Invariants, Kronecker Products
and
Combinatorics of some Remarkable Diophantine systems
by
A. Garsia, G. Musiker, N. Wallach, G. Xin

Abstract. This work lies across three areas of investigation that are by themselves of independent interest. A problem that arose in quantum computing [6], [7] led us to a link that tied these areas together. This link consists of a single formal series with a multifaced interpretation. The deeper exploration of this link yielded results as well as methods for solving some numerical problems in each of these separate areas.

Introduction

Since our work may be of interest to audiences of varied background we will try to keep our notation as elementary as possible and entirely self contained.

The problem in invariant theory that was the point of departure in our investigation is best stated in its simplest and most elementary version. Given two matrices $\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and $\mathbf{B} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$ in $GL[2] := GL(2, \mathbb{C})$, we recall that their tensor product may be written in the block form

$$\mathbf{A} \otimes \mathbf{B} \;\; = \;\; \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} \end{bmatrix}. \qquad \text{I.1}$$

We also recall that the action of a matrix $\mathbf{M} = \|m_{ij}\|_{i,j=1}^{n}$ on a polynomial $P(x) \in \mathbf{R}_n = \mathbb{C}[x_1, x_2, \ldots, x_n]$ may be defined by setting

$$T_M P(x) \;\; = \;\; P(xM), \qquad \text{I.2}$$

where the symbol xM is to be interpreted as multiplication of a row n-vector by an $n \times n$ matrix. This given, we denote by $\mathbf{R}_4^{GL[2]\otimes GL[2]}$ the ring of polynomials in $\mathbf{R}_4$ that are invariant under the action of $\mathbf{A} \otimes \mathbf{B}$ for all pairs $\mathbf{A}, \mathbf{B} \in GL[2]$. In symbols

$$\mathbf{R}_4^{GL[2]\otimes GL[2]} \;\; = \;\; \big\{ P \in \mathbf{R}_4 \, : \, T_{\mathbf{A}\otimes\mathbf{B}} P(x) = P(x) \,\big\}. \qquad \text{I.3}$$

Since the action in I.2 preserves degree and homogeneity, then $\mathbf{R}_4^{GL[2]\otimes GL[2]}$, is graded, and as a vector space it decomposes into the direct sum

$$\mathbf{R}_4^{GL[2]\otimes GL[2]} \;\; = \;\; \bigoplus_{m \geq 0} \mathcal{H}_m\big(\mathbf{R}_4^{GL[2]\otimes GL[2]}\big),$$

where the m^{th} direct summand here denotes the subspace consisting of the $GL[2] \otimes GL[2]$-invariants that are homogenous of degree m. The natural problem then arises to determine the generating function

$$W_2(q) \;\; = \;\; \sum_{m \geq 0} q^m \dim \mathcal{H}_m\big(\mathbf{R}_4^{GL[2]\otimes GL[2]}\big).$$

Now note that using I.1 iteratively we can define the k-fold tensor product

$$\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \cdots \otimes \mathbf{A}_k \, ,$$

thus extend I.3 to its general form

$$\mathbf{R}_{2^k}^{GL[2]\otimes GL[2]\otimes\cdots\otimes GL[2]} \;=\; \big\{ P\in \mathbf{R}_{2^k} \;:\; T_{\mathbf{A}_1\otimes\mathbf{A}_2\otimes\cdots\otimes\mathbf{A}_k} P(x)=P(x)\,\big\} \qquad \text{I.4}$$

and set

$$W_k(q) \;=\; \sum_{m\geq 0} q^m \dim \mathcal{H}_m\big(\mathbf{R}_{2^k}^{GL[2]\otimes GL[2]\otimes\cdots\otimes GL[2]}\,\big). \qquad \text{I.5}$$

Remarkably, to this date only the series $W_2(q), W_3(q), W_4(q), W_5(q)$ are known explicitly. Moreover, although the three series $W_2(q), W_3(q), W_4(q)$ may be hand computed, so far $W_5(q)$ has only been obtained by computer.

The third named author using branching tables calculated to obtain the results in [7] was able to predict the explicit form of $W_5(q)$ by computing a sufficient number of its coefficients. The computation of those branching tables took approximately 50 hours using an array of 9 computers.

The series $W_4(q), W_5(q)$ first appeared in print in works of Luque-Thibon [4] & [5] which were motivated by the same problem of quantum computing. We understand that their computation of $W_5(q)$ was carried out by a brute force use of the partial fraction Algorithm of the fourth named author, and it required several hours with the computers of that time.

The present work was carried out whilst unaware of the work of Luque-Thibon. Our main goal is to acquire a theoretical understanding of the combinatorics underlying such Hilbert series and give a more direct construction of $W_5(q)$ and perhaps bring $W_6(q)$ within reach of present computers.

Fortunately, as is often the case with a difficult problem, the methods that are developed to solve it may be more significant than the problem itself. This is no exception as we shall see.

Let us recall that the pointwise product of two characters $\chi^{(1)}$ and $\chi^{(2)}$ of the symmetric group S_n is also a character of S_n, and we shall denote it here by $\chi^{(1)} * \chi^{(2)}$. This is usually called the "*Kronecker*" product of $\chi^{(1)}$ and $\chi^{(2)}$. An outstanding yet unsolved problem is to obtain a combinatorial rule for the computation of the integer

$$c^{\lambda}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}} \qquad \text{I.6}$$

giving the multiplicity of χ^λ in the Kronecker product $\chi^{\lambda^{(1)}} * \chi^{\lambda^{(2)}} * \cdots * \chi^{\lambda^{(k)}}$. Here χ^λ and each $\chi^{\lambda^{(i)}}$ are irreducible Young characters of S_n. Using the Frobenius map $\mathbf{F}$ that sends the irreducible character χ^λ onto the Schur function S_λ, we can define the Kronecker product of two homogeneous symmetric functions of the same degree f and g by setting

$$f * g \;=\; \mathbf{F}\big((\mathbf{F}^{-1}f) * (\mathbf{F}^{-1}g)\big).$$

With this notation the coefficient in I.6 may also be written in the form

$$c^{\lambda}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}} \;=\; \big\langle s_{\lambda^{(1)}} * s_{\lambda^{(2)}} * \cdots * s_{\lambda^{(k)}} \;,\; s_\lambda \big\rangle, \qquad \text{I.7}$$

where $\langle\;,\;\rangle$ denotes the customary Hall scalar product of symmetric polynomials. The relevancy of all this to the previous problem is a consequence of the following identity.

Theorem I.1

$$W_k(q) \;=\; \sum_{d\geq 0} q^{2d}\big\langle s_{d,d} * s_{d,d} * \cdots * s_{d,d}\;,\; s_{2d}\big\rangle \tag{I.8}$$

where, in each term, the Kronecker product has k factors.

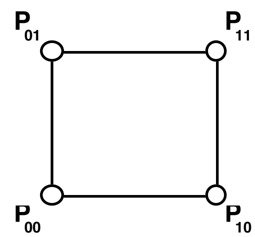
For this reason, here and after we will refer to the task of constructing $W_k(q)$ as the "*Sdd Problem* ".

Using this connection and some auxiliary results on the Kronecker product of symmetric function we derived in [2] that

$$W_2(q) \;=\; \frac{1}{1-q^2}\,,\qquad W_3(q) \;=\; \frac{1}{1-q^4}\,,\qquad W_4(q) \;=\; \frac{1}{(1-q^2)(1-q^4)^2(1-q^6)}. \tag{I.8}$$

Although this approach is worth pursuing (see [2]), the present investigation led us to another surprising facet of this problem.

Again we will start with a special case. We are asked to place integers weights on the vertices of the unit square so that all the sides have equal weights. Denoting the vertices P_{00}, P_{01}, P_{10}, P_{11} (see figure) and their weights p_{00}, p_{01}, p_{10}, p_{11} we are led to the following Diophantine system



$$\mathcal{S}_2 : \left\| \begin{array}{l} p_{00} + p_{01} \;-\; p_{10} - p_{11} = 0 \\ p_{00} - p_{01} \;+\; p_{10} - p_{11} = 0 \end{array}\right. .$$

The general solution to this problem may be expressed as the formal series

$$F_2(y_{00}, y_{01}, y_{10}, y_{11}) \;=\; \sum_{p\in\mathcal{S}_2} y_{00}^{p_{00}} y_{01}^{p_{01}} y_{10}^{p_{10}} y_{11}^{p_{11}} \;=\; \frac{1}{(1-y_{00}y_{11})(1-y_{01}y_{10})}. \tag{I.9}$$

In particular, making the substitution $y_{00} = y_{01} = y_{10} = y_{11} = q$ we derive that the enumerator of solutions by total weight is given by the generating function

$$G_2(q) \;=\; \sum_{d\geq 0} m_d(2) q^{2d} \;=\; \frac{1}{(1-q^2)^2}.$$

with $m_d(2)$ giving the number of solutions of total weight $2d$.

This problem generalizes to arbitrary dimensions. That is we seek to enumerate the distinct ways of placing weights on the vertices of the unit k-dimensional hypercube so that all hyperfaces have the same weight. Denoting by $p_{\epsilon_1\epsilon_2\cdots\epsilon_k}$ the weight we place on the vertex of coordinates $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ we obtain a Diophantine system $\mathcal{S}_k$ of k equations in the 2^k variables $\{p_{\epsilon_1\epsilon_2\cdots\epsilon_k}\}_{\epsilon_i=0,1}$.

For instance, using this notation, for the 3-dimensional cube we obtain the system

$$\mathcal{S}_3 : \left\| \begin{array}{l} p_{000} + p_{001} + p_{010} + p_{011} - p_{100} - p_{101} - p_{110} - p_{111} \;=\; 0 \\ p_{000} + p_{001} - p_{010} - p_{011} + p_{100} + p_{101} - p_{110} - p_{111} \;=\; 0 \\ p_{000} - p_{001} + p_{010} - p_{011} + p_{100} - p_{101} + p_{110} - p_{111} \;=\; 0 \end{array}\right. .$$

In this case the enumerator of solutions by total weight is

$$G_3(q) \;=\; \sum_{d\geq 0} m_d(3) q^{2d} \;=\; \frac{1-q^8}{(1-q^2)^4(1-q^4)^2}.$$

The relevance of all this to the previous problem is a consequence of the following identity

Theorem I.2

Denoting by $m_d(k)$ the number of solutions of the system $\mathcal{S}_k$ of total weight $2d$ and setting

$$G_k(q) \;=\; \sum_{d\geq 0} m_d(k) q^{2d}, \tag{I.10}$$

we have

$$G_k(q) \;=\; \sum_{d\geq 0} q^{2d} \big\langle h_{d,d} * h_{d,d} * \cdots * h_{d,d} \,,\; S_{2d} \big\rangle, \tag{I.11}$$

where, $h_{d,d}$ denotes the homogenous basis element indexed by the two part partition (d,d) , and in each term, the Kronecker product has k factors.

For this reason, we will refer to the task of constructing the series $G_k(q)$ as the "*Hdd Problem* ".

Theorem I.2 shows that the algorithmic machinery of Diophantine analysis may be used in the construction of generating functions of Kronecker coefficients as well as Hilbert series of ring of invariants. More precisely we are referring here to the "*constant term methods*" of MacMahon partition analysis which have been recently translated into computer software by Andrews et al [1] and G. Xin in [8].

