Bases for Cluster Algebras from Surfaces

Gregg Musiker (U. Minnesota), Ralf Schiffler (U. Conn.),
and Lauren Williams (UC Berkeley)

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http//math.umn.edu/∼ musiker/BAD12.pdf

Outline.

1. Introduction to Cluster Algebras.
2. The Positivity and Atomic Bases Conjectures.
3. Examples of Cluster Algebras from Surfaces.
4. Our Results.
5. Recent Developments.

`http//math.umn.edu/~ musiker/BAD12.pdf`

Introduction to Cluster Algebras

In the late 1990's: Fomin and Zelevinsky were studying total positivity and canonical bases of algebraic groups. They noticed recurring combinatorial and algebraic structures.

Let them to define cluster algebras, which have now been linked to quiver representations, Poisson geometry Teichmüller theory, tilting theory, mathematical physics, discrete integrable systems, and other topics.

Cluster algebras are a certain class of commutative rings which have a distinguished set of generators that are grouped into overlapping subsets, called clusters, each having the same cardinality.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

Generators:

Specify an initial finite set of them, a Cluster, $\{x_1, x_2, \ldots, x_{n+m}\}$.

Construct the rest via Binomial Exchange Relations:

$$x_\alpha x'_\alpha = \prod x_{\gamma_i}^{d_i^+} + \prod x_{\gamma_i}^{d_i^-}.$$

The set of all such generators are known as Cluster Variables, and the initial pattern B of exchange relations determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Example: Coordinate Ring of Grassmannian$(2, n+3)$

Let $Gr_{2,n+3} = \{V | V \subset \mathbb{C}^{n+3},\ \dim V = 2\}$ planes in $(n+3)$-space

Elements of $Gr_{2,n+3}$ represented by 2-by-$(n+3)$ matrices of full rank.

Plücker coordinates $p_{ij}(M) =$ det of 2-by-2 submatrices in columns i and j.

The coordinate ring $\mathbb{C}[Gr_{2,n+3}]$ is generated by all the p_{ij}'s for $1 \leq i < j \leq n+3$ subject to the Plücker relations given by the 4-tuples

$$p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk} \text{ for } i < j < k < \ell.$$

Claim. $\mathbb{C}[Gr_{2,n+3}]$ has the structure of a cluster algebra. Clusters are each maximal algebraically independent sets of p_{ij}'s.

Each have size $(2n+3)$ where $(n+3)$ of the variables are frozen and n of them are exchangeable.

Example: Coordinate Ring of Grassmannian$(2, n+3)$

Cluster algebra structure of $Gr_{2,n+3}$ as a triangulated $(n+3)$-gon.

Frozen Variables / Coefficients $\longleftrightarrow$ sides of the $(n+3)$-gon

Cluster Variables $\longleftrightarrow \{p_{ij} : |i-j| \neq 1 \mod (n+3)\} \longleftrightarrow$ diagonals

Seeds $\longleftrightarrow$ triangulations of the $(n+3)$-gon

Clusters $\longleftrightarrow$ Set of p_{ij}'s corresponding to a triangulation

Can exchange between various clusters by flipping between triangulations.

This is called mutation, and we will present a detailed example later.

Another Example: Rank 2 Cluster Algebras

Let $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$, $b, c \in \mathbb{Z}_{>0}$. $(\{x_1, x_2\}, B)$ is a seed for a cluster algebra $\mathcal{A}(b, c)$ of rank 2.

$$\mu_1(B) = \mu_2(B) = -B \text{ and } x_1 x_1' = x_2^c + 1, \quad x_2 x_2' = 1 + x_1^b.$$

Thus the cluster variables in this case are

$$\{x_n : n \in \mathbb{Z}\} \text{ satisfying } x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 \text{ if } n \text{ is odd} \\ x_{n-1}^c + 1 \text{ if } n \text{ is even} \end{cases}.$$

Example ($b = c = 1$): (Finite Type, of Type A_2)

$$x_3 = \frac{x_2 + 1}{x_1}. \quad x_4 = \frac{x_3 + 1}{x_2} = \frac{\frac{x_2+1}{x_1} + 1}{x_2} = \frac{x_1 + x_2 + 1}{x_1 x_2}.$$

$$x_5 = \frac{x_4 + 1}{x_3} = \frac{\frac{x_1+x_2+1}{x_1 x_2} + 1}{(x_2 + 1)/x_1} = \frac{x_1(x_1 + x_2 + 1 + x_1 x_2)}{x_1 x_2 (x_2 + 1)} = \frac{x_1 + 1}{x_2}. \quad x_6 = x_1.$$

