Math 5490
Topics in Applied Mathematics
Introduction to the Mathematics of Climate

Fall 2023
1:25 - 3:20 Tuesdays and Thursdays
Amundson Hall 162

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course website
`www-users.cse.umn.edu/~mcgehee/teaching/Math5490/`

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Review

What determines the Earth's surface temperature?

Conservation of Energy
Heat is a form of energy.
Temperature measures heat.

temperature change ~ energy in – energy out

short wave energy from the Sun

long wave energy from the Earth

Everything else is detail.

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Review

Black-Body Radiation

Stefan-Boltzmann Law

$$F = \sigma T^4$$

watts per square meter

power flux (W/m^2)

kelvin

temperature (K)

0 K = −273°C = "absolute zero"

Stefan-Boltzmann constant

$$\sigma \approx 5.67 \times 10^{-8}\ \mathrm{W/m^2}K^4$$

watt = joule per second = "power"

Reasonable approximation:
Every body in the solar system radiates energy according to this law.

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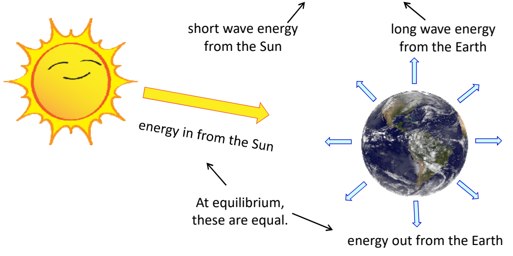
What determines the Earth's surface temperature?

temperature change ~ energy in – energy out


short wave energy from the Sun

long wave energy from the Earth

At equilibrium, these are equal.

energy out from the Earth

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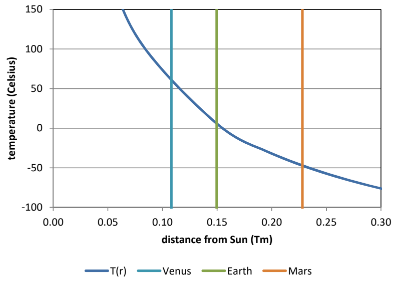
Goldilocks Zone



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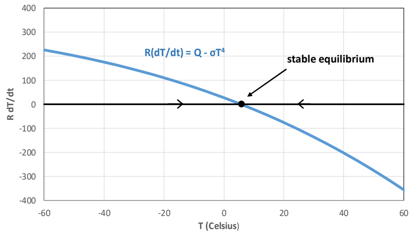
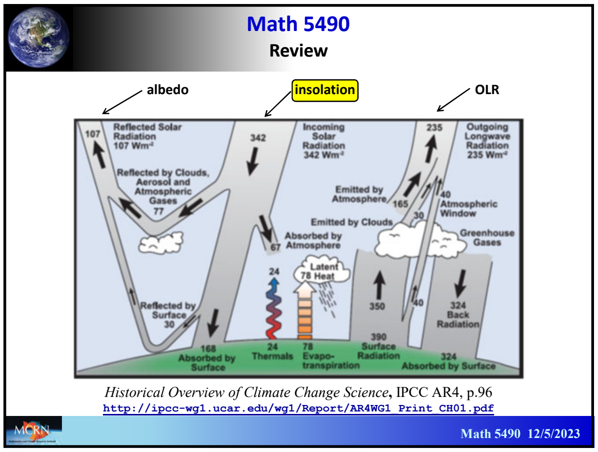
temperature change ~ energy in – energy out

heat capacity → $R\dfrac{dT}{dt} = Q - \sigma T^4$





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Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

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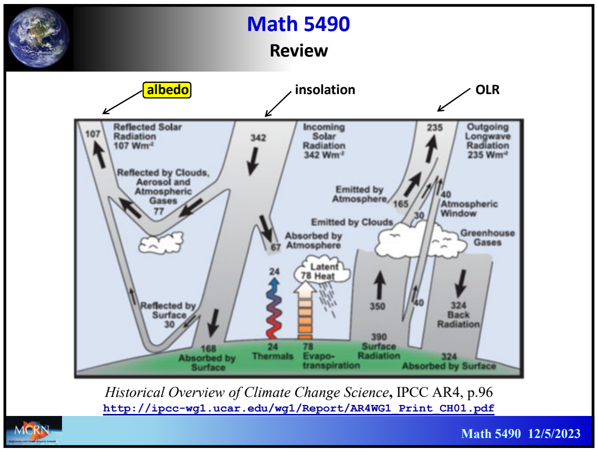

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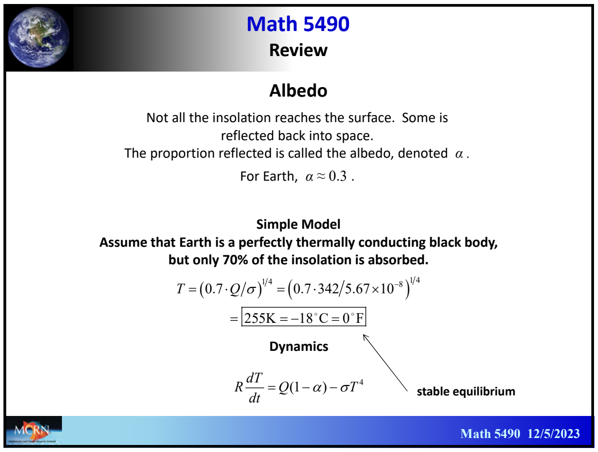

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Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α .

For Earth, $\alpha \approx 0.3$.

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = \left(0.7 \cdot Q/\sigma\right)^{1/4} = \left(0.7 \cdot 342/5.67 \times 10^{-8}\right)^{1/4}$$

$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium

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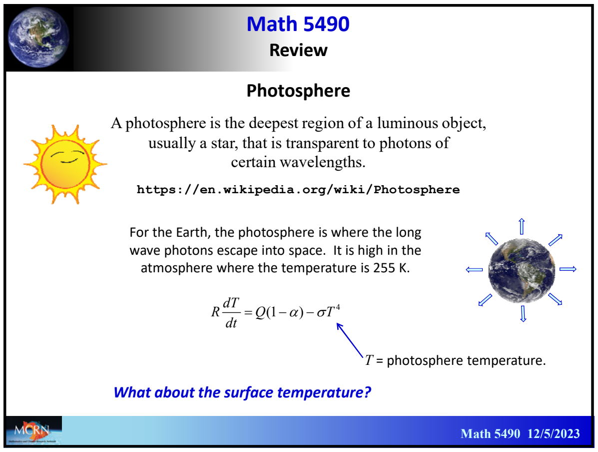

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Photosphere

A photosphere is the deepest region of a luminous object, usually a star, that is transparent to photons of certain wavelengths.

https://en.wikipedia.org/wiki/Photosphere

For the Earth, the photosphere is where the long wave photons escape into space. It is high in the atmosphere where the temperature is 255 K.

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

T = photosphere temperature.

What about the surface temperature?

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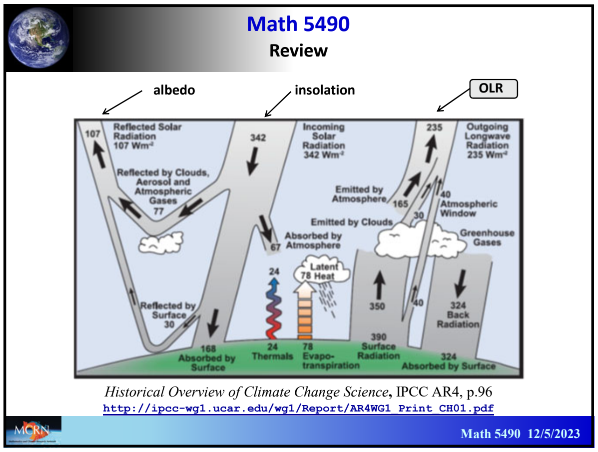

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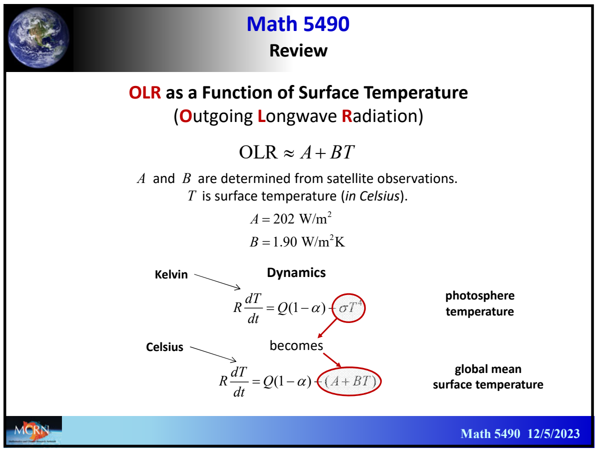

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OLR as a Function of Surface Temperature
(Outgoing Longwave Radiation)

$$\text{OLR} \approx A + BT$$

A and B are determined from satellite observations.
T is surface temperature (*in Celsius*).

$$A = 202 \text{ W/m}^2$$

$$B = 1.90 \text{ W/m}^2\text{K}$$

Dynamics

Kelvin

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

photosphere temperature

becomes

Celsius

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature

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OLR as a Function of Surface Temperature

$$\text{OLR} \approx A + BT$$

Important:
$A+BT$ is **not** a linear approximation to the Stefan-Boltzmann equation.

Dynamics

Kelvin

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

becomes

Celsius

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

different

photosphere temperature

global mean surface temperature

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Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

What's missing?

Earth is not homogeneous. For example, it is warmer at the equator and colder at the poles. The temperature should depend on latitude.