To see what this leads to, we start by noting that using MacMahon's approach the solutions of $\mathcal{S}_2$ may be obtained by the following identity

$$F_2(y_{00},y_{01},y_{10},y_{11}) \;=\; \sum_{p\in\mathcal{S}_2} y_{00}^{p_{00}} y_{01}^{p_{01}} y_{10}^{p_{10}} y_{11}^{p_{11}} a_1^{p_{00}+p_{01}\,-\,p_{10}-p_{11}} a_2^{p_{00}-p_{01}\,+\,p_{10}-p_{11}} \Big|_{a_1^0 a_2^0},$$

where the symbol "$\Big|_{a_1^0 a_2^0}$" denotes the operator of taking the constant term in a_1, a_2. This identity may also be written in the form

$$F_2(y_{00},y_{01},y_{10},y_{11}) \;=\; \frac{1}{(1-y_{00}a_1a_2)(1-y_{01}a_1/a_2)(1-y_{10}a_2/a_1)(1-y_{11}/a_1a_2)} \Big|_{a_1^0 a_2^0}.$$

In particular the enumerator of the solutions of $\mathcal{S}_2$ by total weight may be computed from the identity

$$G_2(q) \;=\; \frac{1}{(1-qa_1a_2)(1-qa_1/a_2)(1-qa_2/a_1)(1-q/a_1a_2)} \Big|_{a_1^0 a_2^0}.$$

More generally we have

$$G_k(q) \;=\; \frac{1}{\prod_{S\subseteq[1,k]} \Big(1 - q\prod_{i\in S} a_i / \prod_{j\notin S} a_j\Big)} \Big|_{a_1^0 a_2^0 \cdots a_k^0}. \tag{I.12}$$

Now, standard methods of Invariant Theory yield that we also have

$$W_k(q) \;=\; \frac{\prod_{i=1}^k (1-\frac{1}{a_i^2})}{\prod_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)}\Big|_{a_1^0a_2^0\cdots a_k^0}. \qquad \text{I.13}$$

A comparison of I.12 and I.13 strongly suggests that a close study of the combinatorics of Diophantine systems such as $\mathcal{S}_k$ should yield a more revealing path to the construction of such Hilbert series. This idea turned out to be fruitful, as we shall see, in that it permitted the solution of a variety of similar problems (see [2], [3]). In particular, we were eventually able to obtain that

$$G_5(\sqrt{q}) \;=\; \frac{Num(5)}{(1-q)^9(1-q^2)^8(1-q^3)^6(1-q^4)^3(1-q^5)}. \qquad \text{I.14}$$

with

$$\begin{aligned}Num(5) = {} & q^{44}+7\,q^{43}+220\,q^{42}+2606\,q^{41}+24229\,q^{40}+169840\,q^{39}+951944\,q^{38}\\ & +4391259\,q^{37}+17128360\,q^{36}+57582491\,q^{35}+169556652\,q^{34}+442817680\,q^{33}\\ & +1036416952\,q^{32}+2192191607\,q^{31}+4219669696\,q^{30}+7433573145\,q^{29}+12041305271\,q^{28}\\ & +18003453305\,q^{27}+24921751416\,q^{26}+32017113319\,q^{25}+38243274851\,q^{24}+42524815013\,q^{23}\\ & +44052440432\,q^{22}+42524815013\,q^{21}+38243274851\,q^{20}+32017113319\,q^{19}+24921751416\,q^{18}\\ & +18003453305\,q^{17}+12041305271\,q^{16}+7433573145\,q^{15}+4219669696\,q^{14}+2192191607\,q^{13}\\ & +1036416952\,q^{12}+442817680\,q^{11}+169556652\,q^{10}+57582491\,q^{9}+17128360\,q^{8}+4391259\,q^{7}\\ & +951944\,q^{6}+169840\,q^{5}+24229\,q^{4}+2606\,q^{3}+220\,q^{2}+7\,q+1\end{aligned}.$$

Surprisingly, the presence of the numerator factor in I.13 absent in I.12 does not increase the complexity of the result, as we see by comparing I.14 to the Luque-Thibon result

$$W_5(\sqrt{q}) \;=\; \frac{P_5(q)}{(1-q^2)^4(1-q^3)(1-q^4)^6(1-q^5)(1-q^6)^5}$$

with

$$\begin{aligned}P_5(q) = {} & q^{54}+q^{52}+16q^{50}+9q^{49}+98q^{48}+154q^{47}+465q^{46}+915q^{45}+2042q^{44}+3794q^{43}+7263q^{42}\\ & +12688q^{41}+21198q^{40}+34323q^{39}+52205q^{38}+77068q^{37}+108458q^{36}+147423q^{35}+191794q^{34}\\ & +241863q^{33}+292689q^{32}+342207q^{31}+386980q^{30}+421057q^{29}+443990q^{28}+451398q^{27}\\ & +443990q^{26}+421057q^{25}+386980q^{24}+342207q^{23}+292689q^{22}+241863q^{21}+191794q^{20} \qquad \text{I.15}\\ & +147423q^{19}+108458q^{18}+77068q^{17}+52205q^{16}+34323q^{15}+21198q^{14}+12688q^{13}\\ & +7263q^{12}+3794q^{11}+2042q^{10}+915q^{9}+465q^{8}+154q^{7}+98q^{6}+9q^{5}+16q^{4}+q^{2}+1\end{aligned}.$$

It should be apparent from the size of the numerators of $W_5(q)$ and $G_5(q)$ that the problem of computing these rational functions explodes beyond $k=4$. In fact it develops that all available computer packages (including "*Omega*" and "*Latte*") fail to directly compute the constant terms in I.12 for $k = 5$. This notwithstanding, we were eventually able to get the partial fraction algorithm of G. Xin to deliver us $G_5(q)$.

This paper covers the variety of techniques we developed in our efforts to compute these remarkable rational functions.

Our efforts in obtaining $W_6(q)$ and $G_6(q)$ are still in progress, so far they only resulted in reducing the computer time required to obtain $W_5(q)$ and $G_5(q)$. Using combinatorial ideas, in conjunction with the partial fraction algorithm of Xin, we developed three essentially distinct algorithms for computing these rational functions as well as other closely related families. The most successful of these algorithms got the computation time for $W_5(q)$ down to ten minutes. The crucial feature of this latter algorithm is an inductive process for successively computing the series $G_k(q)$ and $W_k(q)$, based on a surprising role of divided differences.

The contents are divided into four sections. In the first section we relate these Hilbert series to constant terms and derive a collection of identities to be used in later sections. In the second section we develop the combinatorial model that reduces the computation of our Kronecker products to solutions of Diophantine systems. In the third section we develop the divided difference algorithm for the computation of the complete generating functions yielding $W_k(q)$ and $G_k(q)$. In the fourth and final section, after an illustration of what can be done with bare hands we expand the combinatorial ideas acquired from this experimentation into our three algorithms that yielded $G_5(q)$ and our fast computation of $W_5(q)$.

The reader is referred to the papers of Luque-Thibon [4],[5] and Wallach [7] for an understanding of how these Hilbert series are related to problem arising in the study of quantum computing.

1. Hilbert series of invariants as constant terms.

Let us recall that given two matrices $A = \|a_{ij}\|_{i,i=1}^{m}$ and $B = \|b_{ij}\|_{i,i=1}^{n}$ we use the notation $A \otimes B$ to denote the $nm \times nm$ block matrix

$$A \otimes B \;=\; \|a_{ij}B\|_{i,i=1}^{m}\,. \qquad 1.1$$

For instance, if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \qquad 1.2$$

then

$$A \otimes B = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}. \qquad 1.3$$

Here and in the following the action of an $m \times m$ matrix $A = \|a_{ij}\|_{i,j=1}^{m}$ on a polynomial $P(x) = P(x_1, x_2, \ldots, x_n)$ is denoted $T_A P(x)$ and is defined by setting

$$T_A P(x_1, x_2, \ldots, x_n) \;=\; P\Big(\sum_{i=1}^{m} x_i a_{i1}\,,\, \sum_{i=1}^{m} x_i a_{i2}\,,\, \ldots\,,\, \sum_{i=1}^{m} x_i a_{im}\Big). \qquad 1.4$$

In matrix notation (viewing $x = (x_1, x_2, \ldots, x_n)$ as a row vector) we may simply rewrite this as

$$T_A P(x) \;=\; P\big(xA\big). \qquad 1.5$$

Recall that if G is a group of $m\times m$ matrices we say that $P\in\mathbb{C}[x_1,x_2,\ldots,x_n]$ is "*G-invariant*" if and only if

$$T_A P(x) \;=\; P(x) \qquad \forall \qquad A\in G. \tag{1.6}$$

The subspace of $\mathbb{C}[x]=\mathbb{C}[x_1,x_2,\ldots,x_n]$ of G-invariant polynomials is usually denoted $\mathbb{C}[x]^G$. Clearly, the action in 1.4 preserves homogeneity and degree, thus we have the direct sum decomposition

$$\mathbb{C}[x]^G \;=\; \mathcal{H}_o\big(\mathbb{C}[x]^G\big)\oplus\mathcal{H}_1\big(\mathbb{C}[x]^G\big)\oplus\mathcal{H}_2\big(\mathbb{C}[x]^G\big)\oplus\cdots\oplus\mathcal{H}_d\big(\mathbb{C}[x]^G\big)\oplus\cdots$$

where $\mathcal{H}_d\big(\mathbb{C}[x]^G\big)$ denotes the subspace of G-invariants that are homogeneous of degree d. The "*Hilbert series*" of $\mathbb{C}[x]^G$ is simply given by the formal power series

$$F_G(q) \;=\; \sum_{d\geq 0} q^d \dim\Big(\mathcal{H}_d\big(\mathbb{C}[x]^G\big)\Big). \tag{1.7}$$

Since $\dim\mathcal{H}_d\big(\mathbb{C}[x]^G\big)\leq\dim\Big(\mathcal{H}_d\big(\mathbb{C}[x]\big)\Big)=\binom{d+m-1}{m-1}$ we see that this is a well defined formal power series.

In the case that G is a finite group the Hilbert series $F_G(q)$ is immediately obtained from Molien's formula

$$F_G(q) \;=\; \frac{1}{|G|}\sum_{A\in G}\frac{1}{\det\big(I-qA\big)}.$$

For an infinite group G which possess a unit invariant measure ω this identity becomes

$$F_G(q) \;=\; \int_{A\in G}\frac{1}{\det\big(I-qA\big)}\,d\omega. \tag{1.8}$$

For the present developments we need to specialize all this to the case $G=SL[2]^{\otimes k}$, that is the group of $2^k\times 2^k$ matrices obtained by tensoring a k-tuple of elements of $SL[2]$. More precisely

$$SL[2]^{\otimes k} \;=\; \big\{A_1\otimes A_2\otimes\cdots\otimes A_k \,:\, A_i\in SL[2] \quad\forall\quad i=1,2,\ldots,k\big\}. \tag{1.9}$$

Our first task in this section is to derive the identity in I.12. That is

Theorem 1.1

Setting for $k\geq 1$

$$W_k(q) \;=\; F_{SL[2]^{\otimes k}}(q) \;=\; \sum_{d\geq 0} q^d\dim\Big(\mathcal{H}_d\big(\mathbb{C}[x]^{SL[2]^{\otimes k}}\big)\Big), \tag{1.10}$$

we have

$$W_k(q) \;=\; \frac{\prod_{i=1}^k\big(1-a_i^2\big)}{\prod_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)}\Bigg|_{a_1^0a_2^0\cdots a_k^0}. \tag{1.11}$$

Proof

To keep our exposition within reasonable limits we will need to assume here some well known facts (see [7] for proofs). Since $SL[2]$ has no finite measure the first step is to note that a polynomial $P(x) \in \mathbb{C}[x_1, x_2, \ldots, x_{2^k}]$ is $SL[2]^{\otimes k}$-invariant if and only if it is $SU[2]^{\otimes k}$-invariant. Where as in 1.9

$$SU[2]^{\otimes k} \;\; = \;\; \big\{ A_1 \otimes A_2 \otimes \cdots \otimes A_k \, : \, A_i \in SU[2] \quad \forall \quad i = 1, 2, \ldots, k \big\}. \tag{1.11}$$

In particular we derive that

$$F_{SL[2]^{\otimes k}}(q) \;\; = \;\; F_{SU[2]^{\otimes k}}(q). \tag{1.12}$$

This fact allows us to compute $F_{SL[2]^{\otimes k}}(q)$ using Molien's identity 1.8. Note however that if

$$A \;\; = \;\; A_1 \otimes A_2 \otimes \cdots \otimes A_k$$

and A_i has eigenvalues $t_i, 1/t_i$ then (using plethistic notation) we have

$$\frac{1}{\det\big(I - qA\big)} \;\; = \;\; \sum_{m \geq 0} q^m h_m\big[(t_1 + 1/t_1)(t_2 + 1/t_2) \cdots (t_k + 1/t_k)\big]. \tag{1.13}$$

Denoting by $d\omega_i$ the invariant measure of the i^{th} copy of $SU[2]$ we see that in this case 1.8 reduces to

$$F_{SU[2]^{\otimes k}}(q) = \sum_{m \geq 0} q^m \int_{SU[2]} \cdots \int_{SU[2]} h_m\big[(t_1 + 1/t_1) \cdots (t_k + 1/t_k)\big] d\omega_1 \cdots d\omega_k. \tag{1.14}$$

Now it is well know that if an integrand $f(A)$ of $SU[2]$ is invariant under conjugation then

$$\int_{SU[2]} f(A) d\omega \;\; = \;\; \frac{1}{\pi} \int_{-\pi}^{\pi} f\Big(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \Big) \sin^2 \theta d\theta.$$

This identity reduces 1.14 to

$$F_{SU[2]^{\otimes k}}(q) = \sum_{m \geq 0} q^m \;\, \frac{1}{\pi^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} h_m\big[(e^{i\theta_1} + e^{-i\theta_1}) \cdots (e^{i\theta_k} + e^{-i\theta_k})\big] \sin^2 \theta_1 \cdots \sin^2 \theta_k \;\, d\theta_1 \cdots d\theta_k. \tag{1.15}$$

This given, the identity in 1.11 is an immediate consequence of the following simple fact.