Another Example: Rank 2 Cluster Algebras

Example ($b = c = 2$): (Affine Type, of Type $\widetilde{A}_1$)

$$x_3 = \frac{x_2^2 + 1}{x_1}. \quad x_4 = \frac{x_3^2 + 1}{x_2} = \frac{x_2^4 + 2x_2^2 + 1 + x_1^2}{x_1^2 x_2}.$$

$$x_5 = \frac{x_4^2 + 1}{x_3} = \frac{x_2^6 + 3x_2^4 + 3x_2^2 + 1 + x_1^4 + 2x_1^2 + 2x_1^2 x_2^2}{x_1^3 x_2^2}, \quad \ldots$$

If we let $x_1 = x_2 = 1$, we obtain $\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}$.

The next number in the sequence is $x_7 = \frac{34^2+1}{13} = \frac{1157}{13} = 89$, an integer!

The Positivity Conjecture of Fomin and Zelevinsky

Theorem. (The Laurent Phenomenon FZ 2001) For any cluster algebra defined by initial seed $(\{x_1, x_2, \ldots, x_{n+m}\}, B)$, all cluster variables of $\mathcal{A}(B)$ are Laurent polynomials in $\{x_1, x_2, \ldots, x_{n+m}\}$
(with no coefficient $x_{n+1}, \ldots, x_{n+m}$ in the denominator).

Because of the Laurent Phenomenon, any cluster variable x_α can be expressed as $\frac{P_\alpha(x_1,\ldots,x_{n+m})}{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}$ where $P_\alpha \in \mathbb{Z}[x_1, \ldots, x_{n+m}]$ and the α_i's $\in \mathbb{Z}$.

Conjecture. (FZ 2001) For any cluster variable x_α and any initial seed (i.e. initial cluster $\{x_1, \ldots, x_{n+m}\}$ and initial exchange pattern B), the polynomial $P_\alpha(x_1, \ldots, x_n)$ has nonnegative integer coefficients.

Positivity for Cluster Algebras from Surfaces

Theorem (Fomin-Shapiro-Thurston 2006), (Based on earlier work of Fock-Goncharov and Gekhtman-Shapiro-Vainshtein)

Given a Riemann surface with marked points (S, M), one can define a corresponding cluster algebra $\mathcal{A}(S, M)$.

$$\text{Seed} \leftrightarrow \text{Triangulation } T = \{\tau_1, \tau_2, \ldots, \tau_n\}$$

$$\text{Cluster Variable} \leftrightarrow \text{Arc } \gamma \ (x_i \leftrightarrow \tau_i \in T)$$

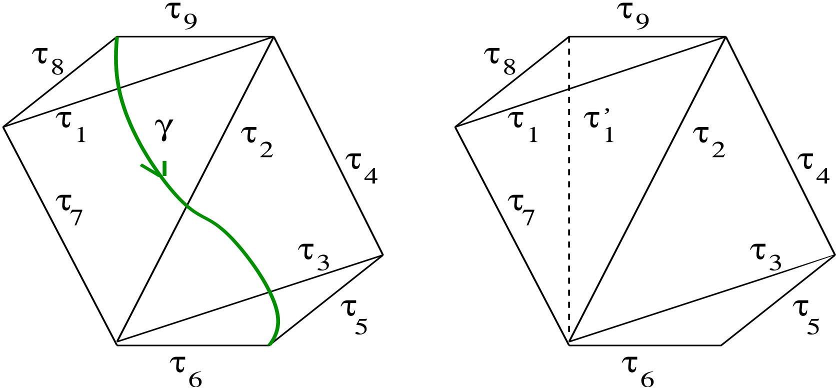
Cluster Mutation (Binomial Exchange Relations) $\leftrightarrow$ Flipping Diagonals.

Theorem. (M-Schiffler-Williams 2009) Let $\mathcal{A}(S, M)$ be any cluster algebra arising from a surface (with or without punctures), where the coefficient system is of geometric type, and let Σ be any initial seed.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients. Proof via explicit combinatorial formulas in terms of graphs.

