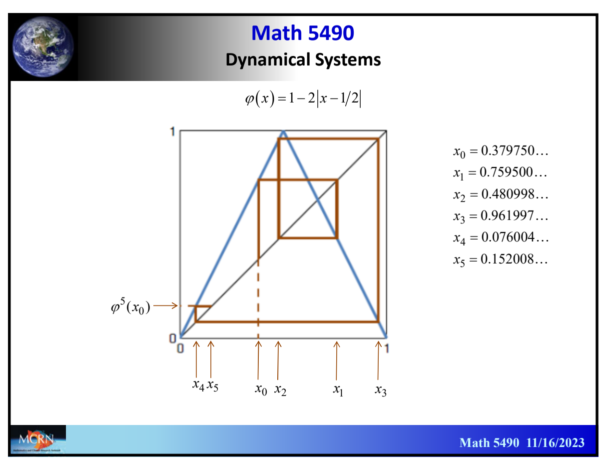
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Dynamical Systems
φ(x) = 1 − 2|x − 1/2|
φ5(x0)
x0 = 0.379750…
x1 = 0.759500…
x2 = 0.480998…
x3 = 0.961997…
x4 = 0.076004…
x5 = 0.152008…
x4 x5 x0 x2 x1 x3
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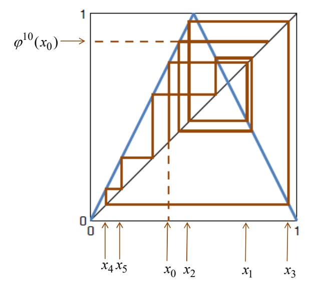
Dynamical Systems

$\varphi(x) = 1 - 2|x - 1/2|$



$x_0 = 0.379750\ldots$
$x_1 = 0.759500\ldots$
$x_2 = 0.480998\ldots$
$x_3 = 0.961997\ldots$
$x_4 = 0.076004\ldots$
$x_5 = 0.152008\ldots$
$\vdots$
$x_{10} = 0.864271\ldots$

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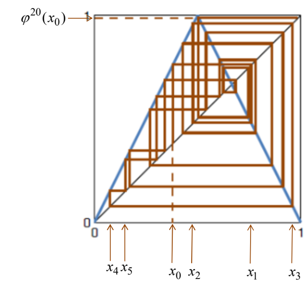
Dynamical Systems

$\varphi(x) = 1 - 2|x - 1/2|$



$x_0 = 0.379750\ldots$
$x_1 = 0.759500\ldots$
$x_2 = 0.480998\ldots$
$x_3 = 0.961997\ldots$
$x_4 = 0.076004\ldots$
$x_5 = 0.152008\ldots$
$\vdots$
$x_{10} = 0.864271\ldots$
$\vdots$
$x_{20} = 0.985695\ldots$

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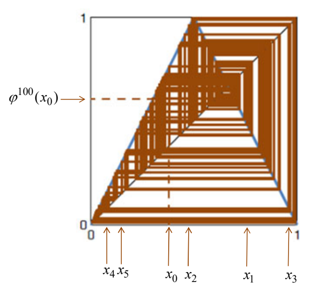
$\varphi(x) = 1 - 2|x - 1/2|$



$x_0 = 0.379750\ldots$
$x_1 = 0.759500\ldots$
$x_2 = 0.480998\ldots$
$x_3 = 0.961997\ldots$
$x_4 = 0.076004\ldots$
$x_5 = 0.152008\ldots$
$\vdots$
$x_{10} = 0.864271\ldots$
$\vdots$
$x_{20} = 0.985695\ldots$
$\vdots$
$x_{100} = 0.602054\ldots$

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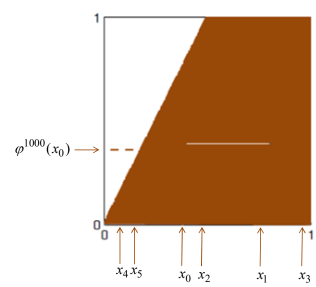
Dynamical Systems