Proposition 1.1

If $Q(a_1, a_2, \ldots, a_k)$ is a Laurent polynomial in $\mathbb{Q}[a_1, a_2, \ldots, a_k; 1/a_1, 1/a_2, \ldots, 1/a_k]$ then

$$\Big(\frac{1}{2\pi}\Big)^k \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} Q[(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_k}] d\theta_1 d\theta_2 \cdots d\theta_k \;\; = \;\; Q(a_1, a_2, \ldots, a_k)\Big|_{a_1^0}\Big|_{a_2^0} \cdots \Big|_{a_k^0}. \tag{1.16}$$

Proof

By linearity, it suffices to consider $Q(a_1, a_2, \ldots, a_k) = a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k}$, in which case 1.16 obviously holds.

Going back to 1.15 the substitution

$$\sin^2\theta_j = \frac{1-\frac{e^{2i\theta_j}+e^{-2i\theta_j}}{2}}{2}$$

reduces the coefficient of q^n to

$$\frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} h_m\big[(e^{i\theta_1}+e^{-i\theta_1})\cdots(e^{i\theta_k}+e^{-i\theta_k})\big]\prod_{i=1}^{k}\Big(1-\frac{e^{2i\theta_j}+e^{-2i\theta_j}}{2}\Big)\,d\theta_1\cdots d\theta_k. \qquad 1.17$$

However the factor

$$h_m\big[(e^{i\theta_1}+e^{-i\theta_1})\cdots(e^{i\theta_k}+e^{-i\theta_k})\big]$$

is invariant under any of the interchanges $e^{i\theta_j}\longleftrightarrow e^{-i\theta_j}$. Thus the integral in 1.22 may be simplified to

$$\frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} h_m\big[(e^{i\theta_1}+e^{-i\theta_1})\cdots(e^{i\theta_k}+e^{-i\theta_k})\big]\prod_{i=1}^{k}\Big(1-e^{2i\theta_j}\Big)\,d\theta_1\cdots d\theta_k. \qquad 1.18$$

Proposition 1.1 then yields that this integral may be computed as the constant term

$$h_m\big[(a_1+1/a_1)(a_1+1/a_1)\cdots(a_1+1/a_1)\big]\prod_{i=1}^{k}\Big(1-a_i^2\Big)\Big|_{a_1^0a_2^0\cdots a_k^0}.$$

Using this in 1.15 we derive that

$$\begin{aligned} F_{SU[2]^{\otimes k}}(q) &= \sum_{m\geq 0} q^m h_m\big[(a_1+1/a_1)(a_1+1/a_1)\cdots(a_1+1/a_1)\big]\prod_{i=1}^{k}\Big(1-a_i^2\Big)\Big|_{a_1^0a_2^0\cdots a_k^0}\\ &= \sum_{m\geq 0} q^m h_m\Big[\sum_{S\subseteq[1,k]}\frac{\prod_{i\in S}a_i}{\prod_{j\notin S}a_j}\Big]\prod_{i=1}^{k}\Big(1-a_i^2\Big)\Big|_{a_1^0a_2^0\cdots a_k^0}\\ &= \Bigg(\prod_{S\subseteq[1,k]}\frac{1}{\Big(1-q\frac{\prod_{i\in S}a_i}{\prod_{j\notin S}a_j}\Big)}\Bigg)\prod_{i=1}^{k}\Big(1-a_i^2\Big)\Big|_{a_1^0a_2^0\cdots a_k^0} \end{aligned}$$

This completes the proof of Theorem 1.1.

Note that if we restrict our action of $SU[2]^{\otimes k}$ to the subgroup of matrices

$$T_2^{\otimes k} = \left\{\begin{bmatrix} t_1 & 0\\ 0 & \bar t_1\end{bmatrix}\otimes\begin{bmatrix} t_2 & 0\\ 0 & \bar t_2\end{bmatrix}\otimes\cdots\otimes\begin{bmatrix} t_k & 0\\ 0 & \bar t_k\end{bmatrix}\ :\ t_r=e^{i\theta_r}\right\}$$

then a similar use of Molien's theorem yields the following result.

Theorem 1.2

The Hilbert series of the ring of invariants $\mathbf{R}_{2^k}^{T_2^{\otimes k}}$ *is given by the constant term*

$$F_{T_2^{\otimes k}}(q) = \frac{1}{\prod_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)}\Bigg|_{a_1^0a_2^0\cdots a_k^0}. \qquad 1.19$$

Proof

The integrand $1/\det(1-qA)$ is the same as in the previous proof and only the Haar measure changes. In this case we must take to $dw = d\theta_1 d\theta_2 \cdots d\theta_k/(2\pi)^k$ in 1.8, and Molien's theorem gives

$$F_{T_2^{\otimes k}}(q) \;=\; \frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{1}{\prod\limits_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}t_i/\prod_{j\notin S}t_j\Big)}d\theta_1 d\theta_2\cdots d\theta_k.$$

Thus 1.19 follows from Proposition 1.1.

Remark 1.2

There is another path leading to the same result that is worth mentioning here since it gives a direct way of connecting Invariants to Diophantine systems. For notational simplicity we will deal with the case $k=3$. Note that the element

$$\begin{bmatrix} t_1 & 0\\ 0 & \bar t_1\end{bmatrix}\otimes\begin{bmatrix} t_2 & 0\\ 0 & \bar t_2\end{bmatrix}\otimes\begin{bmatrix} t_3 & 0\\ 0 & \bar t_3\end{bmatrix}\in T_2^{\otimes 3}$$

is none other than the 8×8 matrix

$$A(t_1,t_2,t_3) \;=\; \begin{bmatrix} t_1t_2t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & t_1t_2/t_3 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & t_1t_3/t_2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & t_1/t_2t_3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & t_2t_3/t_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & t_2/t_1t_3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & t_3/t_1t_2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/t_1t_2t_3 \end{bmatrix}.$$

This gives that for any monomial $x^p = x_1^{p_1}x_2^{p_2}\cdots x_8^{p_8}$ we have

$$A(t_1,t_2,t_3)\,x^p \;=\; t_1^{p_1+p_2+p_3+p_4-p_5-p_6-p_7-p_8}\,t_2^{p_1+p_2-p_3-p_4+p_5+p_6-p_7-p_8}\,t_3^{p_1-p_2+p_3-p_4+p_5-p_6+p_7-p_8}\times x^p.$$

Thus all the monomials are eigenvectors and a polynomial $P(x_1,x_2,\ldots,x_8)$ will be invariant if and only if all its monomials are eigenvectors of eigenvalue 1. It then follows that the Hilbert series $F_{T_2^{\otimes 3}}(q)$ of $Q[x_1,x_2,\ldots,x_8]^{T_2^{\otimes 3}}$ is obtained by q-counting these monomials by total degree. That is q-counting by the statistic $p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8$ the solutions of the Diophantine system

$$\mathcal{S}_3 \;=\; \left\| \begin{array}{l} p_1+p_2+p_3+p_4-p_5-p_6-p_7-p_8=0\\ p_1+p_2-p_3-p_4+p_5+p_6-p_7-p_8=0\\ p_1-p_2+p_3-p_4+p_5-p_6+p_7-p_8=0 \end{array}\right. \qquad 1.20$$

and MacMahon partition analysis gives

$$F_{T_2^{\otimes 3}}(q) \;=\; \frac{1}{1-qa_1a_2a_3}\,\frac{1}{1-qa_1a_2/a_3}\,\frac{1}{1-qa_1a_3/a_2}\,\frac{1}{1-qa_1/a_2a_3}\,\frac{1}{1-qa_2a_3/a_1}\,\frac{1}{1-qa_2/a_1a_3}\,\frac{1}{1-qa_3/a_1a_2}\,\frac{1}{1-qa/a_1a_2a_3}\Big|_{a_1^0a_2^0a_3^0}.$$

This gives another proof of the case $k=3$ of 1.19. It is also clear that the same argument can be used for all $k>3$ as well.

Remark 1.3

Full information about the solutions of our systems is given by the complete generating function

$$F_k(x_1,x_2,\ldots,x_{2^k}) \;=\; \sum_{p\in\mathcal{S}_k} x_1^{p_1}x_2^{p_2}\cdots x_{2^k}^{p_{2^k}}. \qquad 1.21$$

Using the notation adopted for $\mathcal{S}_3$ in 1.20, our system $\mathcal{S}_k$ may be written in vector form

$$p_1V_1 + p_2V_2 + \cdots + p_{2^k}V_{2^k} \;=\; 0, \qquad 1.22$$

where $V_1, V_2, \ldots, V_{2^k}$ are the k-vectors $(\pm1,\pm1,\ldots,\pm1)$ yielding the vertices of the hypercube of semiside 1 centered at the origin. In this notation, MacMahon partition analysis gives that the rational function in 1.21 is obtained by taking the constant term

$$F_k(x_1,x_2,\ldots,x_{2^k}) \;=\; \prod_{i=1}^{2^k}\frac{1}{1-x_iA_i}\Big|_{a_1^0a_2^0\cdots a_k^0} \qquad 1.23$$

with the A_i Laurent monomials in $a_1,a_2,\ldots a_k$ which may be written in the form

$$A_i \;=\; \prod_{i=1}^{k} a_i^{1-2\epsilon_i}$$

where $\epsilon_1\epsilon_2\cdots\epsilon_k$ are the binary digits of $i-1$.

In the same vein the companion rational function $W(x_1,x_2,\ldots,x_{2^k})$ associated to the Sdd problem is obtained by taking the constant term

$$W_k(x_1,x_2,\ldots,x_{2^k}) \;=\; \prod_{j=1}^{k}(1-a_j^2)\prod_{i=1}^{2^k}\frac{1}{1-x_iA_i}\Big|_{a_1^0a_2^0\cdots a_k^0}. \qquad 1.24$$

Of course we have

$$G_k(q) \;=\; F_k(x_1,x_2,\ldots,x_{2^k})\Big|_{x_i=q} \quad\text{and}\quad W_k(q) \;=\; W_k(x_1,x_2,\ldots,x_{2^k})\Big|_{x_i=q}.$$

In section 3 we will show that, at least in principle, these rational functions could be constructed by a succession of elementary steps interspersed by single constant term extractions.