Make T depend on $y = \sin(\text{latitude})$

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation distribution

$$s(y) = \text{distribution across latitudes} \quad \left(\int_0^1 s(y)dy = 1\right)$$

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Latitiude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

$$s(y) = \text{distribution across latitudes} \quad \left(\int_0^1 s(y)dy = 1\right)$$

One can show that

β = obliquity = 23.4°

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi} \sqrt{1 - \left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\, d\theta$$

McGehee & Lehman, *SIAM J. Applied Dynamical Systems* **11** (2) (2012), 684–707.

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241\left(3y^2 - 1\right)$$

Chylek & Coakley, *J. Atmos. Sci.* **32** (1975), 675–679.

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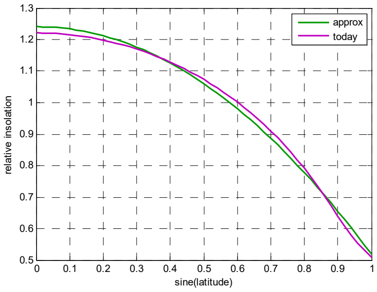
Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.5°



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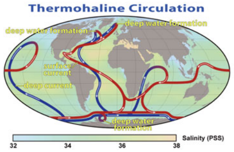
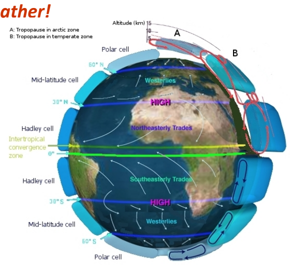
What's Missing?

Weather!





The second law of thermodynamics

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Budyko's Equation

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right) + \boxed{C\left(\overline{T}(t) - T(y,t)\right)}$$

2nd law

global mean temperature $\quad \overline{T}(t) = \int_0^1 T(y,t)dt$

$\left(\overline{T}(t) - T(y,t)\right)$ interpretation

Each point on Earth's surface is trying to assume the global mean temperature. If the temperature at a point is below the the global mean, then it heats up. If the temperature at that point is above the mean, then it cools off.

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Budyko's Equation

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha)-(A+BT)+C(\bar{T}-T)$$

There is still something missing.
What role is played by the ice?

Ice-albedo Feedback

temperature warms
ice melts
albedo decreases
more sunlight absorbed
temperature warms
REPEAT

Why would it stop?

M. I. Budyko, "The effect of solar radiation variations on the climate of the Earth," **Tellus XXI**, 611-619 , 1969.

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Budyko's Equation

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha)-(A+BT)+C(\bar{T}-T)$$

There is still something missing. Where is the ice?

Ice-albedo Feedback

albedo of ice: 0.62
albedo of land and water: 0.32

Assumption: there is a single boundary between ice and no ice occurring at $y=\eta$.

$$\alpha=\alpha(y,\eta)=\begin{cases}\alpha_1=0.32, & y<\eta\\ \alpha_2=0.62, & y>\eta\end{cases}$$

$$R\frac{\partial T}{\partial t}=Qs(y)(1-\alpha(y,\eta)-(A+BT)+C(\bar{T}-T)$$

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Logical Fallacy

Climate changes occurred when humans didn't exist
implies that
humans are not the cause of climate change.

Reasonable Conclusion

Climate changes occurred when humans didn't exist
suggests that
we should try to understand why the climate changed in the past.

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Earth's climate has changed many times in the past.

Paleoclimate

Paleoclimatology is the scientific study of climates predating the invention of meteorological instruments, when no direct measurement data were available. As instrumental records only span a tiny part of Earth's history, the reconstruction of ancient climate is important to understand natural variation and the evolution of the current climate.

`https://en.wikipedia.org/wiki/Paleoclimatology`

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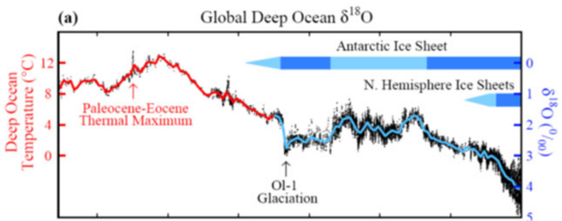
Temperatures in the Cenozoic Era
65 Myr - present



Hansen, et al, Target atmospheric CO2: Where should humanity aim? *Open Atmos. Sci. J.* 2 (2008)

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Temperatures in the Cenozoic Era
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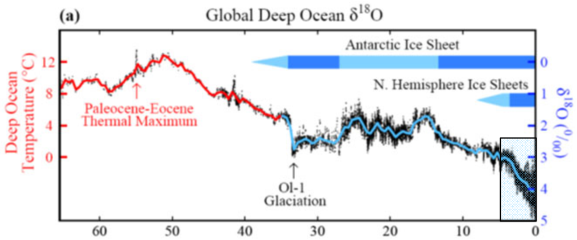
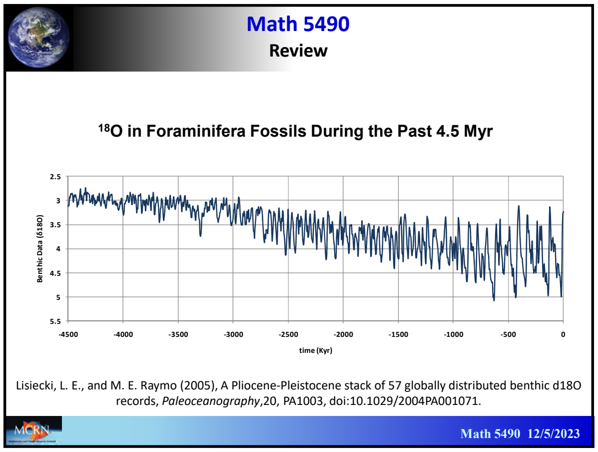


last 4.5 Myr

Hansen, et al, Target atmospheric CO2: Where should humanity aim? *Open Atmos. Sci. J.* 2 (2008)



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^{18}O in Foraminifera Fossils During the Past 4.5 Myr

Lisiecki, L. E., and M. E. Raymo (2005), A Pliocene-Pleistocene stack of 57 globally distributed benthic d18O records, *Paleoceanography*,20, PA1003, doi:10.1029/2004PA001071.

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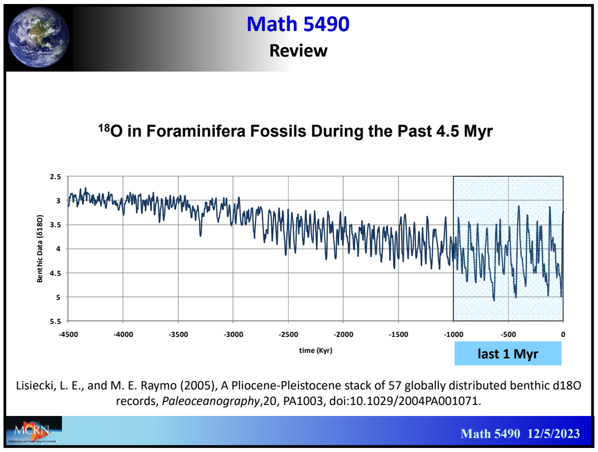

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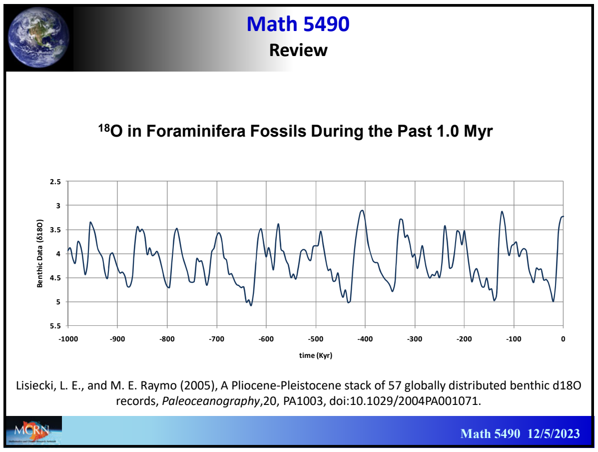

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^{18}O in Foraminifera Fossils During the Past 1.0 Myr

Lisiecki, L. E., and M. E. Raymo (2005), A Pliocene-Pleistocene stack of 57 globally distributed benthic d18O records, *Paleoceanography*,20, PA1003, doi:10.1029/2004PA001071.

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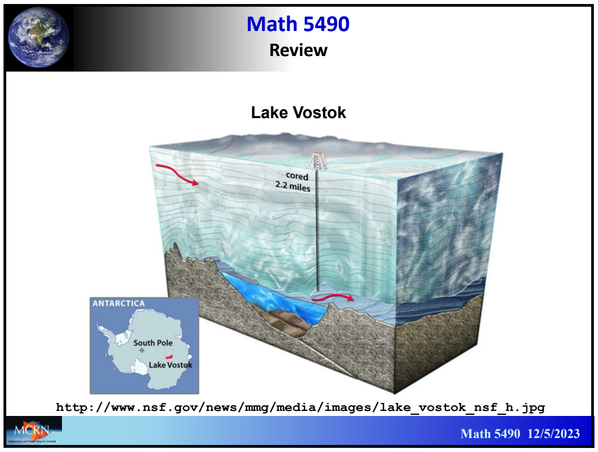

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Lake Vostok

`http://www.nsf.gov/news/mmg/media/images/lake_vostok_nsf_h.jpg`

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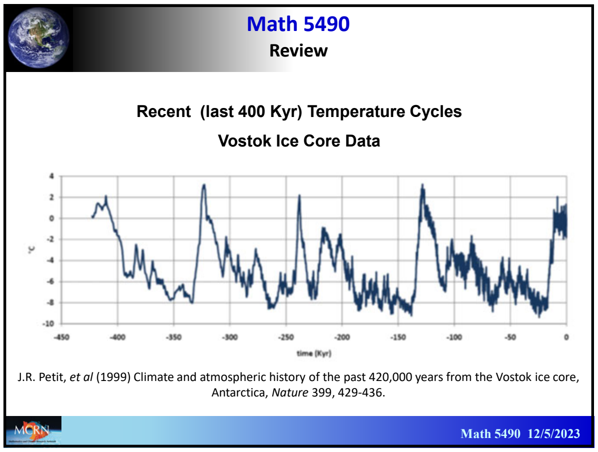

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Recent (last 400 Kyr) Temperature Cycles

Vostok Ice Core Data

J.R. Petit, *et al* (1999) Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica, *Nature* 399, 429-436.