$\varphi(x) = 1 - 2|x - 1/2|$

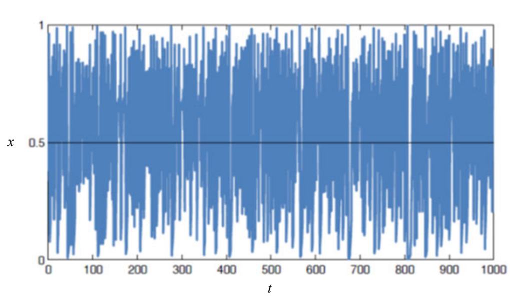


$x_0 = 0.379750\ldots$
$x_1 = 0.759500\ldots$
$x_2 = 0.480998\ldots$
$x_3 = 0.961997\ldots$
$x_4 = 0.076004\ldots$
$x_5 = 0.152008\ldots$
$\vdots$
$x_{10} = 0.864271\ldots$
$\vdots$
$x_{20} = 0.985695\ldots$
$\vdots$
$x_{100} = 0.602054\ldots$

Math 5490 11/16/2023

Math 5490
Dynamical Systems

$\varphi(x) = 1 - 2|x - 1/2|$



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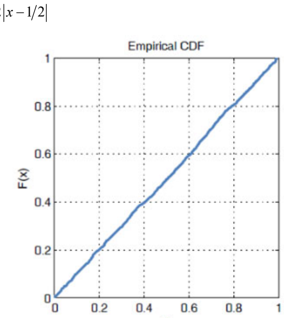
Math 5490
Dynamical Systems

$\varphi(x) = 1 - 2|x - 1/2|$

Sample = first 2000 iterates

x vs the proportion of sample points whose value is below x.

Uniform distribution again.



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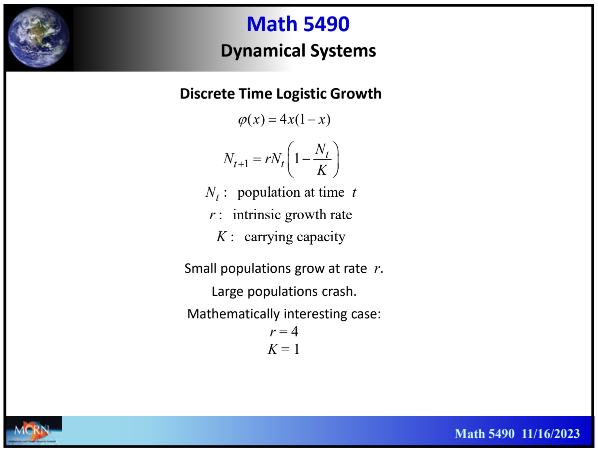

Math 5490
Dynamical Systems
Discrete Time Logistic Growth
$\varphi(x) = 4x(1-x)$
$N_{t+1} = rN_t\left(1 - \frac{N_t}{K}\right)$
N_t : population at time t
r : intrinsic growth rate
K : carrying capacity
Small populations grow at rate r.
Large populations crash.
Mathematically interesting case:
$r = 4$
$K = 1$
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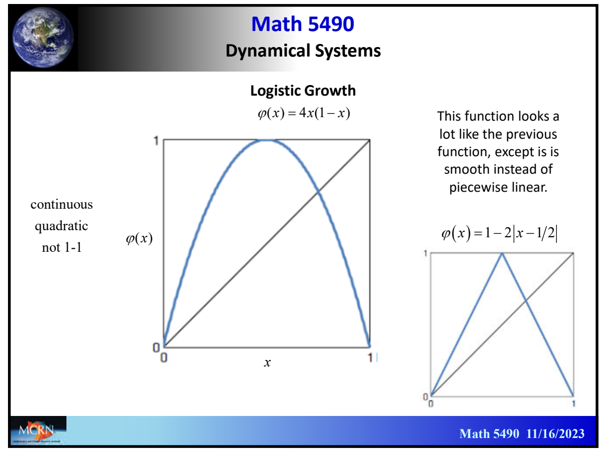

Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
continuous
quadratic
not 1-1
$\varphi(x)$
x
This function looks a lot like the previous function, except is is smooth instead of piecewise linear.
$\varphi(x) = 1 - 2|x - 1/2|$
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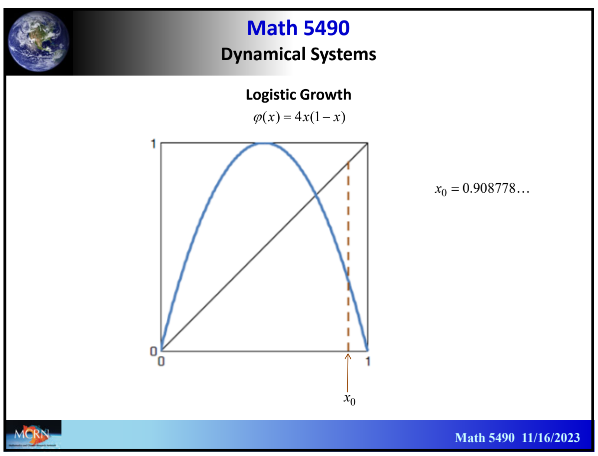
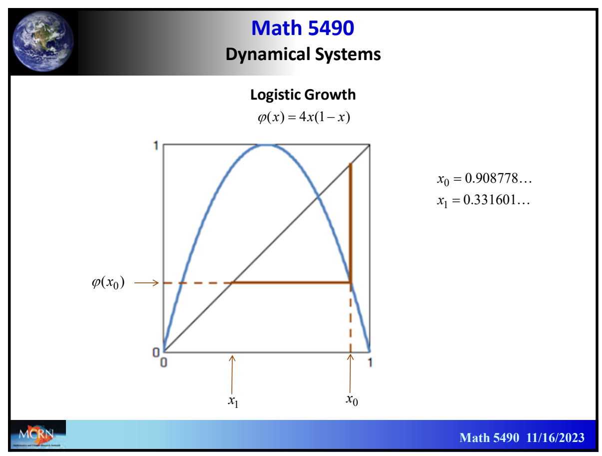
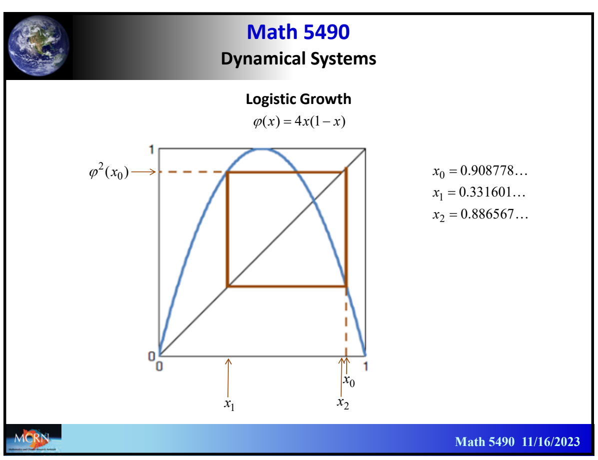


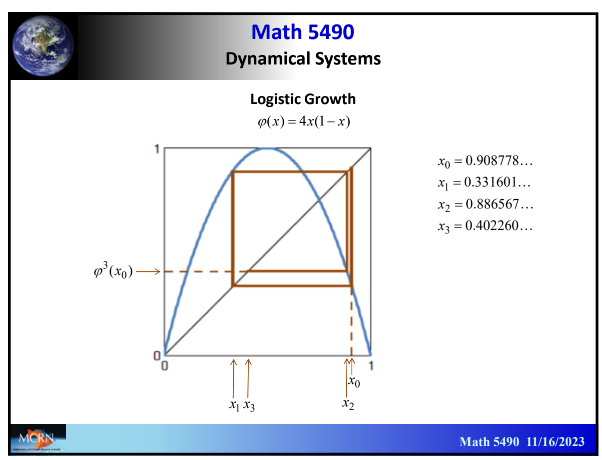
Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
x_0
$x_0 = 0.908778\ldots$
Math 5490 11/16/2023