2. Diophantine systems, Constant terms and Kronecker products

We have seen, by MacMahon partition analysis, that the generating function defined in I.10

$$G_k(q) \;=\; \sum_{d\ge 0} m_d(k) q^{2d},$$

which counts solutions of the diophantine system $\mathcal{S}_k$, is given by the constant term identity in I.12:

$$G_k(q) \;=\; \frac{1}{\prod_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)}\Bigg|_{a_1^0a_2^0\cdots a_k^0}. \qquad 2.1$$

In the last section we proved (Theorem 1.1) that the Hilbert series of invariants

$$W_k(q) \;=\; \sum_{m\ge 0} q^m \dim \mathcal{H}_m\big(\mathbf{R}_{2^k}^{GL[2]\otimes GL[2]\otimes\cdots\otimes GL[2]}\,\big)$$

is given by the constant term

$$W_k(q) \;=\; \frac{\prod_{i=1}^k(1-a_i^2)}{\prod_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)}\Bigg|_{a_1^0a_2^0\cdots a_k^0}. \qquad 2.2$$

A comparison of 2.1 and 2.2 clearly suggests that these two results must be connected. This connection has a beautiful combinatorial underpinning which leads to yet another interpretation of the these remarkable constant terms. The idea is best explained in the simplest case $k=2$. Then 2.2 reduces to

$$W_2(q) \;=\; \frac{1-a_1^2-a_2^2+a_1^2a_2^2}{(1-qa_1a_2)(1-qa_1/a_2)(1-qa_2/a_1)(1-q/a_1a_2)}\Bigg|_{a_1^0a_2^0}.$$

Expanding the inner rational function as product of four formal power series in q we get

$$\begin{aligned}
W_2(q) \;=\; &\sum_{p_{00}\ge 0}\sum_{p_{01}\ge 0}\sum_{p_{10}\ge 0}\sum_{p_{11}\ge 0} q^{p_{00}+p_{01}+p_{10}+p_{11}} a_1^{p_{00}+p_{01}-p_{10}-p_{11}} a_2^{p_{00}-p_{01}+p_{10}-p_{11}}\Big|_{a_1^0a_2^0} +\\
&-\sum_{p_{00}\ge 0}\sum_{p_{01}\ge 0}\sum_{p_{10}\ge 0}\sum_{p_{11}\ge 0} q^{p_{00}+p_{01}+p_{10}+p_{11}} a_1^{p_{00}+p_{01}-p_{10}-p_{11}+2} a_2^{p_{00}-p_{01}+p_{10}-p_{11}}\Big|_{a_1^0a_2^0} +\\
&-\sum_{p_{00}\ge 0}\sum_{p_{01}\ge 0}\sum_{p_{10}\ge 0}\sum_{p_{11}\ge 0} q^{p_{00}+p_{01}+p_{10}+p_{11}} a_1^{p_{00}+p_{01}-p_{10}-p_{11}} a_2^{p_{00}-p_{01}+p_{10}-p_{11}+2}\Big|_{a_1^0a_2^0}\\
&+\sum_{p_{00}\ge 0}\sum_{p_{01}\ge 0}\sum_{p_{10}\ge 0}\sum_{p_{11}\ge 0} q^{p_{00}+p_{01}+p_{10}+p_{11}} a_1^{p_{00}+p_{01}-p_{10}-p_{11}+2} a_2^{p_{00}-p_{01}+p_{10}-p_{11}+2}\Big|_{a_1^0a_2^0}
\end{aligned} \qquad 2.3$$

Now the first term is none other than 2.1 for $k=2$ and thus it counts solutions of the diophantine system

$$\mathcal{S}_2 \;=\; \left\|\begin{aligned} p_{00}+p_{01}-p_{10}-p_{11} &= 0\\ p_{00}-p_{01}+p_{10}-p_{11} &= 0\end{aligned}\right. . \qquad 2.4$$

In the same vein, by MacMahon partition analysis, we see that the second term counts solutions of the system

$$\mathcal{S}_2^{10} \;=\; \left\| \begin{array}{rcl} p_{00}+p_{01}-p_{10}-p_{11} &=& -2 \\ p_{00}-p_{01}+p_{10}-p_{11} &=& 0 \end{array} \right. \;; \qquad 2.5$$

analogously the third and fourth terms are respectively counting solutions of the two following systems

$$\mathcal{S}_2^{01} \;=\; \left\| \begin{array}{rcl} p_{00}+p_{01}-p_{10}-p_{11} &=& 0 \\ p_{00}-p_{01}+p_{10}-p_{11} &=& -2 \end{array} \right. \;, \qquad 2.6$$

$$\mathcal{S}_2^{11} \;=\; \left\| \begin{array}{rcl} p_{00}+p_{01}-p_{10}-p_{11} &=& -2 \\ p_{00}-p_{01}+p_{10}-p_{11} &=& -2 \end{array} \right. \;. \qquad 2.7$$

Applying the same decomposition in the general case we see that the series $W_k(q)$ may be viewed as the end product of an inclusion exclusion process applied to a family of Diophantine systems. To derive some further consequences of this fact, it is more convenient to use another combinatorial model for these systems.

In this alternate model our family of objects consists of the collection $\mathcal{F}_d$ of d-subsets of the $2d$-element set

$$\Omega_{2d} = \{1, 2, 3, \ldots, 2d\}.$$

For a given $A = \{1 \le i_1 < i_2 < \cdots < i_d \le 2d\} \in \mathcal{F}_d$ and σ in the symmetric group S_{2d} we set

$$\sigma A \;=\; \{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_d}\}.$$

This clearly defines an action of S_{2d} on $\mathcal{F}_d$ as well as on the k-fold cartesian product

$$\mathcal{F}_d^{\otimes k} \;=\; \mathcal{F}_d \times \mathcal{F}_d \times \mathcal{F}_d \times \cdots \times \mathcal{F}_d.$$

Theorem 2.1

The number $m_d(k)$ of solutions of the diophantine system $\mathcal{S}_k$ is equal to the number of orbits in the action of S_{2d} on $\mathcal{F}_d^{\otimes k}$.

Proof

It will be sufficient to see this for $k = 2$. Then leaving d generic we can visualize an element of $\mathcal{F}_d \times \mathcal{F}_d$ by the Ven diagram of Fig. 1. There we have depicted the pair (A_1, A_2) as it lies in Ω_{2d}. Using these two sets we can decompose Ω_{2d} into 4 parts we labeled $\mathbf{A_{00}}, \mathbf{A_{01}}, \mathbf{A_{10}}, \mathbf{A_{11}}$. More precisely "$\mathbf{A_{00}}$" labels the set $A_1 \cap A_2$, "$\mathbf{A_{01}}$" labels the set $A_1 \cap A_2^c$, "$\mathbf{A_{10}}$" labels the set $A_1^c \cap A_2$ and "$\mathbf{A_{11}}$" labels the set $A_1^c \cap A_2^c$. Here we use "A_i^c" to denote the complement of A_i in Ω_{2d}. This given, if we let $p_{00}, p_{01}, p_{10}, p_{11}$ denote the respective cardinalities of these sets, the

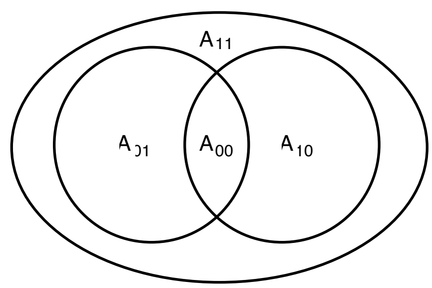


Fig.1

condition that the pair (A_1, A_2) belongs top $\mathcal{F}_d \times \mathcal{F}_d$ yields that we must have

$$\begin{array}{rcl} p_{00}+p_{01}+p_{10}+p_{11} &=& 2d \\ p_{00}+p_{01} = |A_1| &=& d \\ p_{00}+p_{10} = |A_2| &=& d \end{array} \qquad .$$

Note that this system of equations is equivalent to the system

$$\begin{aligned} p_{00}+p_{01}+p_{10}+p_{11} &= 2d \\ p_{00}+p_{01}-p_{10}-p_{11} &= 0 \\ p_{00}-p_{01}+p_{10}-p_{11} &= 0 \end{aligned} .$$

It easily seen that, given any solution $(p_{00},p_{01},p_{10},p_{11})$ of this system, we can immediately construct a pair of subsets $(A_1,A_2)\in\mathcal{F}_d\times\mathcal{F}_d$ by simply filling the sets $\mathbf{A_{00}},\mathbf{A_{01}},\mathbf{A_{10}},\mathbf{A_{11}}$ in the diagram of Fig 1 with $p_{00},p_{01},p_{10},p_{11}$ respective elements from the set Ω_{2d}. Moreover, any two such fillings can be seen to be images of each other under suitable permutations of S_{2d}. In other words by this construction we obtain a bijection between the orbits of $\mathcal{F}_d\times\mathcal{F}_d$ under $\mathcal{S}_{2d}$ and the solutions of the system $\mathcal{S}_2$ we have previously encountered. This proves the theorem for $k=2$. The general case follows by an entirely analogous argument.

Theorem 2.1 immediately leads to a

Proof of Theorem I.2

We are to show that

$$m_k(d) \;=\; \big\langle h_{d,d}*h_{d,d}*\cdots*h_{d,d}\;,\;s_{2d}\big\rangle. \qquad 2.8$$

It is well known that a transitive action of a group G on a set Ω is equivalent to the action of G on the left G-cosets of the stabilizer of any element of Ω. In our case, if we take this element to be subset $\{1,2,\ldots,d\}$ of Ω_{2d} then the stabilizer is the Young subgroup $S_{\{1,\ldots,d\}}\times S_{\{d+1,\ldots,2d\}}$ of S_{2d} and thus the Frobenius characteristic of this action is the homogeneous basis element $h_{d,d}=h_dh_d$. It follows then that the Frobenius characteristic of the action of S_{2d} on the k-tuples $(A_1,A_2,\ldots,A_k)$ of d-subsets of Ω_{2d} is given by the k-fold Kronecker product

$$h_{d,d}*h_{d,d}*\cdots*h_{d,d}.$$

Thus the scalar product

$$\big\langle h_{d,d}*h_{d,d}*\cdots*h_{d,d}\;,\;s_{2d}\big\rangle$$

yields the multiplicity of the trivial under this action. But it is well known, and easy to see that this multiplicity is also equal to the number of orbits under this action. Thus Theorem 2.1 gives 2.8.

We can now give a

Proof of Theorem I.1

Again we will only need to do it for $k=2$. To this end note that by Theorem I.2 the number of solutions of the system in 2.4 is given by the scalar product

$$\big\langle h_{d,d}*h_{d,d}\;,\;s_{2d}\big\rangle. \qquad 2.9$$

In the same vein we see that the number of solutions to the system in 2.4 may be viewed as the number of orbits in the action of S_{2d} on the pairs of subsets (A_1,A_2) of Ω_{2d} where A_2 and its complement A_2^c have the same cardinality but A_1 has one more element than its complement A_1^c. We have seen that the Frobenius characteristic of the action of S_{2d} on subsets of cardinality d is $h_{d,d}$. On the other hand the action of S_{2d} on sets of cardinality $d+1$ is equivalent to the action of S_{2d} on left cosets of $S_{1,2\ldots,d+1}\times S_{d+2\ldots,2d}$ yielding that the Frobenius characteristic for this action is $h_{d+1}h_{d-1}$. Thus the Frobenius characteristic of the action of S_{2d} on such pairs must be the Kronecker product

$$h_{d+1}h_{d-1} * h_d h_d.$$

It follows then that the number of solutions of the system in 2.5 is given by the scalar product

$$\big\langle h_{d+1}h_{d-1} * h_d h_d \ , \ s_{2d}\big\rangle. \qquad 2.10$$

The same reasoning gives that the number of solutions of the systems in 2.6 and 2.7 are given by the scalar products

$$\big\langle h_d h_d * h_{d+1}h_{d-1} \ , \ s_{2d}\big\rangle \quad \text{and} \quad \big\langle h_{d+1}h_{d-1} * h_{d+1}h_{d-1} \ , \ s_{2d}\big\rangle. \qquad 2.11$$

It follows then that the coefficient of q^{2d} in the formal power series resulting from the alternating sum in 2.3 is none other than the same alternating sum of the scalar products in 2.9. 2.10 and 2.11. That is

$$\begin{aligned} W_2(q)\Big|_{q^{2d}} &= \big\langle h_d h_d * h_d h_d \ , \ s_{2d}\big\rangle - \big\langle h_{d+1}h_{d-1} * h_d h_d \ , \ s_{2d}\big\rangle - \big\langle h_d h_d * h_{d+1}h_{d-1} \ , \ s_{2d}\big\rangle + \big\langle h_{d+1}h_{d-1} * h_{d+1}h_{d-1} \ , \ s_{2d}\big\rangle \\ &= \big\langle \big(h_d h_d - h_{d+1}h_{d-1}\big) * \big(h_d h_d - h_{d+1}h_{d-1}\big) \ , \ s_{2d}\big\rangle \ = \ \big\langle s_{d,d} * s_{d,d} \ , \ s_{2d}\big\rangle. \end{aligned}$$

This gives

$$W_2(q) \ = \ \sum_{d\geq 0} q^{2d}\big\langle s_{d,d} * s_{d,d} \ , \ s_{2d}\big\rangle.$$

An entirely analogous argument proves the general identity in I.8.