Example of Hexagon (Type A_3 or $\mathbb{C}[Gr_{2,6}]$)

Consider the triangulated hexagon (S, M) by the triangulation T.



$$\begin{aligned}
x_1 x_1' &= (x_7 x_9) + x_2(x_8) \\
x_2 x_2'' &= x_3(x_9) + x_1'(x_4) \\
x_3 x_3''' &= x_2''(x_6) + x_1'(x_5)
\end{aligned}$$

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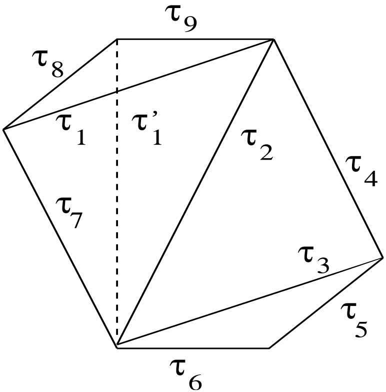

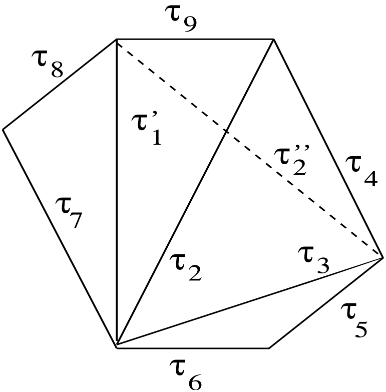

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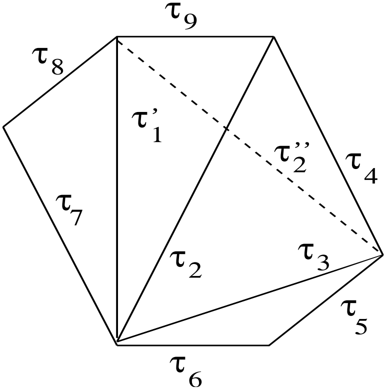

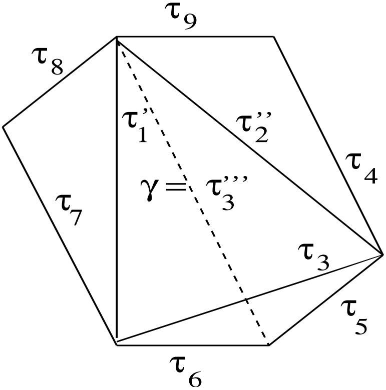

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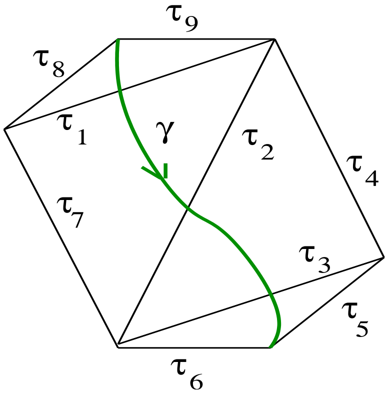
Example of Hexagon (Type A_3 or $\mathbb{C}[Gr_{2,6}]$)

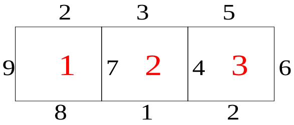


By using the Ptolemy exchange relations on τ_1, τ_2, then τ_3, we obtain

$$\begin{aligned} x_3''' &= x_\gamma = \frac{1}{x_1 x_2 x_3}\Big(x_2^2(x_5 x_8) + x_2(x_5 x_7 x_9) + x_2(x_4 x_6 x_8) \\ &+ (x_4 x_6 x_7 x_9) + x_1 x_3 (x_6 x_9) \Big). \end{aligned}$$

Example of Hexagon (Type A_3 or $\mathbb{C}[Gr_{2,6}]$)

Consider the graph $G_{T_H,\gamma} =$

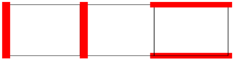


$G_{T_H,\gamma}$ has five perfect matchings $(x_4, x_5, \ldots, x_9 = 1)$:

$(x_9)x_1x_3(x_6)$,

$(x_9x_7x_4x_6)$,

$x_2(x_8)(x_4x_6)$,



$(x_9x_7)x_2(x_5)$,

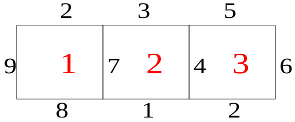
$x_2(x_8)x_2(x_5)$.

$$\frac{x_1x_3y_1y_2y_3+y_1y_3+x_2y_3+x_2y_1+x_2^2}{x_1x_2x_3}$$

A perfect matching $M \subseteq E$ is a set of distinguished edges so that every vertex of V is covered exactly once. The weight of a matching M is the product of the weights of the constituent edges, i.e. $x(M) = \prod_{e \in M} x(e)$.