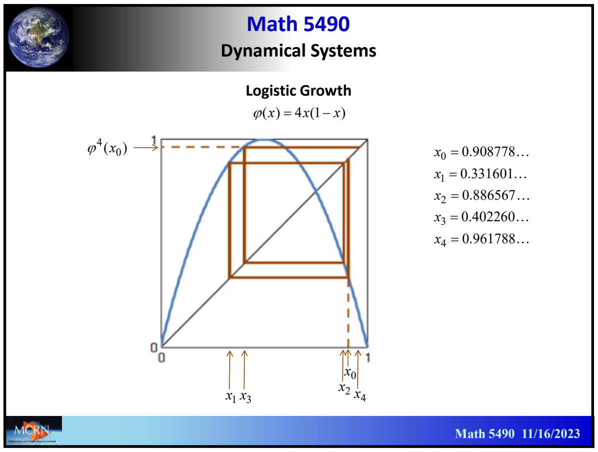
Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
$\varphi(x_0)$
x_1
x_0
$x_0 = 0.908778\ldots$
$x_1 = 0.331601\ldots$
Math 5490 11/16/2023

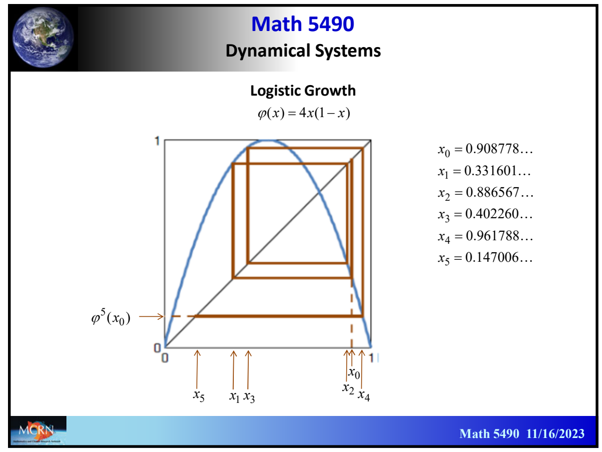


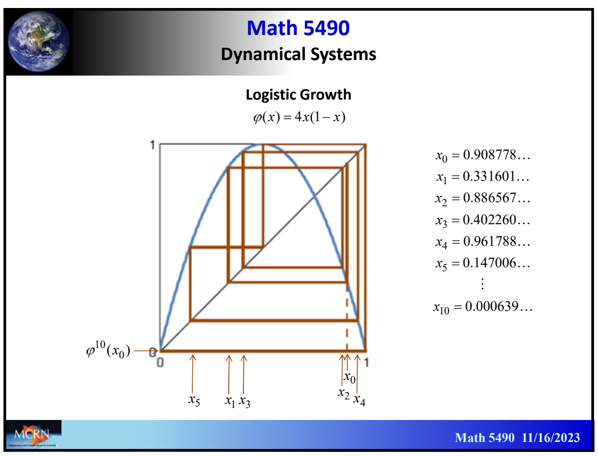
Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
$\varphi^2(x_0)$
x_1
x_0
x_2
$x_0 = 0.908778\ldots$
$x_1 = 0.331601\ldots$
$x_2 = 0.886567\ldots$
Math 5490 11/16/2023