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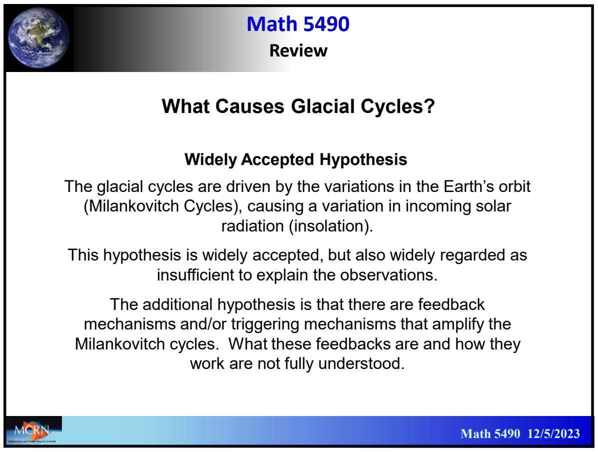

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What Causes Glacial Cycles?

Widely Accepted Hypothesis

The glacial cycles are driven by the variations in the Earth's orbit (Milankovitch Cycles), causing a variation in incoming solar radiation (insolation).

This hypothesis is widely accepted, but also widely regarded as insufficient to explain the observations.

The additional hypothesis is that there are feedback mechanisms and/or triggering mechanisms that amplify the Milankovitch cycles. What these feedbacks are and how they work are not fully understood.

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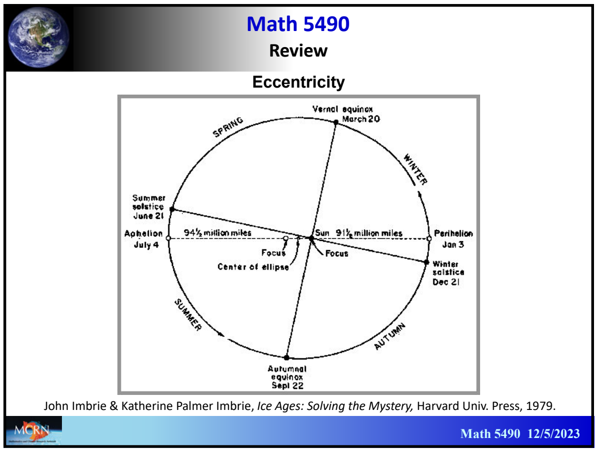

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Eccentricity

John Imbrie & Katherine Palmer Imbrie, *Ice Ages: Solving the Mystery,* Harvard Univ. Press, 1979.

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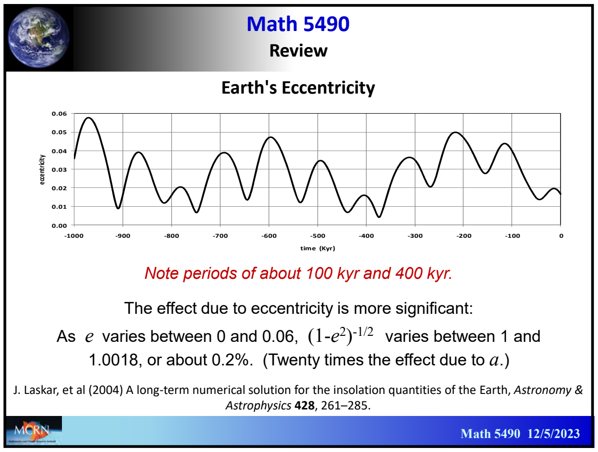

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Earth's Eccentricity

Note periods of about 100 kyr and 400 kyr.

The effect due to eccentricity is more significant:

As e varies between 0 and 0.06, $(1\text{-}e^2)^{-1/2}$ varies between 1 and 1.0018, or about 0.2%. (Twenty times the effect due to a.)

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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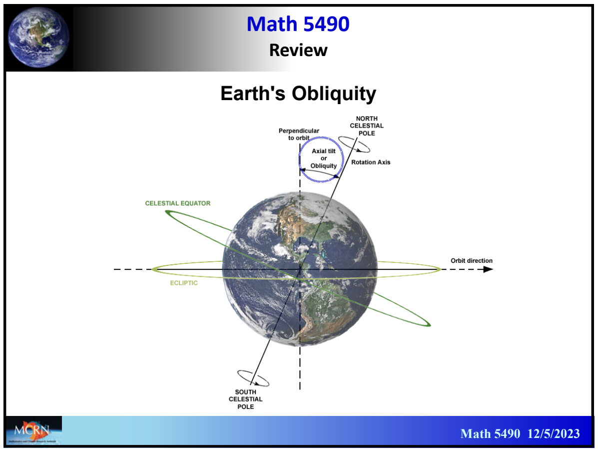

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Earth's Obliquity

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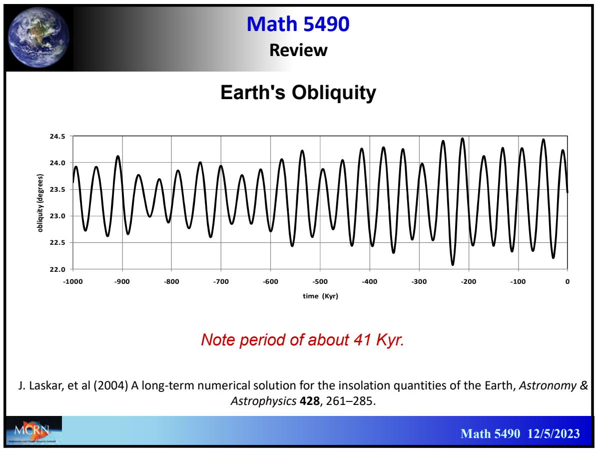

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Earth's Obliquity

Note period of about 41 Kyr.

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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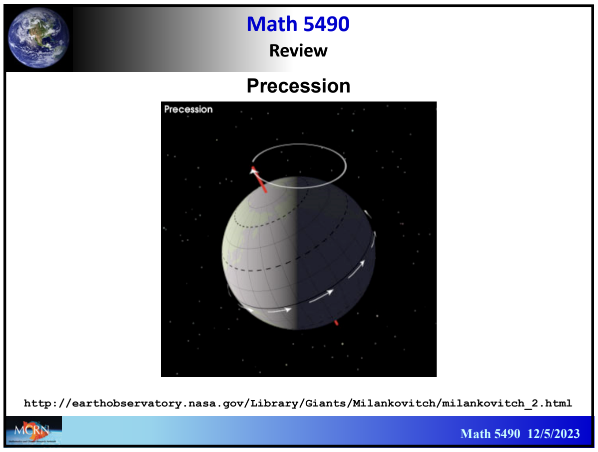

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Precession

`http://earthobservatory.nasa.gov/Library/Giants/Milankovitch/milankovitch_2.html`

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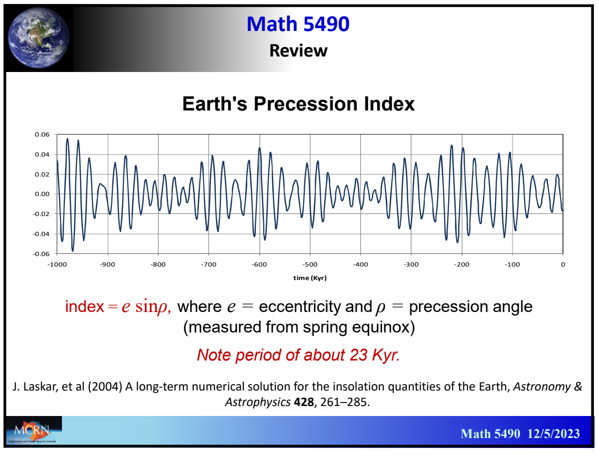

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Earth's Precession Index

index $= e \sin\rho$, where e = eccentricity and ρ = precession angle (measured from spring equinox)

Note period of about 23 Kyr.

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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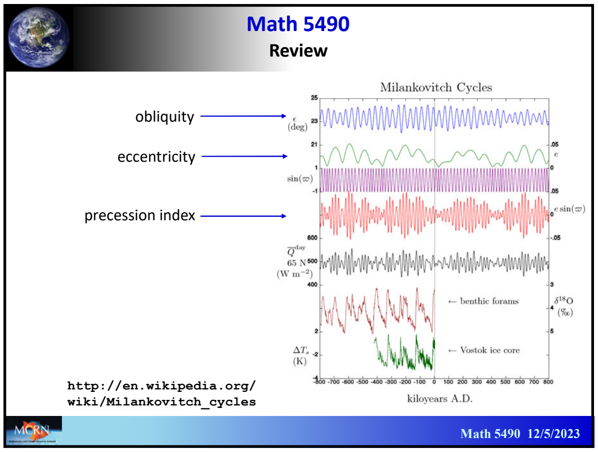

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obliquity

eccentricity

precession index

`http://en.wikipedia.org/wiki/Milankovitch_cycles`

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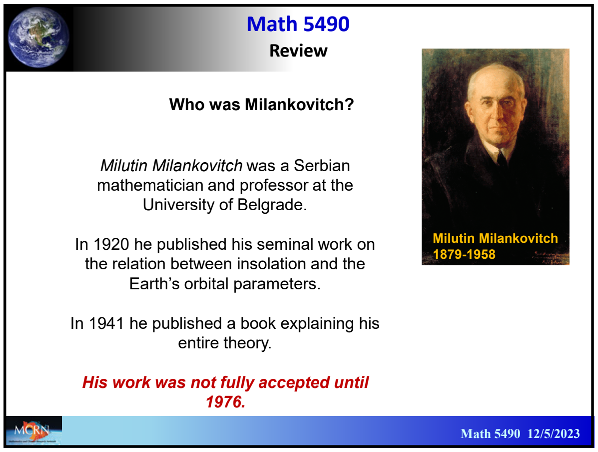

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Who was Milankovitch?