Example of Hexagon (Type A_3 or $\mathbb{C}[Gr_{2,6}]$)

Consider the graph $G_{T_H,\gamma} =$



$G_{T_H,\gamma}$ has five perfect matchings $(x_4, x_5, \ldots, x_9 = 1)$:

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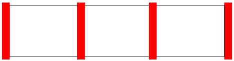

$(x_9x_7x_4x_6)$,

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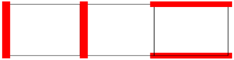


$(x_9x_7)x_2(x_5)$,

$x_2(x_8)x_2(x_5)$.

$$\frac{x_1x_3y_1y_2y_3 + y_1y_3 + x_2y_3 + x_2y_1 + x_2^2}{x_1x_2x_3}$$

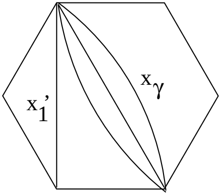
These five monomials exactly match those appearing in the numerator of the expansion of x_γ. The denominator of $x_1x_2x_3$ corresponds to the labels of the three tiles. The y_i's correspond to principal coefficients (heights).

Cluster Monomials

Definition: A cluster monomial is a product of cluster variables all coming from the same cluster.

For example, in the case of a cluster algebra from a surface (e.g. a cluster algebra of type A_n corresponding to an $(n+3)$-gon):

A cluster monomial corresponds to a multidissection D (i.e. a partial triangulation allowing multiple copies of edges). We define $x_D = \prod_{\gamma \in D} x_\gamma$.

E.g. $D =$ 

, $x_D = x_1' x_\gamma^3 = (\frac{1+x_2}{x_1})(\frac{x_1x_3+1+2x_2+x_2^2}{x_1x_2x_3})^3$ in the coordinates of the triangulation T.

Since each x_γ is positive with respect to any cluster, i.e. initial triangulation, so is each x_D.

The Positive Cone of a Cluster Algebra and Atomic Bases

Definition: Given a cluster algebra $\mathcal{A}$, let $\mathcal{C}_{\mathcal{A}}$ denote the positive cone of $\mathcal{A}$, i.e. the subset of elements that are positive Laurent polynomials when expanded with respect to any initial cluster and seed.

Corollary (M-Schiffler-Williams 2009) For a cluster algebra from a surface, any cluster monomial is in the positive cone $\mathcal{C}_{\mathcal{A}}$.

Definition: A basis $\mathcal{B}$ for a cluster algebra $\mathcal{A}$ is an Atomic Basis if:

1. Every element of $\mathcal{B}$ is in the positive cone $\mathcal{C}_{\mathcal{A}}$.
2. $\mathcal{B}$ is a basis in the usual sense, i.e. linearly independent and a spanning set for $\mathcal{A}$.
3. $\mathcal{B}$ includes the set of cluster monomials.
4. $\mathcal{B}$ is the set of positive indecomposable elements of $\mathcal{C}_{\mathcal{A}}$.

Remark: Positivity of cluster variables (i.e. monomials) not always known and other potential obstacles to existence of atomic bases.

Atomic Bases

Theorem (Cerulli Irelli 2011; Dupont-Thomas 2011)
The set of cluster monomials is an atomic basis for a cluster algebra of type A_n.

Remark: Cerulli Irelli's proof is representation theoretic and extends to all finite skew-symmetric type (ADE).

Dupont-Thomas's proof is combinatorial and they have a related result for affine $\widetilde{A}_n$ that we will state momentarily.

Cluster Monomials of Rank Two Cluster Algebras

In the case of a rank two cluster algebra $\mathcal{A}(b, c)$ with $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$:

$$\text{Clusters} \longleftrightarrow \{x_n, x_{n+1}\} \text{ for } n \in \mathbb{Z}.$$

$$\text{Cluster Monomials} \longleftrightarrow x_n^k x_{n+1}^\ell \text{ for } k, \ell \geq 0.$$

Again, each cluster monomial is a positive Laurent polynomial with respect to **any** choice of initial cluster.