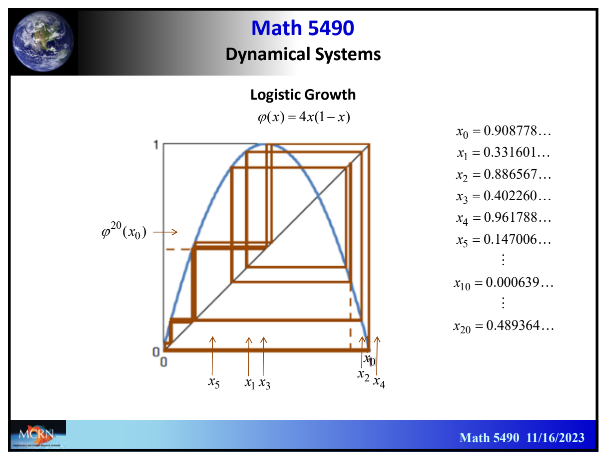


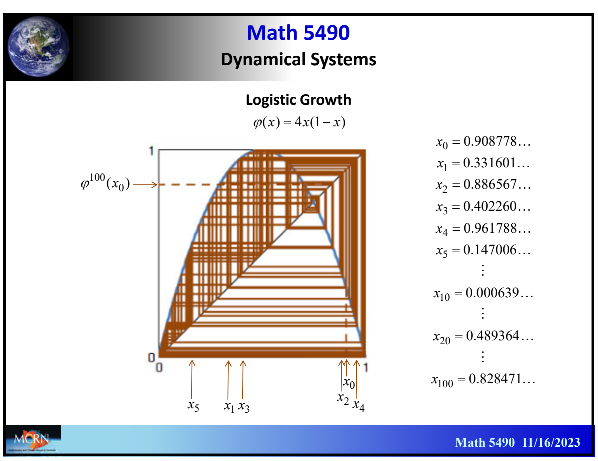
Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
$\varphi^3(x_0)$
x_1 x_3
x_0
x_2
$x_0 = 0.908778\ldots$
$x_1 = 0.331601\ldots$
$x_2 = 0.886567\ldots$
$x_3 = 0.402260\ldots$
Math 5490 11/16/2023

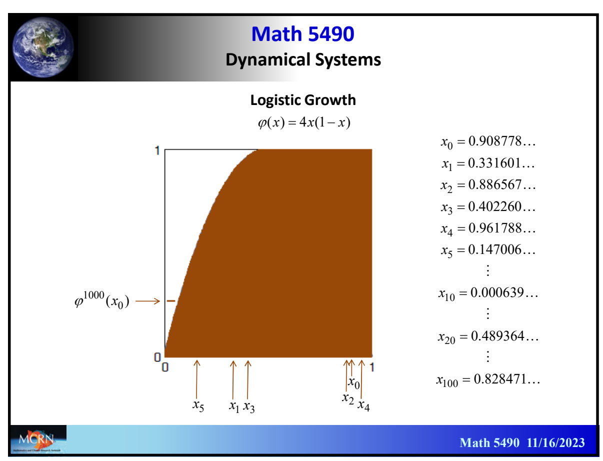

Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
$\varphi^4(x_0)$
$x_0 = 0.908778\ldots$
$x_1 = 0.331601\ldots$
$x_2 = 0.886567\ldots$
$x_3 = 0.402260\ldots$
$x_4 = 0.961788\ldots$
x_1 x_3 x_0 x_2 x_4
Math 5490 11/16/2023


Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
$\varphi^5(x_0)$
$x_0 = 0.908778\ldots$
$x_1 = 0.331601\ldots$
$x_2 = 0.886567\ldots$
$x_3 = 0.402260\ldots$
$x_4 = 0.961788\ldots$
$x_5 = 0.147006\ldots$
x_5 x_1 x_3 x_0 x_2 x_4
Math 5490 11/16/2023


Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
$\varphi^{10}(x_0)$
$x_0 = 0.908778\ldots$
$x_1 = 0.331601\ldots$
$x_2 = 0.886567\ldots$
$x_3 = 0.402260\ldots$
$x_4 = 0.961788\ldots$
$x_5 = 0.147006\ldots$
$\vdots$
$x_{10} = 0.000639\ldots$
x_5 x_1 x_3 x_0 x_2 x_4
Math 5490 11/16/2023


Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
$\varphi^{20}(x_0)$
$x_0 = 0.908778\ldots$
$x_1 = 0.331601\ldots$
$x_2 = 0.886567\ldots$
$x_3 = 0.402260\ldots$
$x_4 = 0.961788\ldots$
$x_5 = 0.147006\ldots$
$\vdots$
$x_{10} = 0.000639\ldots$
$\vdots$
$x_{20} = 0.489364\ldots$
x_5 x_1 x_3 x_0 x_2 x_4
Math 5490 11/16/2023

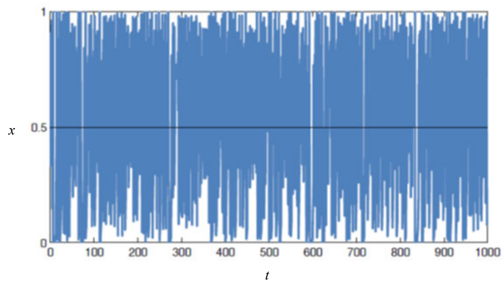

Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
$\varphi^{100}(x_0)$
$x_0 = 0.908778\ldots$
$x_1 = 0.331601\ldots$
$x_2 = 0.886567\ldots$
$x_3 = 0.402260\ldots$
$x_4 = 0.961788\ldots$
$x_5 = 0.147006\ldots$
$\vdots$
$x_{10} = 0.000639\ldots$
$\vdots$
$x_{20} = 0.489364\ldots$
$\vdots$
$x_{100} = 0.828471\ldots$
x_5 x_1 x_3 x_0 x_2 x_4
Math 5490 11/16/2023


Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = 4x(1-x)$
$\varphi^{1000}(x_0)$
$x_0 = 0.908778\ldots$
$x_1 = 0.331601\ldots$
$x_2 = 0.886567\ldots$
$x_3 = 0.402260\ldots$
$x_4 = 0.961788\ldots$
$x_5 = 0.147006\ldots$
$\vdots$
$x_{10} = 0.000639\ldots$
$\vdots$
$x_{20} = 0.489364\ldots$
$\vdots$
$x_{100} = 0.828471\ldots$
x_5 x_1 x_3 x_0 x_2 x_4
Math 5490 11/16/2023

Math 5490
Dynamical Systems

Logistic Growth

$\varphi(x) = 4x(1-x)$



Math 5490
Dynamical Systems

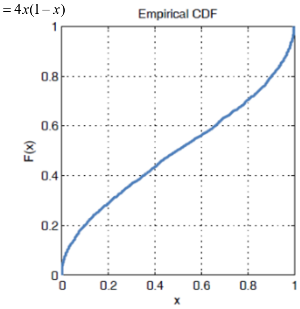
Logistic Growth

$\varphi(x) = 4x(1-x)$

Sample = first 2000 iterates

x vs the proportion of sample points whose value is below x.

Continuous distribution, but not uniform.



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Dynamical Systems

Discrete Time Logistic Growth

$$N_{t+1} = rN_t\left(1 - \frac{N_t}{K}\right)$$

N_t : population at time t

r : intrinsic growth rate

K : carrying capacity

Continuous Time Logistic Growth

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

Measure population as fraction of carrying capacity.

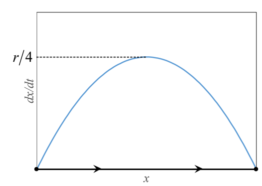
$$N = Kx \quad \Rightarrow \quad K\frac{dx}{dt} = \frac{dN}{dt} = rKx(1-x)$$

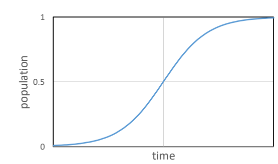
$$\frac{dx}{dt} = rx(1-x)$$

Math 5490
Dynamical Systems

Continuous Time Logistic Growth

$$\frac{dx}{dt} = rx(1-x)$$





For any positive r, all solutions move asymptotically from 0 to 1.