Milutin Milankovitch was a Serbian mathematician and professor at the University of Belgrade.

In 1920 he published his seminal work on the relation between insolation and the Earth's orbital parameters.

In 1941 he published a book explaining his entire theory.

His work was not fully accepted until 1976.

Milutin Milankovitch
1879-1958

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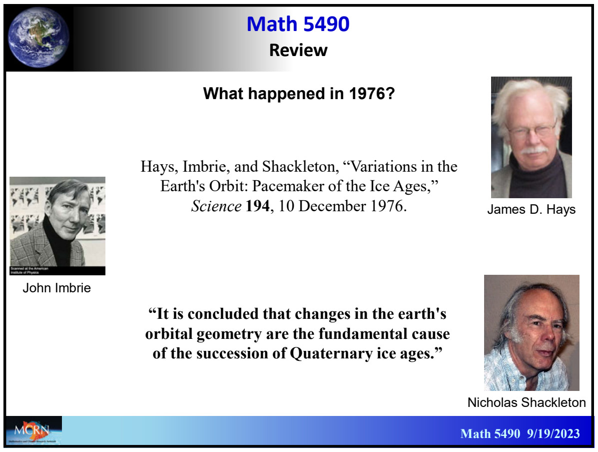

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What happened in 1976?

Hays, Imbrie, and Shackleton, "Variations in the Earth's Orbit: Pacemaker of the Ice Ages," *Science* **194**, 10 December 1976.

James D. Hays

John Imbrie

"It is concluded that changes in the earth's orbital geometry are the fundamental cause of the succession of Quaternary ice ages."

Nicholas Shackleton

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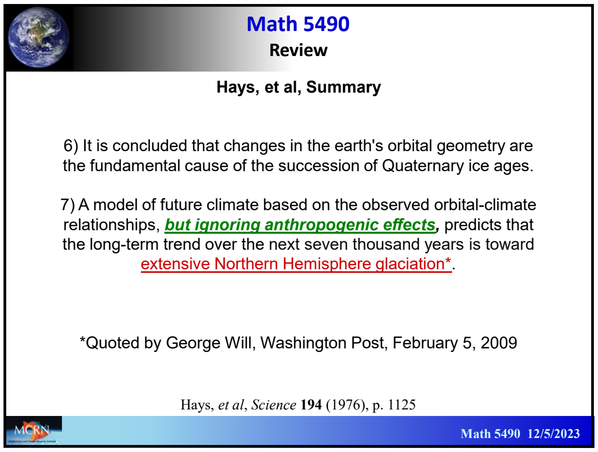

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Hays, et al, Summary

6) It is concluded that changes in the earth's orbital geometry are the fundamental cause of the succession of Quaternary ice ages.

7) A model of future climate based on the observed orbital-climate relationships, ***but ignoring anthropogenic effects***, predicts that the long-term trend over the next seven thousand years is toward extensive Northern Hemisphere glaciation*.

*Quoted by George Will, Washington Post, February 5, 2009

Hays, *et al*, *Science* **194** (1976), p. 1125

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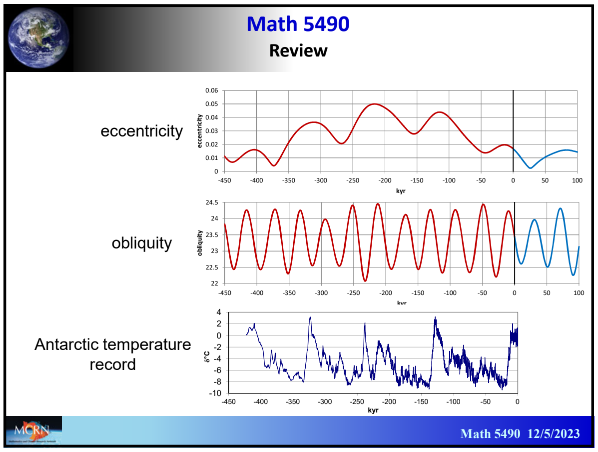

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eccentricity

obliquity

Antarctic temperature record

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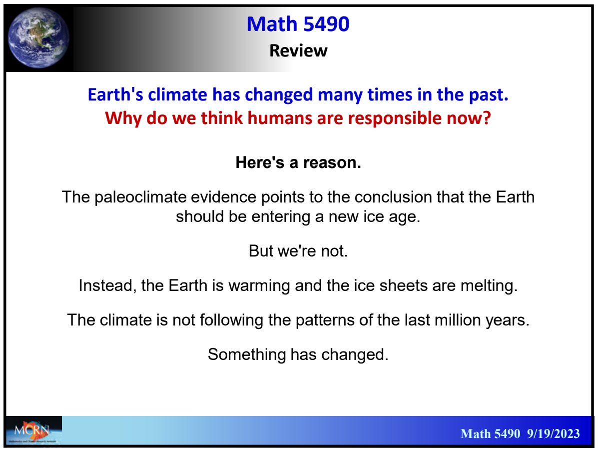

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Review

Earth's climate has changed many times in the past.
Why do we think humans are responsible now?

Here's a reason.

The paleoclimate evidence points to the conclusion that the Earth should be entering a new ice age.

But we're not.

Instead, the Earth is warming and the ice sheets are melting.

The climate is not following the patterns of the last million years.

Something has changed.

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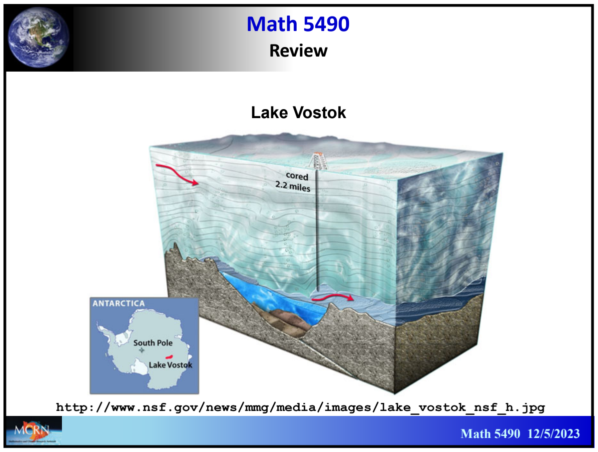
Earth's Carbon Cycle

Earth's climate has changed many times in the past.
Why do we think humans are responsible now?

Why do we think that atmospheric CO_2 has anything to do with climate change?

Why do we think that the increase in atmospheric CO_2 has anything to do with human activity?

Why do we think that atmospheric CO_2 has anything to do with surface temperature?



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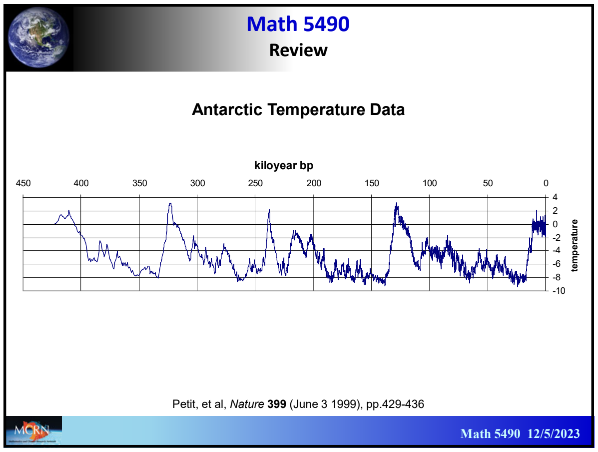
Lake Vostok

`http://www.nsf.gov/news/mmg/media/images/lake_vostok_nsf_h.jpg`



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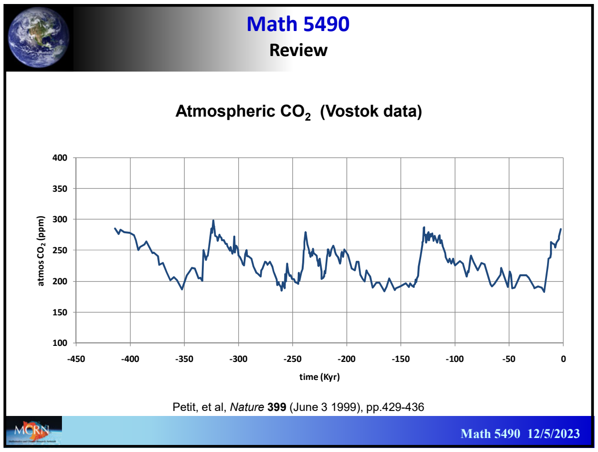
Antarctic Temperature Data