However, let us now return to the case $(b = c = 2)$: In $\mathcal{A}(2, 2)$, we have cluster variables $x_0 = \frac{x_1^2+1}{x_2}$ and $x_3 = \frac{x_2^2+1}{x_1}$. Consider the element $z = x_0 x_3 - x_1 x_2$. If we expand z in the cluster $\{x_1, x_2\}$, we obtain

$$z = \frac{x_1^2 x_2^2 + x_1^2 + x_2^2 + 1}{x_1 x_2} - \frac{x_1^2 x_2^2}{x_1 x_2} = \frac{x_1^2 + x_2^2 + 1}{x_1 x_2}.$$

By symmetry, z expands as $\frac{x_n^2 + x_{n+1}^2 + 1}{x_n x_{n+1}}$ w.r.t. any cluster $\{x_n, x_{n+1}\}$.

Atomic Bases for the Cluster Algebra $\mathcal{A}(2,2)$

Consequently, z is in the positive cone, but we either have

(i) z is not in the positive span of the cluster monomials, or

(ii) the cluster monomials are not linearly independent .

Either way, for $\mathcal{A}(2,2)$, the **cluster monomials are not an atomic basis**.

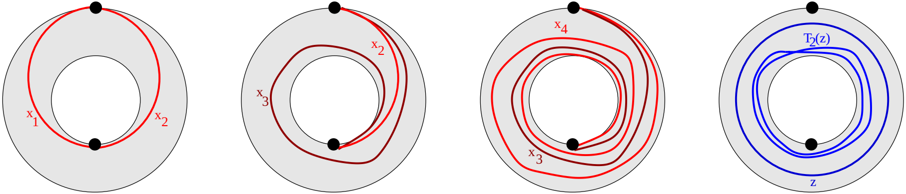
Definition: Let $T_n(x)$ denote the (normalized) Chebyshev polynomial of the first kind, satisfying $T_0(x) = 2$, $T_1(x) = x$, and $T_n(x) = xT_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$.

Theorem (Sherman-Zelevinsky 2004): The set

$$\{\text{cluster monomials}\} \cup \{\, T_n(z) : n \geq 1\}$$

is an atomic basis for $\mathcal{A}(2,2)$.

The cluster algebra $\mathcal{A}(2,2)$ can also be viewed as coming from an annulus with a marked point on each boundary:

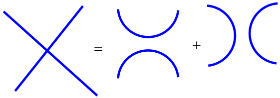


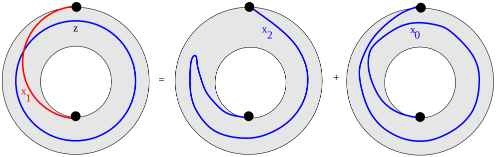
Cluster variables x_n's correspond to arcs that wind around the annulus,

and $T_n(z)$'s correspond to bracelets that avoid marked points but are closed loops in the interior of the annulus.

Why closed loops make sense as cluster algebra elements

For general cluster algebras from surfaces, can consider closed loops γ. The corresponding cluster algebra element x_γ can be computed using the Skein Relations:





Consequently, w.r.t. any triangulation T, x_γ is a well-defined (not obvious from above description) Laurent polynomial.

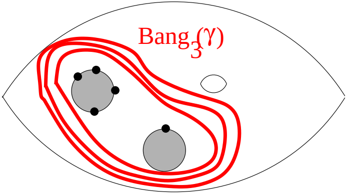
$$(\mathcal{A}(2,2) \text{ Example: } x_1 z = x_2 + x_0 \text{ implies } z = \frac{x_2 + \frac{x_1^2+1}{x_2}}{x_1} = \frac{x_2^2 + x_1^2 + 1}{x_1 x_2}.)$$

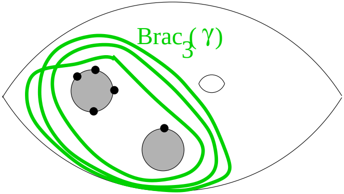
Remark: Defining z as the closed loop for other annulli, Dupont-Thomas prove for all annuli (cluster algebras of affine type $\widetilde{A}_n$) that $\{\text{cluster monomials}\} \cup \{T_n(z) : n \geq 1\}$ is an atomic basis.

(The proof uses a mix of combinatorics and representation theory.)

The example of the annulus motivates the following definitions:

Definition: Given a closed loop γ in a marked surface (S, M), with no self-crossings, that is disjoint from the boundary of S and M, we define a bangle $Bang_k\gamma$ to be the union of k copies of γ, up to isotopy.





Easy Claim: $x_{Bang_k\gamma} = x_\gamma^k$.