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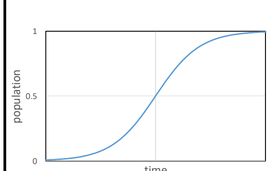
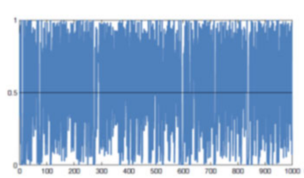
Dynamical Systems

Continuous Time vs Discrete Time

$r = 4$





very different

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Dynamical Systems

Logistic Growth

$\varphi(x) = rx(1-x)$

Fixed Point

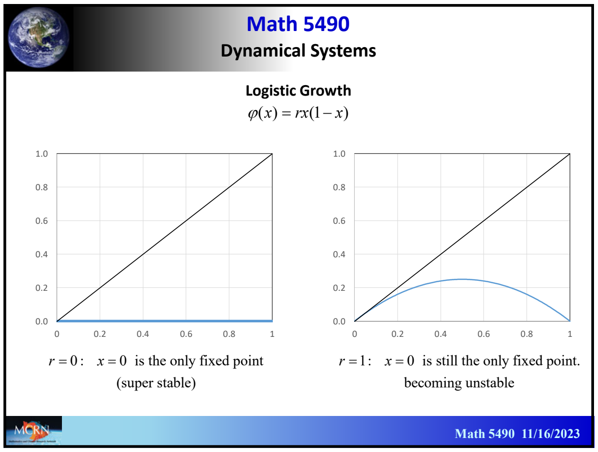
$\varphi(x) = x$

anolog of rest point

$rx(1-x) = x \quad \Rightarrow \quad x = 0 \quad \text{or} \quad r(1-x) = 1$

$x = 0$ is always a fixed point.