3. Enter divided difference operators

There is a truly remarkable approach to the solutions of a variety of constant term problems which exhibit the same types of symmetries of the Hdd and Sdd problems. We will introduce the approach in some simple cases first. We define as the "*double*" of the Diophantine system

$$\mathcal{S}_2 \ = \ \left\| \begin{array}{l} p_1 + p_2 - p_3 - p_4 = 0 \\ p_1 - p_2 + p_3 - p_4 = 0 \end{array} \right. \qquad 3.1$$

the system

$$\mathcal{SS}_2 \ = \ \left\| \begin{array}{l} p_1 + p_2 - p_3 - p_4 \ + \ p_5 + p_6 - p_7 - p_8 = 0 \\ p_1 - p_2 + p_3 - p_4 \ + \ p_5 - p_6 + p_7 - p_8 = 0 \end{array} \right. . \qquad 3.2$$

As we can easily see we have simply repeated twice each linear form and appropriately increased the indices of the variables. Now suppose that we are in possession of the complete generating function of $\mathcal{S}_2$, that is

$$F_{\mathcal{S}_2}(x_1,x_2,x_3,x_4) \ = \ \sum_{p\in\mathcal{S}_2} x_1^{p_1}x_2^{p_2}x_3^{p_3}x_4^{p_4}. \qquad 3.3$$

Then the complete generating function of $\mathcal{SS}_2$ is simply given by

$$F_{\mathcal{SS}_2}(x_1,x_2,\ldots,x_8) \ = \ \delta_{1,5}\delta_{2,6}\delta_{3,7}\delta_{4,8}F_{\mathcal{S}_2}(x_1,x_2,x_3,x_4), \qquad 3.4$$

where for any pair of indices (i,j) we let $\delta_{i,j}$ denote the divided difference operator

$$\delta_{i,j} \;=\; \frac{1}{x_i - x_j}(1 - s_{i,j}) \tag{3.5}$$

with $s_{i,j}$ denoting the transposition that interchanges the pair x_i, x_j. This is proved as follows.

By MacMahon partition analysis we have

$$F_{\mathcal{S}_2}(x_1,x_2,x_3,x_4) \;=\; \frac{1}{(1-x_1a_1a_2)}\frac{1}{(1-x_2a_1/a_2)}\frac{1}{(1-x_3a_2/a_1)}\frac{1}{(1-x_4/a_1a_2)}\Big|_{a_1^0a_2^0}. \tag{3.6}$$

Now note that since

$$\begin{aligned}\delta_{1,5}\frac{1}{(1-x_1a_1a_2)} &= \left(\frac{1}{(1-x_1a_1a_2)} - \frac{1}{(1-x_5a_1a_2)}\right)\frac{1}{x_1-x_5}\\ &= \left(\frac{1-x_5a_1a_2\;-1+x_1a_1a_2}{(1-x_1a_1a_2)(1-x_5a_1a_2)}\right)\frac{1}{x_1-x_5} \;=\; \frac{a_1a_2}{(1-x_1a_1a_2)(1-x_5a_1a_2)},\end{aligned}$$

we obtain similarly

$$\begin{aligned}\delta_{2,6}\frac{1}{(1-x_2a_1/a_2)} &= \frac{a_1/a_2}{(1-x_2a_1/a_2)(1-x_6a_1/a_2)},\\ \delta_{3,7}\frac{1}{(1-x_3a_2/a_1)} &= \frac{a_2/a_1}{(1-x_3a_2/a_1)(1-x_7a_2/a_1)},\\ \delta_{4,8}\frac{1}{(1-x_4/a_1a_2)} &= \frac{1/a_1a_2}{(1-x_4/a_1a_2)(1-x_8/a_1a_2)}.\end{aligned}$$

Thus applying the operator $\delta_{1,5}\delta_{2,6}\delta_{3,7}\delta_{4,8}$ to both sides of 3.6 gives

$$\begin{aligned}\delta_{1,5}\delta_{2,6}\delta_{3,7}\delta_{4,8}F_{\mathcal{S}_2}(x_1,x_2,x_3,x_4) \;=&\\ \frac{1}{(1-x_1a_1a_2)(1-x_2a_1/a_2)(1-x_3a_2/a_1)(1-x_4/a_1a_2)}&\\ \overline{(1-x_5a_1a_2)(1-x_6a_1/a_2)(1-x_7a_2/a_1)(1-x_8/a_1a_2)}\Big|_{a_1^0a_2^0}&.\end{aligned} \tag{3.7}$$

Now we can easily recognize that 3.7 is precisely the constant term that MacMahon partition analysis would yield for the system $\mathcal{SS}_2$. This proves 3.4.

Note that to obtain the equality in 3.7 we have used the simple fact that the divided difference operator and the constant term operator do commute. This is the fundamental property which is at the root of the present algorithm. This example should make it evident that we have here a general result that may be stated as follows

Theorem 3.1

If $F_{\mathcal{S}}(x_1,x_2,\ldots,x_n)$ is the complete generating function of a Diophantine system $\mathcal{S}$ then the rational function

$$F_{\mathcal{SS}}(x_1,x_2,\ldots,x_n) \;=\; \delta_{1,n+1}\delta_{2,n+2}\cdots\delta_{n,2n}F_{\mathcal{S}}(x_1,x_2,\ldots,x_n)$$

is the complete generating function of the system $\mathcal{SS}$ obtained by doubling $\mathcal{S}$.

This result combined with the next simple observation yields a powerful algorithm for computing a variety of complete generating functions.

Theorem 3.2

Let $F_{\mathcal{S}}(x_1, x_2, \ldots, x_n)$ be the complete generating function of a Diophantine system $\mathcal{S}$ then the complete generating function $F_{\mathcal{S}\mathcal{E}}(x_1, x_2, \ldots, x_n)$ of the system $\mathcal{S}\mathcal{E}$ obtained by adding the equation

$$\mathcal{E} \quad = \quad r_1 p_1 + r_2 p_2 + \cdots + r_n p_n = 0 \tag{3.8}$$

to $\mathcal{S}$ is obtained by taking the constant term

$$F_{\mathcal{S}\mathcal{E}}(x_1, x_2, \ldots, x_n) \quad = \quad F_{\mathcal{S}}(a^{r_1}x_1, a^{r_2}x_2, \ldots, a^{r_n}x_n)\Big|_{a^0}. \tag{3.9}$$

Proof

By assumption

$$F_{\mathcal{S}}(x_1, x_2, \ldots, x_n) \quad = \quad \sum_{p\in\mathcal{S}} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}.$$

Now we have

$$\begin{aligned} F_{\mathcal{S}}(a^{r_1}x_1, a^{r_2}x_2, \ldots, a^{r_n}x_n)\Big|_{a^0} &= \sum_{p\in\mathcal{S}} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} a^{r_1p_1+r_2p_2+\cdots+r_np_n}\Big|_{a^0} \\ &= \sum_{p\in\mathcal{S}\mathcal{E}} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \\ &= F_{\mathcal{S}\mathcal{E}}(x_1, x_2, \ldots, x_n). \end{aligned}$$

These two results provide us with an algorithm for (at least in principle) computing all the Hdd series

$$G_k(q) \quad = \quad \sum_{n\geq 0} \big\langle h_{d,d} * \cdots * h_{d,d} \ , \ s_{2d}\big\rangle q^{2d}$$

as well as the Sdd series

$$W_k(q) \quad = \quad \sum_{n\geq 0} \big\langle s_{d,d} * \cdots * s_{d,d} \ , \ s_{2d}\big\rangle q^{2d}.$$

The algorithm for the Hdd series proceeds as follows.

Step 1

$\mathbf{a_1}$) Compute the complete generating function for the trivial system $\mathcal{S}_1 : \ \| p_1 - p_2 = 0$. That is compute the constant term

$$F_{\mathcal{S}_1}(x_1, x_2) = \frac{1}{(1-x_1a)(1-x_2/a)}\ \Big|_{a^0}.$$

Step 2

$\mathbf{a_2}$) Compute the complete generating function for the double $\mathcal{S}\mathcal{S}_1 : \ \| p_1 - p_2 + p_3 - p_4 = 0$. That is

$$F_{\mathcal{S}\mathcal{S}_1}(x_1, x_2, x_3, x_4) \quad = \quad \delta_{1,3}\delta_{2,4} F_{\mathcal{S}_1}(x_1, x_2).$$

$\mathbf{b_2}$) Then, by Theorem 3.2, the complete generating function for the system $\mathcal{S}_2 : \ \Big\| {p_1+p_2-p_3-p_4=0 \atop p_1-p_2+p_3-p_4=0}$ is given by the constant term

$$F_{\mathcal{S}_2}(x_1, x_2, x_3, x_4) \quad = \quad F_{\mathcal{S}\mathcal{S}_1}(ax_1, ax_2, x_3/a, x_4/a)\Big|_{a^0}.$$

Step k

$\mathbf{a_k}$) Compute the complete generating function for the double $\mathcal{SS}_{k-1}$ of the system $\mathcal{S}_{k-1}$. That is

$$F_{\mathcal{SS}_{k-1}}(x_1,\ldots,x_{2^k}) \;=\; \delta_{1,1+2^{k-1}}\cdots\delta_{2^{k-1},2^k}F_{\mathcal{S}_{k-1}}(x_1,\ldots,x_{2^{k-1}}).$$

$\mathbf{b_k}$) Then by Theorem 3.2, the complete generating function for the system $\mathcal{S}_k$ is given by the constant term

$$F_{\mathcal{S}_k}(x_1,x_2,\ldots,x_{2^k}) \;=\; F_{\mathcal{SS}_{k-1}}(ax_1,ax_2,\ldots,ax_{2^{k-1}},x_{2^{k-1}+1}/a,\ldots,x_{2^k}/a)\Big|_{a^0}.$$

This sequence of steps can be terminated by replacing step $\mathbf{b_k}$) by

$\mathbf{b'_k}$) The q-generating function $G_k(q)$ is given by the constant term

$$G_k(q) \;=\; F_{\mathcal{SS}_{k-1}}(aq,aq,\ldots,aq,q/a,\ldots,q/a)\Big|_{a^0}.$$

The first three steps can be carried out by hand, for step 4 we need a computer, and to carry out step 5 by computer we have to introduce one more tool as we shall see. Unfortunately Step 6 appears beyond reach at the moment.

It will be instructive to see what steps 1,2,3 give.