Definition Given γ a closed loop, as above, a bracelet $Brac_k\gamma$ is a single closed loop obtained by concatenating γ with itself exactly k times, creating $(k-1)$ self-intersections.

Harder Claim: $x_{Brac_k\gamma} = T_k(x_\gamma)$.

Atomic Basis Conjecture for Surfaces

Define a generalized cluster monomial of a surface (S, M) to be a collection of closed loops and mulit-dissections without any pairwise- or self-crossings. Then replace each k-bangle by a k-bracelet.

Conjecture: (Fomin-Shapiro-Thurston, based on earlier work of Fock-Goncharov)
This resulting collection $\{x_\alpha\}$, the bracelets collection is an **atomic basis** for $\mathcal{A}(S, M)$.

Progress Towards This:

Theorem (M-Schiffler-Williams 2011) For a cluster algebra from a surface (with proper coefficients) without punctures (internal marked points) and at least two marked points, the bangles collection and bracelets collection are both in fact bases and all of their elements are in the positive cone.

What's still open

Recall:

Definition: A basis $\mathcal{B}$ for a cluster algebra $\mathcal{A}$ is an Atomic Basis if:

1. Every element of $\mathcal{B}$ is in the positive cone $\mathcal{C}_{\mathcal{A}}$.
2. $\mathcal{B}$ is a basis in the usual sense, i.e. linearly independent and a spanning set for $\mathcal{A}$.
3. $\mathcal{B}$ includes the set of cluster monomials.
4. $\mathcal{B}$ is the set of positive indecomposable elements of $\mathcal{C}_{\mathcal{A}}$.

Our Theorem settles properties (1), (2), and (3). However, proving that the Bracelets Basis is the set of positive indecomposables is tricky.

For surfaces, known in cases A_n (disk with $(n-3)$ points marked on boundary), D_n (once-punctured disk) and $\widetilde{A}_n$ (annulus). but wide open otherwise.

Remark: Cerulli Irelli also has a proof for the $\widetilde{A}_2$ case, an example also covered by Dupont-Thomas's more recent result.

Recent Developments

Further Evidence for Conjecture:

Theorem (Thurston 2012) The bracelets basis $\mathcal{B}$ is **strongly positive**, i.e. any product of elements of the basis decomposes as a linear combination of elements of $\mathcal{B}$ with positive structure constants.

Remark: The bangles basis on the other hand, does **not** have the strong positivity property.

Example: In $\mathcal{A}(2,2)$, the product x_1x_6 decomposes as

$$x_1x_6 = x_3x_4 + z^3 - z$$

in the bangles basis, although it decomposes as

$$x_1x_6 = x_3x_4 + T_3(z) + 2z$$

in the bracelets basis. (Thanks to Pylyavskyy.)

Recent Developments (Continued)

Remark: Our proof only proved linear independence in the presence of principal coefficients or another full-rank coefficient system. (We parametrize the basis elements by g-vectors, thus showing that each element of $\mathcal{B}$ has its own leading term.)

Geiss-Labardini-Schröer recently used categorification to prove that

(i) The Bangles Basis equals the Generic Basis constructed via generic representations and the Caledro-Chapoton map.

(ii) This method shows linear independence even in the coefficient-free case.

Recent Developments (Continued)

Remark: We prove spanning by showing that all closed loops lie in the cluster algebra when there are at least two marked points.

Corollary (M-Schiffler-Williams 2011) Combining with results of Fock and Goncharov, we show that the upper cluster algebra and cluster algebra coincide for unpunctured surfaces with at least two marked points.

Theorem (Muller 2012) An independent proof of the coincidence of these algebras.

For genus $g \geq 1$ surfaces with exactly one marked point, even finding a candidate for an atomic basis is still open.

Thank You For Listening

Bases for Cluster Algebras from Surfaces (with Ralf Schiffler and Lauren Williams), `arXiv:math.CO/1110.4364` (To appear in Compos. Math.)

Matrix Formulae and Skein Relations for Cluster Algebras from Surfaces (with Lauren Williams), `arXiv:math.CO/1108.3382`
(To appear in Int. Math. Res. Notices)

Positivity for Cluster Algebras from Surfaces (with Ralf Schiffler and Lauren Williams), *Adv. Math.*, **227** (2011), Issue 6, 2241–2308
`arXiv:math.CO/0906.0748`

Slides Available at `http//math.umn.edu/`$\sim$ `musiker/BAD12.pdf`