$r > 1 \quad \Rightarrow \quad x = 1 - \frac{1}{r}$ is a fixed point.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Dynamical Systems
Logistic Growth
$\varphi(x) = rx(1-x)$
$r = 0$: $x = 0$ is the only fixed point
(super stable)
$r = 1$: $x = 0$ is still the only fixed point.
becoming unstable
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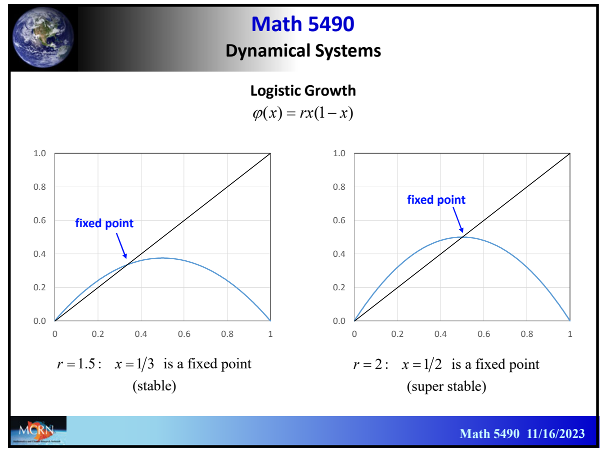
Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = rx(1-x)$
fixed point
fixed point
$r = 1.5$: $x = 1/3$ is a fixed point
(stable)
$r = 2$: $x = 1/2$ is a fixed point
(super stable)
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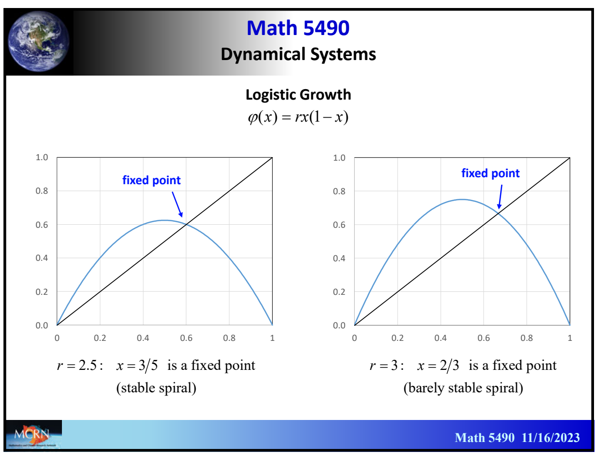
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Dynamical Systems
Logistic Growth
$\varphi(x) = rx(1-x)$
fixed point
fixed point
$r = 2.5$: $x = 3/5$ is a fixed point
(stable spiral)
$r = 3$: $x = 2/3$ is a fixed point
(barely stable spiral)
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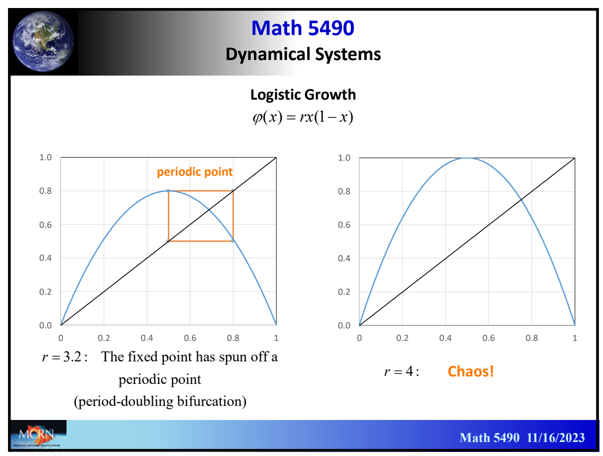
Math 5490
Dynamical Systems
Logistic Growth
$\varphi(x) = rx(1-x)$
periodic point
$r = 3.2$: The fixed point has spun off a
periodic point
(period-doubling bifurcation)
$r = 4$: Chaos!
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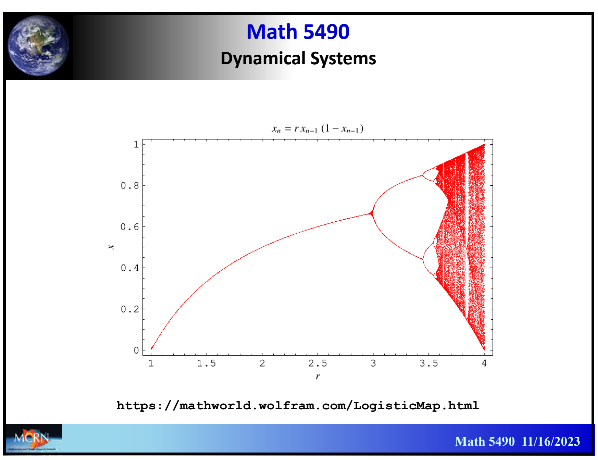
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Dynamical Systems
$x_n = r\,x_{n-1}(1 - x_{n-1})$
https://mathworld.wolfram.com/LogisticMap.html
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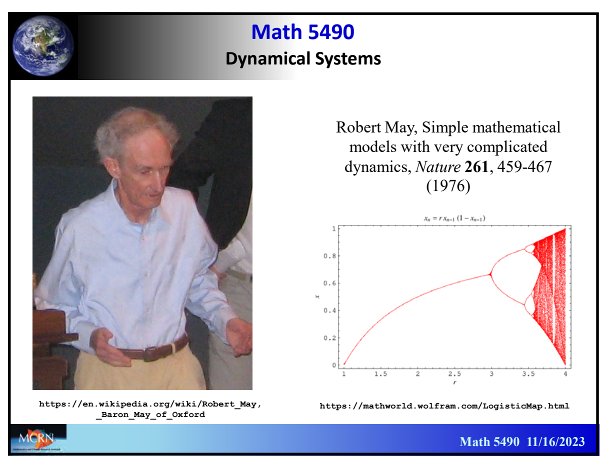
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Dynamical Systems
Robert May, Simple mathematical models with very complicated dynamics, Nature 261, 459-467 (1976)
https://en.wikipedia.org/wiki/Robert_May,_Baron_May_of_Oxford
https://mathworld.wolfram.com/LogisticMap.html
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