Petit, et al, *Nature* **399** (June 3 1999), pp.429-436

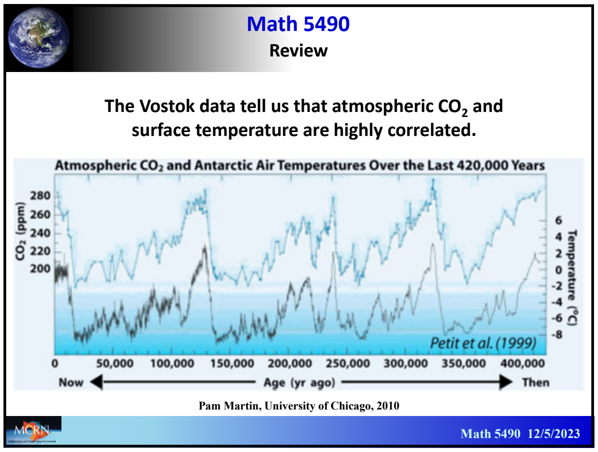


Math 5490

Review

Atmospheric CO_2 (Vostok data)

Petit, et al, *Nature* **399** (June 3 1999), pp.429-436

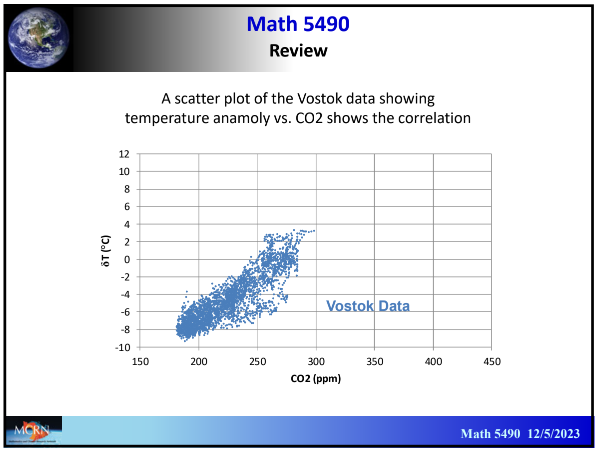


Math 5490

Review

The Vostok data tell us that atmospheric CO_2 and surface temperature are highly correlated.

Pam Martin, University of Chicago, 2010



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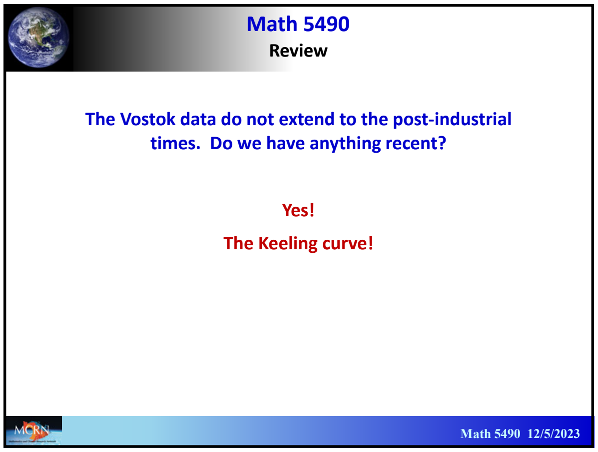
Review

A scatter plot of the Vostok data showing temperature anamoly vs. CO2 shows the correlation



Math 5490
Review

The Vostok data do not extend to the post-industrial times. Do we have anything recent?

Yes!

The Keeling curve!

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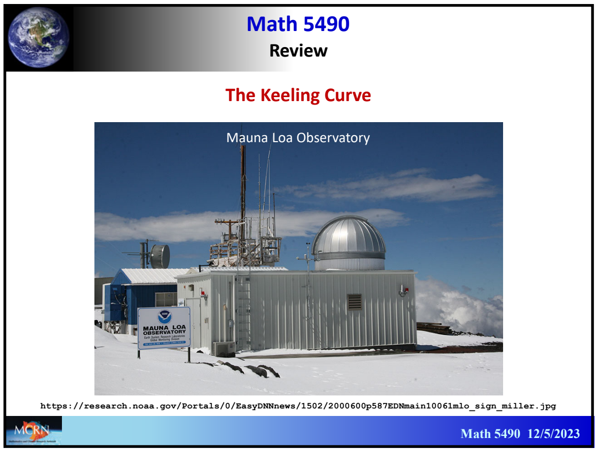

Math 5490
Review

The Keeling Curve

https://research.noaa.gov/Portals/0/EasyDNNnews/1502/2000600p587EDNmain10061mlo_sign_miller.jpg

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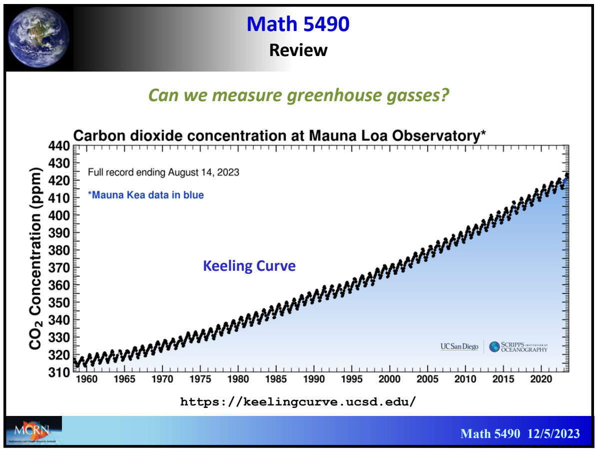

Math 5490
Review

Can we measure greenhouse gasses?

https://keelingcurve.ucsd.edu/

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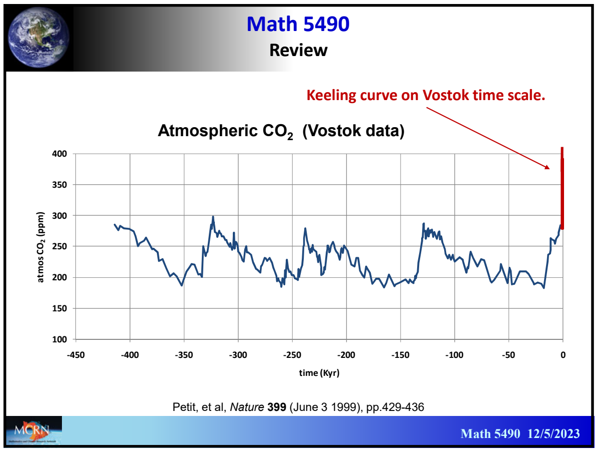

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Keeling curve on Vostok time scale.

Petit, et al, *Nature* **399** (June 3 1999), pp.429-436

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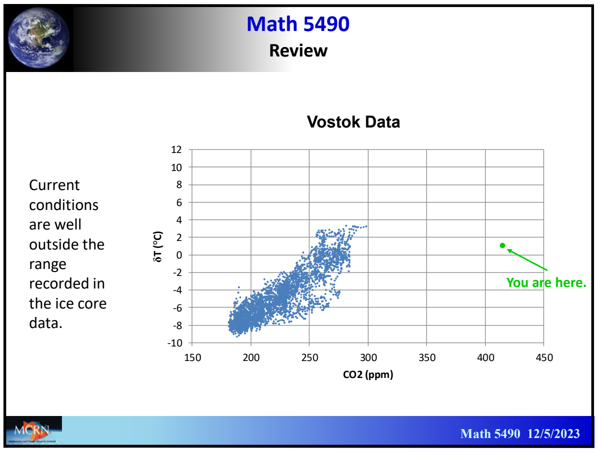

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Current conditions are well outside the range recorded in the ice core data.

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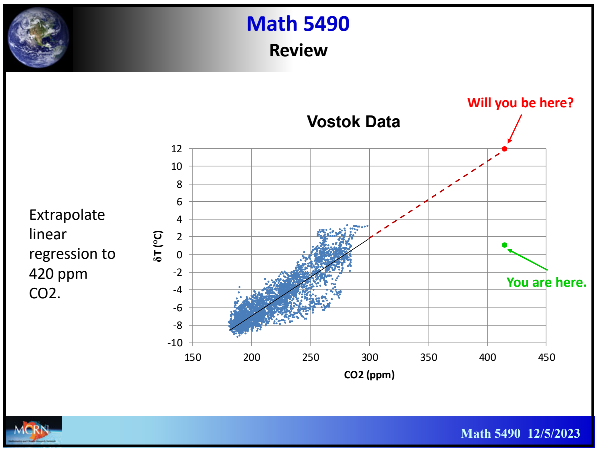

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Extrapolate linear regression to 420 ppm CO2.

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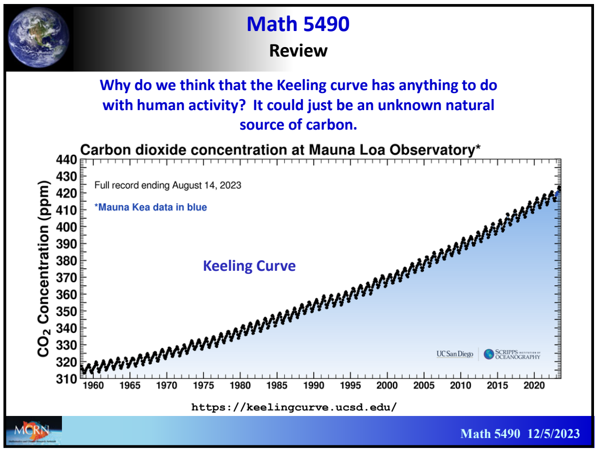

Math 5490
Review

Why do we think that the Keeling curve has anything to do with human activity? It could just be an unknown natural source of carbon.

https://keelingcurve.ucsd.edu/

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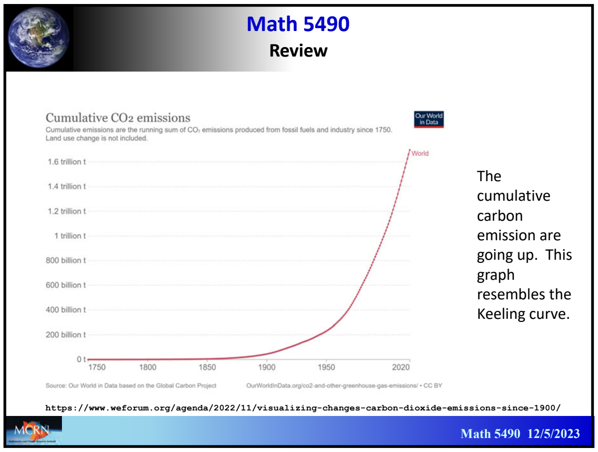