Step 1 :

$\mathbf{a_1}$)

$$F_{\mathcal{S}1}(x_1,,x_2)=\frac{1}{1-x_1x_2}.$$

Step 2 :

$\mathbf{a_2}$)

$$F_{\mathcal{SS}_1}(x_1,x_2,x_3,x_4)=\frac{(1-x_1x_2x_3x_4)}{(1-x_1x_2)(1-x_2x_3)(1-x_1x_4)(1-x_3x_4)}.$$

$\mathbf{b_2}$)

$$\begin{aligned}F_{\mathcal{S}_2}(x_1,x_2,x_3,x_4) \;&=\; \frac{(1-x_1x_2x_3x_4)}{(1-a^2x_1x_2)(1-x_2x_3)(1-x_1x_4)(1-x_3x_4/a^2)}\Big|_{a^0}\\ &=\; \frac{1}{(1-x_2x_3)(1-x_1x_4)}.\end{aligned}$$

Step 3 :

$\mathbf{a_3}$)

$$\begin{aligned}F_{\mathcal{SS}_2}(x_1,\ldots,x_8) \;&=\; \frac{(1-x_1x_4x_5x_8)(1-x_2x_3x_6x_7)}{(1-x_1x_8)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)}\\ &\qquad\times\frac{1}{(1-x_1x_4)(1-x_2x_3)(1-x_6x_7)(1-x_5x_8)}.\end{aligned}$$

$\mathbf{b_3}$)

$$\begin{aligned}F_{\mathcal{S}_3}(x_1,\ldots,x_8) \;&=\; \frac{(1-x_1x_4x_5x_8)(1-x_2x_3x_6x_7)}{(1-x_1x_8)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)}\\ &\qquad\times\frac{1}{(1-a^2x_1x_4)(1-a^2x_2x_3)(1-x_6x_7/a^2)(1-x_5x_8/a^2)}\Big|_{a^0}.\end{aligned} \qquad 3.10$$

We can compute this constant term in many ways. In particular we could use one of the MacMahon identities given by Andrews in [1]. But it is interesting to point out that our divided difference algorithm has already provided us (in step $\mathbf{a_2}$)) a formula we can use in step $\mathbf{b_3}$). In fact, the output of step $\mathbf{a_2}$)

$$F_{\mathcal{SS}_1}(x_1,x_2,x_3,x_4) \;=\; \frac{(1-x_1x_2x_3x_4)}{(1-x_1x_2)(1-x_2x_3)(1-x_1x_4)(1-x_3x_4)}$$

is the complete generating function of the system $p_1-p_2+p_3-p_4=0$, so by MacMahon partition analysis we should also have

$$F_{\mathcal{SS}_1}(x_1,x_2,x_3,x_4) \;=\; \frac{1}{(1-ax_1)(1-x_2/a)(1-ax_3)(1-x_4/a)}\Big|_{a^0}.$$

This implies that

$$\begin{aligned}
&\frac{1}{(1-a^2x_1x_4)(1-a^2x_2x_3)(1-x_6x_7/a^2)(1-x_5x_8/a^2)}\Big|_{a^0}\\
&\qquad= \frac{(1-x_1x_2x_3x_4)}{(1-x_1x_2)(1-x_2x_3)(1-x_1x_4)(1-x_3x_4)}\Bigg|_{\substack{x_1\to x_1x_4\\ x_3\to x_2x_3\\ x_2\to x_6x_7\\ x_4\to x_5x_8}}\\
&\qquad= \frac{(1-x_1x_2x_3x_4x_5x_6x_7x_8)}{(1-x_1x_4x_6x_7)(1-x_6x_7x_2x_3)(1-x_1x_4x_5x_8)(1-x_2x_3x_5x_8)}.
\end{aligned}$$

Using this in 3.10 gives

$$\begin{aligned}
F_{\mathcal{S}_3}(x_1,\ldots,x_8) \;&=\; \frac{(1-x_1x_4x_5x_8)(1-x_2x_3x_6x_7)}{(1-x_1x_8)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)}\\
&\qquad\times\frac{(1-x_1x_2x_3x_4x_5x_6x_7x_8)}{(1-x_1x_4x_6x_7)(1-x_6x_7x_2x_3)(1-x_1x_4x_5x_8)(1-x_2x_3x_5x_8)}\\
&=\; \frac{1}{(1-x_1x_8)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)}\times\frac{(1-x_1x_2x_3x_4x_5x_6x_7x_8)}{(1-x_1x_4x_6x_7)(1-x_2x_3x_5x_8)}.
\end{aligned}$$

Replacing all the x_i by the single variable q, we thus obtain that

$$G_1(q)=\frac{1}{1-q^2}\;,\quad G_2(q)=\frac{1}{(1-q^2)^2}\;,\quad G_3(q)=\frac{1-q^8}{(1-q^2)^4(1-q^4)^2}\;=\;\frac{1+q^4}{(1-q^2)^4(1-q^4)}. \qquad 3.11$$

Using the computer to carry out **Step 4** with $\mathbf{b_4'}$) replacing $\mathbf{b_4}$) gives

$$G_4(q)=\frac{1+q^2+21q^4+36q^6+74q^8+86q^{10}+74q^6+36q^{14}+21q^{16}+q^{18}+q^{20}}{(1-q^2)^7(1-q^4)^4(1-q^6)}.$$

We shall see later what else has to be done to obtain $G_5(q)$.

It is worth noting that our divided difference algorithm can also be adapted to compute the first 5 Sdd series as well. In fact, again due to the fact that divided difference operators commute with the constant term operators, we can also show that all the complete Sdd series can (in principle) be obtained by the following algorithm.

Step 1

$\mathbf{a_1}$) Compute the complete generating function for the Sdd problem for $k = 1$. That is, compute the constant term

$$W_1(x_1, x_2) = \frac{1-a^2}{(1-x_1 a)(1-x_2/a)}\Big|_{a^0}.$$

Step 2

$\mathbf{a_2}$) Compute

$$WW_1(x_1, x_2, x_3, x_4) \;=\; \delta_{1,3}\delta_{2,4} W_1(x_1, x_2).$$

$\mathbf{b_2}$) To obtain the complete generating function for the Sdd problem for $k = 2$ compute the constant term

$$W_2(x_1, x_2, x_3, x_4) \;=\; WW_1(ax_1, ax_2, x_3/a, x_4/a)(1-a^2)\Big|_{a^0}.$$

Step k

$\mathbf{a_k}$) Compute

$$WW_{k-1}(x_1, \ldots, x_{2^k}) \;=\; \delta_{1,1+2^{k-1}} \cdots \delta_{2^{k-1},2^k} W_{k-1}(x_1, \ldots, x_{2^{k-1}}).$$

$\mathbf{b_k}$) To obtain the complete generating function for the Sdd problem for k compute the constant term

$$W_k(x_1, x_2, \ldots, x_{2^k}) \;=\; WW_{k-1}(ax_1, ax_2, \ldots, ax_{2^{k-1}}, x_{2^{k-1}+1}/a, \ldots, x_{2^k}/a)(1-a^2)\Big|_{a^0}.$$

This sequence of steps can be terminated by replacing step **$\mathbf{b_k}$)** by

$\mathbf{b'_k}$) To obtain the generating function $W_k(q)$ compute the constant term

$$W_k(q) \;=\; WW_{k-1}(aq, aq, \ldots, aq, q/a, \ldots, q/a)(1-a^2)\Big|_{a^0}.$$

Unlike the Hdd case only steps **$\mathbf{a_1}$)** and **$\mathbf{a_2}$)** can be carried out by hand, though steps 3 and 4 are routine they are too messy to do by hand. But step 5 again needs further tricks to be carried out by computer. Step 6 appears beyond reach at the moment.

It will be instructive to see what some of these steps give.

Step 1 :

$\mathbf{a_1}$)

$$W_1(x_1, x_2) = \frac{1-x_2^2}{1-x_1 x_2}.$$

Step 2 :

$\mathbf{a_2}$)

$$WW_1(x_1, \ldots, x_4) = \frac{1 - x_2^2 - x_2x_4 - x_4^2 + x_1x_2^2x_4 + x_2^2x_3x_4 - x_1x_2x_3x_4 + x_1x_2x_4^2 + x_2x_3x_4^2 - x_1x_2^2x_3x_4^2}{(1-x_1x_2)(1-x_3x_2)(1-x_1x_4)(1-x_3x_4)}.$$

b₂)

$$W_2(x_1,x_2,x_3,x_4) \;=\; \frac{1-x_2x_4-x_3x_4+x_4^2}{(1-x_1x_4)(1-x_2x_3)}.$$

This gives

$$W_2(q) \;=\; \frac{1}{1-q^2}.$$

Step 3 :

a₃)

$$WW_2(x_1,x_2,\ldots,x_8) \;=\; \frac{(large \quad numerator)}{(1-x_1x_4)(1-x_1x_8)(1-x_2x_3)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)(1-x_5x_8)(1-x_6x_7)}.$$

b₃) Produces

$$W_3(x_1,x_2,\ldots,x_8) \;=\; \frac{(large \quad numerator)}{(1-x_1x_8)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)(1-x_1x_4x_6x_7)(1-x_2x_3x_5x_8)}.$$

(In the next section we will see that **a₃**) and **b₃**) can be considerably improved)

b′₃)

Notwithstanding the complexity of the previous results it turns out that to obtain $W_3(q)$ we need only compute the constant term

$$W_3(q) \;=\; \frac{1}{(1-q^2)} \times \frac{1-a^2}{(1-q^2a^2)(1-q^2/a^2)}\bigg|_{a^0}. \qquad 3.12$$

To this end we start by determining the coefficients A and B in the partial fraction decomposition

$$\frac{(1-a^2)a^2}{(1-q^2a^2)(a^2-q^2)} \;=\; \frac{1}{q^2} + \frac{A}{1-q^2a^2} + \frac{B}{a^2-q^2}$$

obtaining

$$A \;=\; \frac{(1-a^2)a^2}{(a^2-q^2)}\bigg|_{a^2=1/q^2} \;=\; \frac{(1-1/q^2)/q^2}{(1/q^2-q^2)} \;=\; -\frac{1}{q^2(1+q^2)},$$

and

$$B \;=\; \frac{(1-a^2)a^2}{(1-q^2a^2)}\bigg|_{a^2=q^2} \;=\; \frac{(1-q^2)q^2}{(1-q^4)} \;=\; \frac{q^2}{(1+q^2)},$$

(the exact value of B is not needed) and we can write

$$\frac{1-a^2}{(1-q^2a^2)(1-q^2/a^2)} \;=\; \frac{1}{q^2} - \frac{1}{q^2(1+q^2)} \times \frac{1}{(1-a^2q^2)} + \frac{1}{(1+q^2)} \times \frac{q^2/a^2}{1-q^2/a^2}.$$

Thus taking constant terms gives

$$\frac{1-a^2}{(1-q^2a^2)(1-q^2/a^2)}\bigg|_{a^0} \;=\; \frac{1}{q^2} - \frac{1}{q^2(1+q^2)} + 0 \;=\; \frac{1}{1+q^2}.$$

Using this in 3.12 we finally obtain

$$W_3(q) \;=\; \frac{1}{1-q^4}. \qquad 3.13$$

a_4)

$$WW_4(x_1,x_2,\ldots,x_{16}) \;=\; (\textit{too large for typesetting})$$

b'_4) Notwithstanding the complexity of the previous result it turns out that to obtain $W_4(q)$ we need only compute the constant term

$$W_4(q) \;=\; \frac{(1+q^4)(1+q^6)}{(1-q^2)(1-q^4)^2} \times \frac{1-a^2}{(1-a^2q^4)(1-q^4/a^2)(1-a^4q^4)(1-q^4/a^4)}\bigg|_{a^0}$$

To illustrate the power and flexibility of the partial fraction algorithm we will carry this out by hand. The reader is referred to [2] for a brief tutorial on the use of this algorithm. In the next few lines we will strictly adhere to the notation and terminology given in [2].

To begin we note that we need only calculate the constant term

$$C(x) \;=\; \frac{1-a}{(1-ax)(1-x/a)(1-a^2x)(1-x/a^2)}\bigg|_{a^0}, \qquad 3.14$$

since we can write

$$W_4(q) \;=\; \frac{(1+q^4)(1+q^6)}{(1-q^2)(1-q^4)^2} \times C(q^4). \qquad 3.15$$

Now we have

$$\frac{1}{(1-a^2x)(1-x/a^2)} \;=\; \frac{a^2}{(1-a^2x)(a^2-x)} \;=\; \frac{1}{1-x^2}\frac{1}{1-a^2x} + \frac{1}{1-x^2}\frac{x/a^2}{1-x/a^2}.$$

Thus 3.14 may be rewritten in the form

$$\begin{aligned} C(x) \;&=\; \frac{(1-a)}{(1-ax)(1-x/a)}\left(\frac{1}{1-x^2}\frac{1}{1-a^2x} + \frac{1}{1-x^2}\frac{x/a^2}{1-x/a^2}\right)\bigg|_{a^0} \\ &=\; \frac{1}{1-x^2}\left(\frac{(1-a)}{(1-ax)(1-x/a)}\frac{1}{1-a^2x}\bigg|_{a^0} + \frac{(1-a)}{(1-ax)(1-x/a)}\frac{x/a^2}{1-x/a^2}\bigg|_{a^0}\right). \end{aligned} \qquad 3.16$$

Note that in the first constant term we have only one dually contributing term and on the second we have only one contributing term. This gives

$$\frac{(1-a)}{(1-ax)(1-x/a)}\frac{1}{1-a^2x}\bigg|_{a^0} \;=\; \frac{(1-a)}{(1-ax)}\frac{1}{1-a^2x}\bigg|_{a=x} \;=\; \frac{(1-x)}{(1-x^2)}\frac{1}{1-x^3} \qquad 3.17$$

and

$$\frac{(1-a)}{(1-ax)(1-x/a)}\frac{x/a^2}{1-x/a^2}\bigg|_{a^0} = \frac{(1-a)}{(1-x/a)}\frac{x/a^2}{1-x/a^2}\bigg|_{a=1/x} = \frac{-(1-x)}{(1-x^2)}\frac{x^2}{1-x^3}. \qquad 3.18$$

Using 3.17 and 3,18 in 3.16 we get

$$\begin{aligned} C(x) \;&=\; \frac{1}{1-x^2}\left(\frac{(1-x)}{(1-x^2)}\frac{1}{1-x^3} - \frac{(1-x)}{(1-x^2)}\frac{x^2}{1-x^3}\right) \\ &=\; \frac{1-x}{(1-x^2)(1-x^3)} \end{aligned}.$$

Thus

$$C(q^4) \;=\; \frac{1-q^4}{(1-q^8)(1-q^{12})}$$

and 3.15 gives

$$W_4(q) \;=\; \frac{(1+q^4)(1+q^6)}{(1-q^2)(1-q^4)^2}\times\frac{1-q^4}{(1-q^8)(1-q^{12})} \;=\; \frac{1}{(1-q^2)(1-q^4)^2(1-q^6)}. \tag{3.19}$$

We will see in section 4 what needs to be done to carry out step $\mathbf{b_5'}$ on the computer.