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Review

The cumulative carbon emission are going up. This graph resembles the Keeling curve.

https://www.weforum.org/agenda/2022/11/visualizing-changes-carbon-dioxide-emissions-since-1900/

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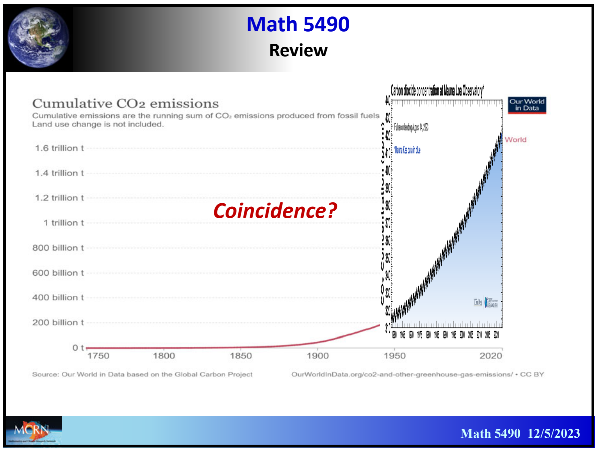

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Coincidence?

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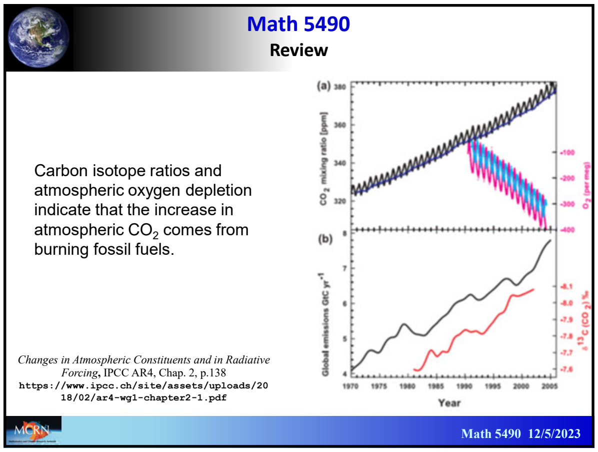

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Carbon isotope ratios and atmospheric oxygen depletion indicate that the increase in atmospheric CO_2 comes from burning fossil fuels.

Changes in Atmospheric Constituents and in Radiative Forcing, IPCC AR4, Chap. 2, p.138 https://www.ipcc.ch/site/assets/uploads/2018/02/ar4-wg1-chapter2-1.pdf

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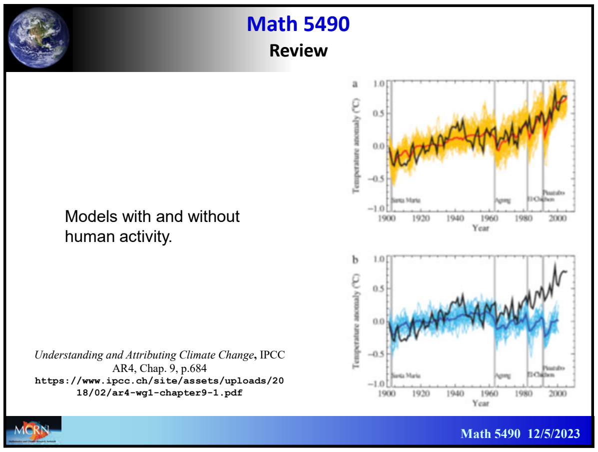

Math 5490
Review

Models with and without human activity.

Understanding and Attributing Climate Change, IPCC AR4, Chap. 9, p.684 https://www.ipcc.ch/site/assets/uploads/2018/02/ar4-wg1-chapter9-1.pdf

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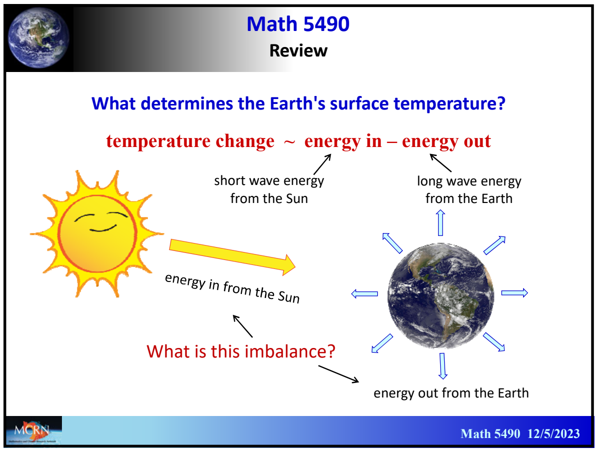

Math 5490
Review

What determines the Earth's surface temperature?

temperature change ~ energy in – energy out

short wave energy from the Sun

long wave energy from the Earth

energy in from the Sun

What is this imbalance?

energy out from the Earth

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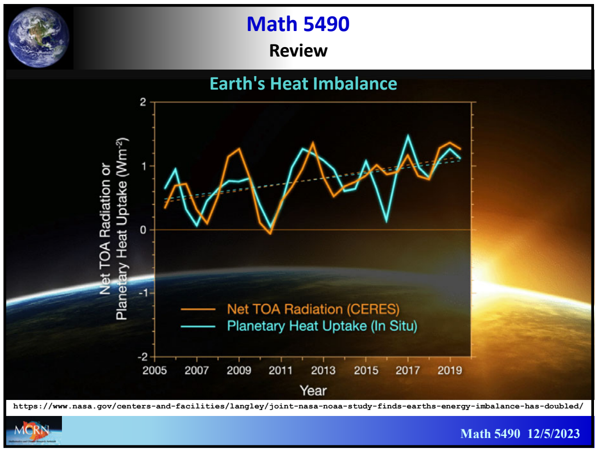

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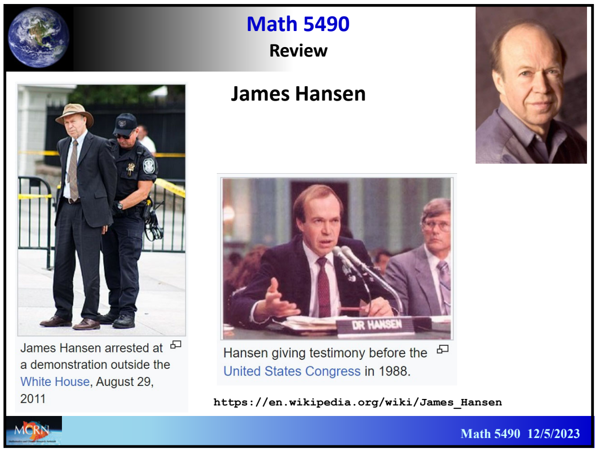
https://www.nasa.gov/centers-and-facilities/langley/joint-nasa-noaa-study-finds-earths-energy-imbalance-has-doubled/



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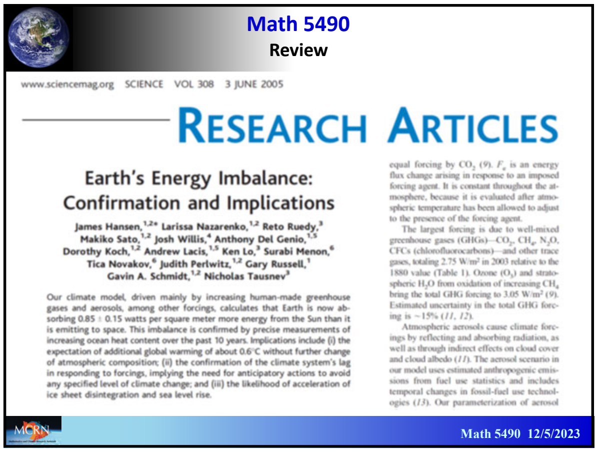
James Hansen

James Hansen arrested at a demonstration outside the White House, August 29, 2011

Hansen giving testimony before the United States Congress in 1988.

https://en.wikipedia.org/wiki/James_Hansen



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Review

www.sciencemag.org SCIENCE VOL 308 3 JUNE 2005

RESEARCH ARTICLES

Earth's Energy Imbalance: Confirmation and Implications

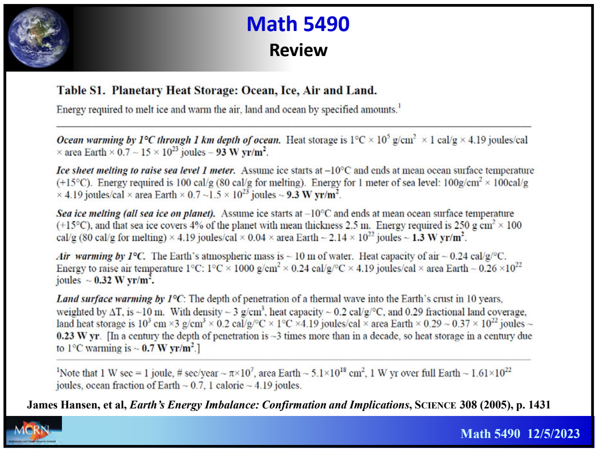
James Hansen,[1,2*] Larissa Nazarenko,[1,2] Reto Ruedy,[3] Makiko Sato,[1,2] Josh Willis,[4] Anthony Del Genio,[1,5] Dorothy Koch,[1,2] Andrew Lacis,[1,5] Ken Lo,[3] Surabi Menon,[6] Tica Novakov,[6] Judith Perlwitz,[1,2] Gary Russell,[1] Gavin A. Schmidt,[1,2] Nicholas Tausnev[3]

Our climate model, driven mainly by increasing human-made greenhouse gases and aerosols, among other forcings, calculates that Earth is now absorbing 0.85 ± 0.15 watts per square meter more energy from the Sun than it is emitting to space. This imbalance is confirmed by precise measurements of increasing ocean heat content over the past 10 years. Implications include (i) the expectation of additional global warming of about 0.6°C without further change of atmospheric composition; (ii) the confirmation of the climate system's lag in responding to forcings, implying the need for anticipatory actions to avoid any specified level of climate change; and (iii) the likelihood of acceleration of ice sheet disintegration and sea level rise.

equal forcing by CO_2 (*9*). F_e is an energy flux change arising in response to an imposed forcing agent. It is constant throughout the atmosphere, because it is evaluated after atmospheric temperature has been allowed to adjust to the presence of the forcing agent.