The identities

$$W_2(q)=\frac{1}{(1-q^2)}\ ,\quad W_3(q)=\frac{1}{(1-q^4)}\ ,\quad W_4(q)=\frac{1}{(1-q^2)(1-q^4)^2(1-q^6)}\ , \tag{3.20}$$

have also been derived in [2] by symmetric function methods from the relation

$$W_k(q)=\frac{\prod_{i=1}^k(1-a_i^2)}{\prod\limits_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)}\Bigg|_{a_1^0a_2^0\cdots a_k^0}=\sum_{d\geq 0}q^{2d}\big\langle s_{d,d}*s_{d,d}*\cdots *s_{d,d}\ ,\ s_{2d}\big\rangle. \tag{3.21}$$

In fact, all three results in 3.20 are immediate consequences of the following deeper symmetric function identity. (for a proof see section 2 in [2].)

Theorem 3.3

$$s_{d,d}*s_{d,d} \;=\; \sum_{\lambda\vdash 2d} s_\lambda\,\chi(\lambda\in EO_4) \tag{3.22}$$

where EO_4 denotes the set of partitions of length 4 whose parts are ≥ 0 and all even or all odd.

Remark 3.1

Note that the Kronecker product identity

$$\big\langle s_{d,d}*s_{d,d}*s_{d,d}*s_{d,d}*s_{d,d}\ ,\ s_{2d}\big\rangle \;=\; \big\langle s_{d,d}*s_{d,d}*s_{d,d}\ ,\ s_{d,d}*s_{d,d}\big\rangle.$$

suggests obtaining $W_5(q)$ by means of a combinatorial interpretation of the coefficients of the Schur function expansion of the Kronecker product $s_{d,d}*s_{d,d}*s_{d,d}$. However, to this date no formula has been given for these coefficients, combinatorial or otherwise.

4. Solving the Hdd problem for $\mathbf{k = 5}$.

Our initial efforts at solving the Hdd an Sdd problems were entirely carried out by computer experimentation. After obtaining quite easily the series $G_2(q)$, $G_3(q)$, $G_4(q)$ and $W_2(q)$, $W_3(q)$, $W_4(q)$, all the computer packages available to us failed to directly deliver $G_5(q)$ and $W_5(q)$.

In this section we will give a brief view of the combinatorial and manipulatorial gyrations we had to perform to extract $G_5(q)$ and $W_5(q)$ out of our computers first after several hours of computer time and then reducing computation times down to a few minutes.

The computer data obtained for the Hdd problem for $k = 2, 3, 4$ were combinatorially so revealing that we have been left with a strong impression that this problem should have a very beautiful combinatorial general solution. Only time will tell if this will ever be the case. To stimulate further research we will begin by reviewing our initial computer and manual combinatorial findings.

Recall that we denoted by $\mathcal{F}_d$ the collection of all d-subsets of a $2d$ element set Ω_{2d}. We also showed (Theorem 2.1) that the coefficient $m_d(k)$ in the series

$$G_k(q) \;=\; \sum_{d\leq 0} q^{2d} m_d(k) \tag{4.1}$$

counts the number of orbits under the action of the symmetric group $\mathcal{S}_{2d}$ on the k-fold cartesian product $\mathcal{F}_d \times \mathcal{F}_d \times \cdots \times \mathcal{F}_d$. Denoting by $(A_1, A_2, \ldots, A_k)$ a generic element of this cartesian product, then each orbit is uniquely determined by the 2^k cardinalities

$$p_{\epsilon_1,\epsilon_2,\cdots,\epsilon_k} \;=\; \left| A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_k^{\epsilon_k} \right|$$

where for each $1 \leq i \leq k$ we set

$$A_i^{\epsilon_i} \;=\; \begin{cases} A_i & \text{if } \epsilon_i = 0, \\ {}^cA_i & \text{if } \epsilon_i = 1. \end{cases} \qquad\qquad (\text{here } {}^cA_i = \Omega_{2d}/A_i).$$

It is also convenient to set

$$A_{\epsilon_1,\epsilon_2,\cdots,\epsilon_k} \;=\; A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_k^{\epsilon_k}. \tag{4.2}$$

This given we have seen that the condition that $(A_1, A_2, \ldots, A_k) \in \mathcal{F}_d \times \mathcal{F}_d \times \cdots \times \mathcal{F}_d$ is equivalent to the Diophantine system

$$\mathcal{S}_k \;=\; \left\| \begin{array}{lll} \sum_{\epsilon_1=0}^1 \sum_{\epsilon_2=0}^1 \cdots \sum_{\epsilon_k=0}^1 (1-2\epsilon_1) p_{\epsilon_1,\epsilon_2,\cdots,\epsilon_k} & = & 0, \\ \sum_{\epsilon_1=0}^1 \sum_{\epsilon_2=0}^1 \cdots \sum_{\epsilon_k=0}^1 (1-2\epsilon_2) p_{\epsilon_1,\epsilon_2,\cdots,\epsilon_k} & = & 0, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots & & \vdots \\ \sum_{\epsilon_1=0}^1 \sum_{\epsilon_2=0}^1 \cdots \sum_{\epsilon_k=0}^1 (1-2\epsilon_k) p_{\epsilon_1,\epsilon_2,\cdots,\epsilon_k} & = & 0, \end{array} \right. \tag{4.3}$$

together with the condition that Ω_{2d} has cardinality $2d$, that is

$$\sum_{\epsilon_1=0}^1 \sum_{\epsilon_2=0}^1 \cdots \sum_{\epsilon_k=0}^1 p_{\epsilon_1,\epsilon_2,\cdots,\epsilon_k} \;=\; 2d. \tag{4.4}$$

There are several algorithms available to solve such a system. See for instance [6, Chapter 4.6].The algorithm we used for our computer experimentations is the MacMahon algorithm which has been recently implemented in MATHEMATICA by Andrews, Paule and Riese and in MAPLE by G. Xin using the partial fraction method of computing constant terms.
The former can be downloaded from the web site.

http://www.risc.uni-linz.ac.at/research/combinat/software/Omega/

and the latter from the web site

http://www.combinatorics.net.cn/homepage/xin/maple/ell2.rar"

For computer implementation we found more convenient to use the alternate notation adopted in section 1 (Remark 1.3). That is

$$\mathcal{S}_k \;=\; \| p_1 V_1 \,+\, p_2 V_2 \,+\, \cdots \,+\, p_{2^k} V_{2^k} \;=\; 0\;. \tag{4.5}$$

This gives

$$\mathcal{S}_2 = \left\| \begin{matrix} p_1 + p_2 - p_3 - p_4 & = & 0 \\ p_1 - p_2 + p_3 - p_4 & = & 0 \end{matrix} \right.$$

and

$$\mathcal{S}_3 \;=\; \left\| \begin{matrix} p_1 \,+\, p_2 \,+\, p_3 \,+\, p_4 \,-\, p_5 \,-\, p_6 \,-\, p_7 \,-\, p_8 & = & 0 \\ p_1 \,+\, p_2 \,-\, p_3 \,-\, p_4 \,+\, p_5 \,+\, p_6 \,-\, p_7 \,-\, p_8 & = & 0 \\ p_1 \,-\, p_2 \,+\, p_3 \,-\, p_4 \,+\, p_5 \,-\, p_6 \,+\, p_7 \,-\, p_8 & = & 0 \end{matrix} \right. . \tag{4.6}$$

These algorithms may yield quite a bit more than the number of solutions of such a system. For instance, in our case letting $\mathcal{C}_k$ denote the collection of solutions of the system $\mathcal{S}_k$, the "Omega package" of Andrews, Paule and Riese should, in principle, yield the formal power series

$$F_k(x_1, x_2, \ldots, x_{2^k}) \;=\; \sum_{(p_1, p_2, \ldots, p_{2^k}) \in\, \mathcal{C}_k} x_1^{p_1} x_2^{p_2} \cdots x_{2^k}^{p_{2^k}}\,.$$

It follows from the general theory of Diophantine systems that $F_k(x_1, x_2, \ldots, x_{2^k})$ is always the Taylor series of a rational function.

Now for $\mathcal{S}_2$ the Omega package gives

$$F_2(x_1, x_2, x_3, x_4) \;=\; \frac{1}{(1 - x_1 x_4)(1 - x_2 x_3)} \tag{4.7}$$

and for $\mathcal{S}_3$ the Omega package gives

$$F_3(x_1, x_2, \ldots, x_8) = \frac{1 - x_2 x_3 x_5 x_8 x_1 x_4 x_6 x_7}{(1 - x_1 x_8)(1 - x_2 x_7)(1 - x_3 x_6)(1 - x_4 x_5)(1 - x_2 x_3 x_5 x_8)(1 - x_1 x_4 x_6 x_7)}. \tag{4.8}$$

But this is as far as this package went in our computers. However we could go further by giving up full information about the solutions and only ask for the series

$$G_k(q) \;=\; F_k(x_1, x_2, \ldots, x_{2^k})\big|_{x_i = q}.$$

which, as we have seen, may be computed using the identity in I.12:

$$G_k(q) \;=\; \frac{1}{\prod_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)}\Big|_{a_1^0a_2^0\cdots a_k^0}. \qquad 4.9$$

For example, the program *Latte* by De Loera, Hemmecke, Tauzer, Yoshida, et. al., which is available at *http://www.math.ucdavis.edu /˜ latte/* computed the $G_4(q)$ series in approximately 30 seconds. However, this is as far as `Latte` went on our machines. We should also mention that all three series in 3.11 as well as $G_4(q)$ and three series in 3.20 can be obtained in only a few seconds, from the software of G. Xin by computing the corresponding constant terms in I.12 and I.13.

To get our computers to deliver $G_5(q)$ and $W_5(q)$ in a matter of minutes a divide and conquer strategy had to be adopted. More precisely, these rational functions were obtained by decomposing the constant terms I.12 and I.13 as sums of constant terms. This decomposition had its origin from an effort to find a human proof of the identities in 4.7 and 4.8. More importantly, the surprising simplicity of 4.7 and 4.8 required a combinatorial explanation. Our findings there provided the combinatorial tools that were used in our first computations of $G_5(q)$ and $W_5(q)$. This given, before describing our work on these series, we will show how to deal with 4.7 and 4.8 entirely by hand.