The largest forcing is due to well-mixed greenhouse gases (GHGs)—CO_2, CH_4, N_2O, CFCs (chlorofluorocarbons)—and other trace gases, totaling 2.75 W/m^2 in 2003 relative to the 1880 value (Table 1). Ozone (O_3) and stratospheric H_2O from oxidation of increasing CH_4 bring the total GHG forcing to 3.05 W/m^2 (*9*). Estimated uncertainty in the total GHG forcing is ~15% (*11*, *12*).

Atmospheric aerosols cause climate forcings by reflecting and absorbing radiation, as well as through indirect effects on cloud cover and cloud albedo (*11*). The aerosol scenario in our model uses estimated anthropogenic emissions from fuel use statistics and includes temporal changes in fossil-fuel use technologies (*13*). Our parameterization of aerosol



Math 5490
Review

Table S1. Planetary Heat Storage: Ocean, Ice, Air and Land.

Energy required to melt ice and warm the air, land and ocean by specified amounts.[1]

Ocean warming by 1°C through 1 km depth of ocean. Heat storage is $1°C \times 10^5$ g/cm^2 $\times$ 1 cal/g $\times$ 4.19 joules/cal $\times$ area Earth $\times$ 0.7 ~ 15×10^{23} joules ~ **93 W yr/m^2**.

Ice sheet melting to raise sea level 1 meter. Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C). Energy required is 100 cal/g (80 cal/g for melting). Energy for 1 meter of sea level: 100g/cm^2 $\times$ 100cal/g $\times$ 4.19 joules/cal $\times$ area Earth $\times$ 0.7 ~1.5×10^{23} joules ~ **9.3 W yr/m^2**.

Sea ice melting (all sea ice on planet). Assume ice starts at –10°C and ends at mean ocean surface temperature (+15°C), and that sea ice covers 4% of the planet with mean thickness 2.5 m. Energy required is 250 g cm^2 $\times$ 100 cal/g (80 cal/g for melting) $\times$ 4.19 joules/cal $\times$ 0.04 $\times$ area Earth ~ 2.14×10^{22} joules ~ **1.3 W yr/m^2**.

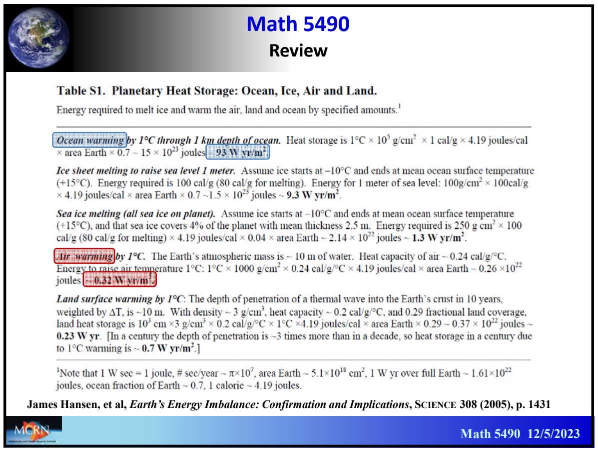
Air warming by 1°C. The Earth's atmospheric mass is ~ 10 m of water. Heat capacity of air ~ 0.24 cal/g/°C. Energy to raise air temperature 1°C: 1°C $\times$ 1000 g/cm^2 $\times$ 0.24 cal/g/°C $\times$ 4.19 joules/cal $\times$ area Earth ~ 0.26×10^{22} joules ~ **0.32 W yr/m^2.**

Land surface warming by 1°C: The depth of penetration of a thermal wave into the Earth's crust in 10 years, weighted by ΔT, is ~10 m. With density ~ 3 g/cm^3, heat capacity ~ 0.2 cal/g/°C, and 0.29 fractional land coverage, land heat storage is 10^3 cm $\times$3 g/cm^3 $\times$ 0.2 cal/g/°C $\times$ 1°C $\times$4.19 joules/cal $\times$ area Earth $\times$ 0.29 ~ 0.37×10^{22} joules ~ **0.23 W yr**. [In a century the depth of penetration is ~3 times more than in a decade, so heat storage in a century due to 1°C warming is ~ **0.7 W yr/m^2**.]

---

[1] Note that 1 W sec = 1 joule, # sec/year ~ $\pi\times10^7$, area Earth ~ 5.1×10^{18} cm^2, 1 W yr over full Earth ~ 1.61×10^{22} joules, ocean fraction of Earth ~ 0.7, 1 calorie ~ 4.19 joules.

James Hansen, et al, *Earth's Energy Imbalance: Confirmation and Implications*, SCIENCE 308 (2005), p. 1431



Math 5490
Review

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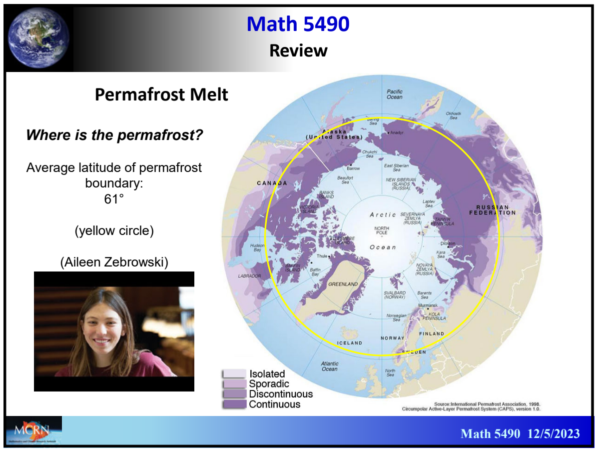

Math 5490
Review

Permafrost Melt

Where is the permafrost?

Average latitude of permafrost boundary:
61°

(yellow circle)

(Aileen Zebrowski)

Source:International Permafrost Association, 1998. Circumpolar Active-Layer Permafrost System (CAPS), version 1.0.

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Review

How much carbon would be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?

Math 5490
Review

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

$$\overline{T} = \int_0^1 T(y)dy$$

surface temperature

sin(latitude)

heat capacity

insolation

albedo

OLR

heat transport

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 + s_2\left(3y^2 - 1\right), \quad \text{where} \quad s_2 = -0.241$$

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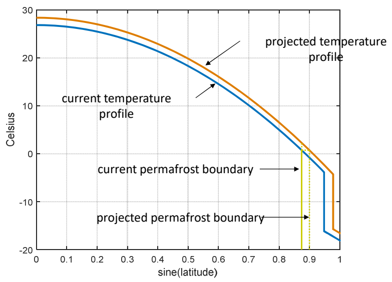
Equilibrium Temperature Profile



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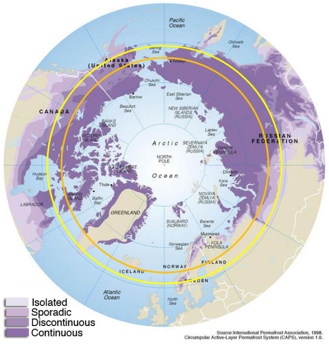
Review

Where is the permafrost?

Average latitude of permafrost boundary: 61°
(yellow circle)

Projected permafrost boundary
(orange circle)



Source:International Permafrost Association, 1998.
Circumpolar Active-Layer Permafrost System (CAPS), version 1.0.

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Review

How much carbon could be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?

Recall that the surface area is proportional to y, the sine of the latitude.

Current permafrost boundary: $y_p = \sin(61^\circ)C \approx 0.875$

Proportion of globe cover by permafrost: $1 - y_p = 0.125$

$\Delta y \approx 0.027$

Proportion of permafrost melted: $\dfrac{0.027}{0.125} = 0.216$

Amount of carbon released: $0.216 \times 1400 = \boxed{302\ \text{GtC}}$

Total fossil fuel emissions since 1751: 580 GtC

To hold the GMT at 2°C, we will have to withdraw 300 GtC from the atmosphere as the permafrost melts.

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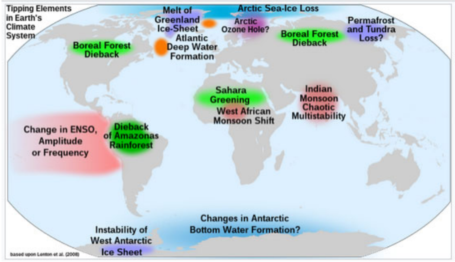
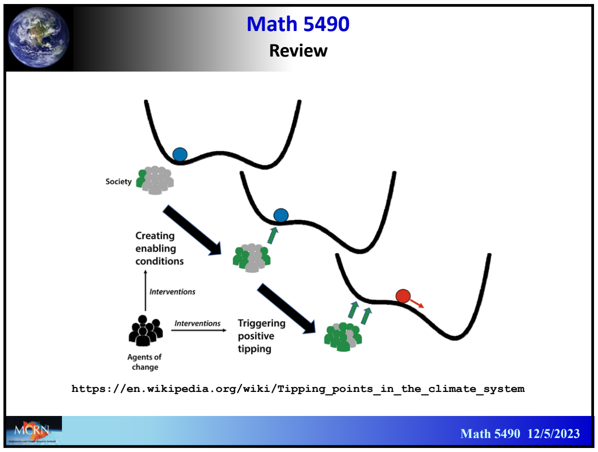


`https://en.wikipedia.org/wiki/Tipping_points_in_the_climate_system`



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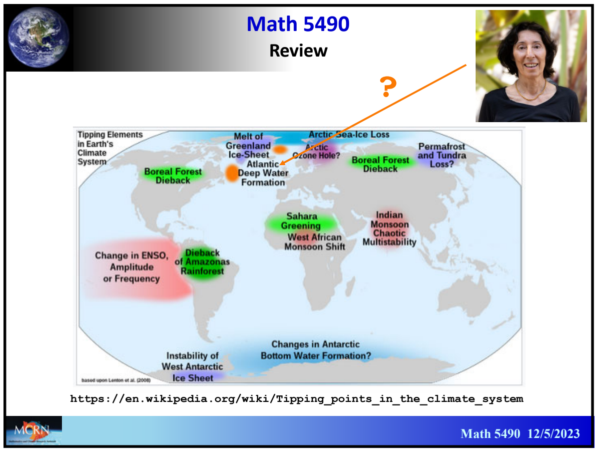

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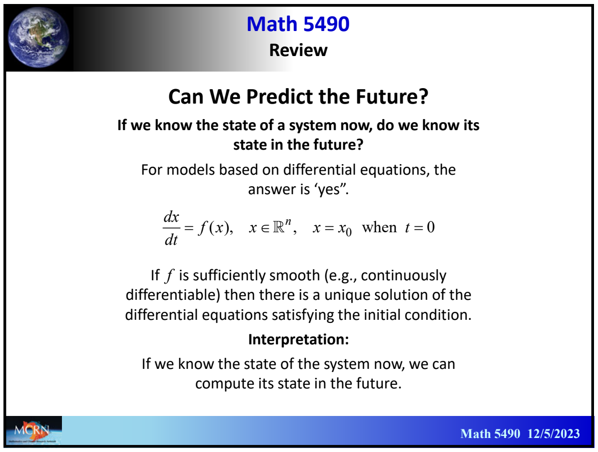

Math 5490
Review

Can We Predict the Future?

If we know the state of a system now, do we know its state in the future?

For models based on differential equations, the answer is 'yes".

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad x = x_0 \text{ when } t = 0$$

If f is sufficiently smooth (e.g., continuously differentiable) then there is a unique solution of the differential equations satisfying the initial condition.

Interpretation:

If we know the state of the system now, we can compute its state in the future.

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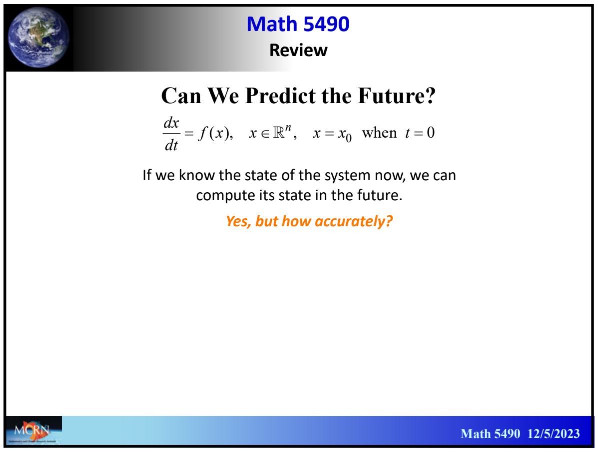

Math 5490
Review

Can We Predict the Future?

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad x = x_0 \text{ when } t = 0$$

If we know the state of the system now, we can compute its state in the future.

Yes, but how accurately?

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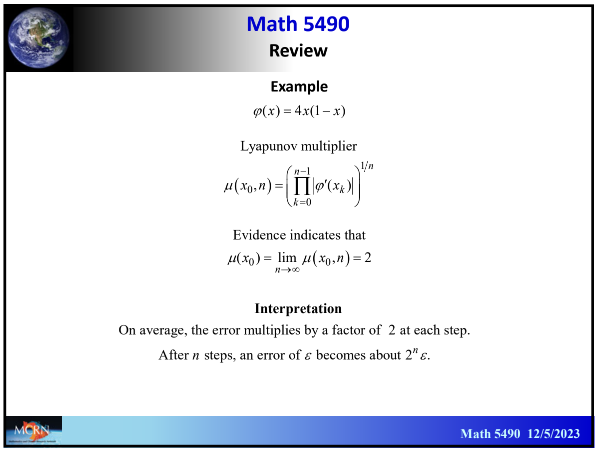

Math 5490
Review

Example

$$\varphi(x) = 4x(1-x)$$

Lyapunov multiplier

$$\mu(x_0,n) = \left(\prod_{k=0}^{n-1} |\varphi'(x_k)|\right)^{1/n}$$

Evidence indicates that

$$\mu(x_0) = \lim_{n\to\infty} \mu(x_0,n) = 2$$

Interpretation

On average, the error multiplies by a factor of 2 at each step.

After n steps, an error of ε becomes about $2^n\varepsilon$.

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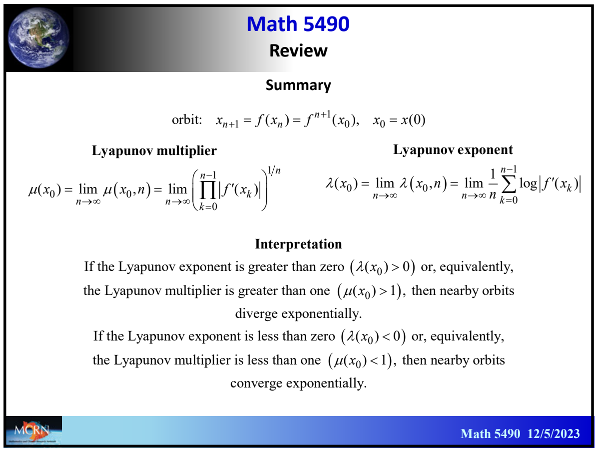

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Review

Summary

orbit: $x_{n+1} = f(x_n) = f^{n+1}(x_0), \quad x_0 = x(0)$

Lyapunov multiplier

$$\mu(x_0) = \lim_{n\to\infty} \mu(x_0,n) = \lim_{n\to\infty}\left(\prod_{k=0}^{n-1} |f'(x_k)|\right)^{1/n}$$

Lyapunov exponent

$$\lambda(x_0) = \lim_{n\to\infty} \lambda(x_0,n) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log|f'(x_k)|$$

Interpretation

If the Lyapunov exponent is greater than zero $(\lambda(x_0) > 0)$ or, equivalently, the Lyapunov multiplier is greater than one $(\mu(x_0) > 1)$, then nearby orbits diverge exponentially.

If the Lyapunov exponent is less than zero $(\lambda(x_0) < 0)$ or, equivalently, the Lyapunov multiplier is less than one $(\mu(x_0) < 1)$, then nearby orbits converge exponentially.

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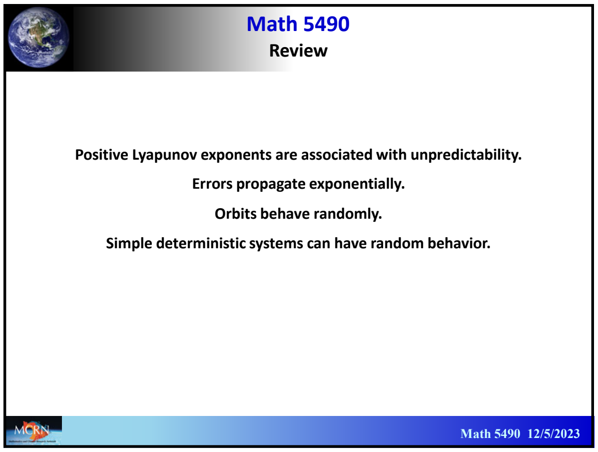

Math 5490

Review

Positive Lyapunov exponents are associated with unpredictability.

Errors propagate exponentially.

Orbits behave randomly.

Simple deterministic systems can have random behavior.

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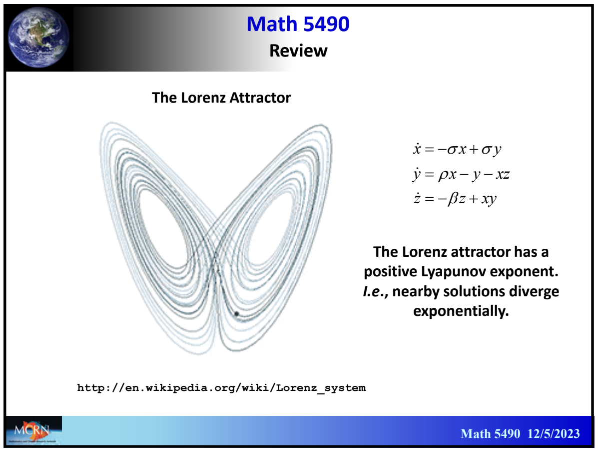

Math 5490

Review

The Lorenz Attractor

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = -\beta z + xy$$

The Lorenz attractor has a positive Lyapunov exponent. *I.e.*, nearby solutions diverge exponentially.

`http://en.wikipedia.org/wiki/Lorenz_system`

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