Beginning with

$$\mathcal{S}_2 = \left\| \begin{array}{lcl} p_1+p_2-p_3-p_4 &=& 0 \\ p_1-p_2+p_3-p_4 &=& 0 \end{array}\right. \qquad 4.10$$

we immediately notice that

$$(1,0,0,1) \quad \text{and} \quad (0,1,1,0) \qquad 4.11$$

are solutions. Moreover, setting

$$a=\min(p_1,p_4) \quad \text{and} \quad b=\min(p_2,p_3), \qquad 4.12$$

we can easily see that the difference

$$(q_1,q_2,q_3,q_4) \;=\; (p_1,p_2,p_3,p_4)-(a,b,b,a)=(p_1-a,p_2-b,p_3-b,p_4-a)$$

must also be a solution. Now from 4.11 we derive that

$$\min(q_1,q_4)=0 \quad \text{and} \quad \min(q_2,q_3)=0,$$

which gives us four possibilities for (q_1,q_2,q_3,q_4):

$$(0,0,x,y)\;,\quad (0,x,0,y)\;,\quad (x,0,y,0)\;,\quad (x,y,0,0), \qquad 4.13$$

for some non negative integers x,y. Testing the first equation of $\mathcal{S}_2$ immediately forces the first and last in 4.13 to identically vanish. Similarly, the second equation of $\mathcal{S}_2$ yields that the second and third in 4.13 must also identically vanish. This proves that the general solution of $\mathcal{S}_2$ is of the form (a,b,b,a). We thus obtain the full generating function of solutions of $\mathcal{S}_2$:

$$F_2(x_1,x_2,x_3,x_4) \;=\; \sum_{a\ge 0}\sum_{b\ge 0} x_1^a x_2^b x_3^b x_4^a \;=\; \frac{1}{(1-x_1x_4)(1-x_2x_3)}.$$

This proves 4.7.

It turns out that we can deal with $\mathcal{S}_3$ in a similar manner. Again we begin by noticing the four "*symmetric*" solutions

$$(1,0,0,0,0,0,0,1)\,,\quad (0,1,0,0,0,0,1,0)\,,\quad (0,0,1,0,0,1,0,0)\,,\quad (0,0,0,1,1,0,0,0).$$

Next we set

$$a=\min(p_1,p_8),\;\; b=\min(p_2,p_7),\;\; c=\min(p_3,p_6),\;\; d=\min(p_4,p_5),$$

and by subtraction we get a solution

$$(q_1,q_2,q_3,q_4,q_5,q_6,q_7,q_8) \;=\; (p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)-(a,b,c,d,d,c,b,a) \tag{4.14}$$

with the property that

$$\min(q_i,q_{9-i})=0 \qquad \text{for } \; 1\le i\le 4. \tag{4.15}$$

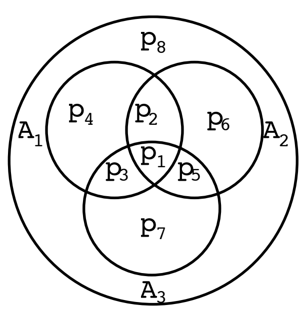
It will be good here and after to call the set

$$\{i\in[1,n]\,:\,p_i\ge 1\}$$

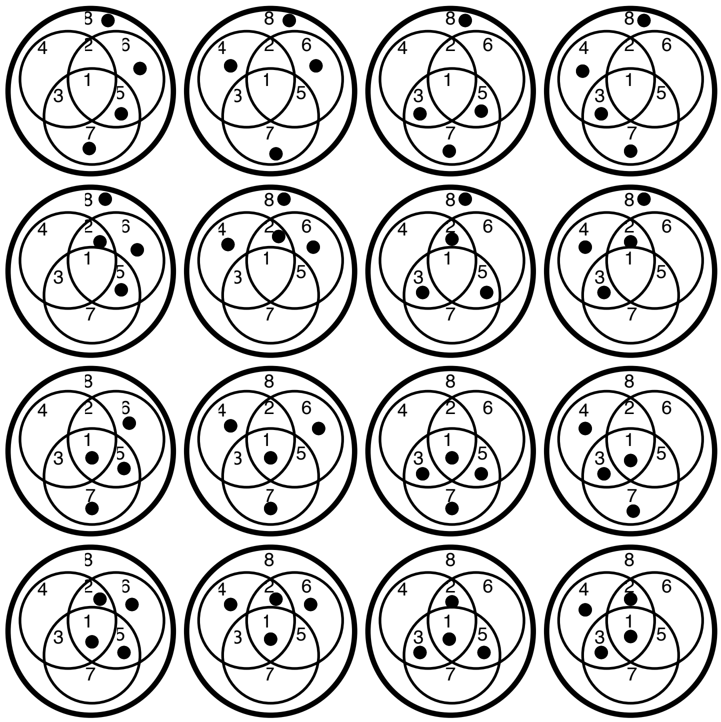
the "*support*" of the composition $(p_1,p_2,\ldots,p_n)$. This given, we derive that the resulting composition in 4.14 will necessarily have its support contained in at least one of the following 16 patterns.

$$\begin{array}{llll}
(0,0,0,0,*,*,*,*)\,, & (0,0,0,*,0,*,*,*)\,, & (0,0,*,0,*,0,*,*)\,, & (0,0,*,*,0,0,*,*)\,,\\
(0,*,0,0,*,*,0,*)\,, & (0,*,0,*,0,*,0,*)\,, & (0,*,*,0,*,0,0,*)\,, & (0,*,*,*,0,0,0,*)\,,\\
(*,0,0,0,*,*,*,0)\,, & (*,0,0,*,0,*,*,0)\,, & (*,0,*,0,*,0,*,0)\,, & (*,0,*,*,0,0,*,0)\,,\\
(*,*,0,0,*,*,0,0)\,, & (*,*,0,*,0,*,0,0)\,, & (*,*,*,0,*,0,0,0)\,, & (*,*,*,*,0,0,0,0)\,,
\end{array} \tag{4.16}$$

Unlike the case $k=2$ not all of these patterns force a trivial solution. To find out which it is helpful to resort to a Venn diagram imagery. To this end recall that a solution of S_3 gives the cardinalities of the 8 regions of the Venn diagram of three d-subsets A_1,A_2,A_3 of a set of cardinality $2d$ (see figure).



In the following figure, each pattern, is represented by a Venn diagram where in each region $A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap A_3^{\epsilon_3}$ that corresponds to a $*$ in the pattern we placed a black dot. That means that only the regions with a dot may have ≥ 0 cardinality. The miracle is that all but the two patterns $(0,*,*,0,*,0,0,*)$ and $(*,0,0,*,0,*,*,0)$ can be quickly excluded by a reasoning that only uses the positions of the dots in the Venn diagram. In fact, in each of the excluded cases, we show that it is impossible to replace the dots by ≥ 0 integers in such a manner that the three sets A_1, A_2, A_3 and their complements ${}^cA_1$, ${}^cA_2$, ${}^cA_3$ end up having the same cardinality.



The reasoning is so cute that we are compelled to present it here in full. In what follows the j^{th} diagram in the i^{th} row will be referred to as "D_{ij}":

(1) D_{11}, D_{14}, D_{22} D_{33}, D_{41} and D_{44} can be immediately excluded because one of A_1, A_2, A_3, A_1^c, A_2^c or A_3^c would be empty.

(2) In D_{21} the dot next to 8 should give the cardinality of A_2^c (say d) and then the dot next to the 2 should also give d. But that forces the dots next to 5 and 6 to be 0, leaving A_3 empty. The same reasoning applies to D_{12}, D_{13}, D_{24} ,D_{31} ,D_{34} D_{42} , D_{43}.

That leaves only the two diagrams D_{23} and D_{32} which clearly correspond to the two above mentioned patterns. Now we see that for D_{32} we must have the equalities

$$p_1 + p_4 \;=\; p_1 + p_6 \;=\; p_1 + p_7 \;=\; p_6 + p_7.$$

This forces $p_1 = p_4 = p_6 = p_7$. In summary this pattern can only support the composition $(u,0,0,u,0,u,u,0)$. The same reasoning yields that the diagram D_{23} can only support the composition $(0,v,v,0,v,0,0,v)$. It follows that the general solution of $\mathcal{S}_3$ must be of the form

$$(a,b,c,d,d,c,b,a)+(u,v,v,u,v,u,u,v).$$

Now recall that after the subtraction of a symmetric solution we are left with an "*asymmetric* " solution satisfying the inequalities in 4.15. Thus to avoid over counting we must impose the condition $u\,v=0$. This leaves only three possibilities $u=v=0$, $u>0, v=0$ or $u=0, v>0$. Thus

$$\begin{aligned}F_3(x_1,x_2,\ldots,x_8) &= \sum_{a\geq 0}\sum_{b\geq 0}\sum_{c\geq 0}\sum_{d\geq 0}(x_1x_8)^a(x_2x_7)^b(x_3x_6)^c(x_4x_5)^d\Big(1+\sum_{u\geq 1}(x_1x_4x_6x_7)^u+\sum_{v\geq 1}(x_2x_3x_5x_8)^v\Big)\\ &= \frac{1}{(1-x_1x_8)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)}\left(1+\frac{x_1x_4x_6x_7}{1-x_1x_4x_6x_7}+\frac{x_2x_3x_5x_8}{1-x_2x_3x_5x_8}\right).\end{aligned}$$

which is only another way of writing 4.8.

It is easy to see that the decomposition of a solution into a sum of a symmetric plus an asymmetric solution can be carried out in full generality. In fact, note that if $0\leq i\leq 2^k-1$ has binary digits $\epsilon_1\epsilon_2\cdots\epsilon_k$ then the binary digits of 2^k-1-i are $\overline{\epsilon}_1\overline{\epsilon}_2\cdots\overline{\epsilon}_k$ (with $\overline{\epsilon}=1-\epsilon$). Thus we see from 4.5 that in each equation p_i and p_{2^k+1-i} appear with opposite signs. This shows that for each $k\geq 2$ the system $\mathcal{S}_k$ has 2^{k-1} symmetric solutions, which may be symbolically represented by the monomials

$$x_1x_{2^k}\ ,\quad x_2x_{2^k-1}\ ,\quad x_3x_{2^k-2}\ ,\ \ldots\ ,\ x_{2^{k-1}}x_{2^{k-1}+1}.$$

Proceeding as we did for $\mathcal{S}_2$ and $\mathcal{S}_3$ we arrive at a unique decomposition of each solution of $\mathcal{S}_k$ into a sum

$$(p_1,p_2,\ldots,p_{2^k})\ =\ (u_1,u_2,\ldots,u_2,u_1)\ +\ (q_1,q_2,\ldots,q_{2^k})$$

with the first summand symmetric and the second asymmetric, that is

$$u_i\ =\ u_{2^k+1-i}\qquad\text{and}\qquad q_i\,q_{2^k+1-i}\ =\ 0\qquad(\text{for } 1\leq i\leq 2^{k-1})$$

and thereby obtain a factorization of $F_k(x)$ in the form

$$F_k(x)\ =\ \Big(\prod_{i=1}^{2^{k-1}}\frac{1}{1-x_ix_{2^k+1-i}}\Big)F_k^A(x) \tag{4.17}$$

with $F_k^A(x)$ denoting the complete generating function of the asymmetric solutions.

This given it is tempting to try to apply, in the general case, the same process we used for $k=3$ and obtain the rational function $F_k^A(x)$ by selecting the patterns that do contain the support of an asymmetric solution. Note that the total number of asymmetric patterns to be examined is $2^{2^{k-1}}$ which is already 256 for $k=4$. For $k=5$ the number grows to $65,536$ and doing this by hand is out of the question. Moreover, it is easy to see, by going through a few cases, that even for $k=4$ the geometry of the Venn Diagrams is so intricate that the only way that we can find out if a given pattern contains the support of a solution is to solve the corresponding reduced system.

Nevertheless, using some inherent symmetries of the problem, the complexity of the task can be substantially reduced to permit the construction of $G_5(q)$ by computer. To describe how this was done we need some notation.

We will start with the complete generating function of the system $\mathcal{S}_k$ as given in Remark 1.3, that is

$$F_k(x_1, x_2, \ldots, x_{2^k}) \;=\; \prod_{i=1}^{2^k} \frac{1}{1 - x_i A_i} \Big|_{a_1^0 a_2^0 \cdots a_k^0}. \qquad 4.18$$

We have also seen that the A_i may be written in the form

$$A_i \;=\; \prod_{i=1}^{k} a_i^{1-2\epsilon_i} \qquad 4.19$$

where $\epsilon_1\epsilon_2\cdots\epsilon_2$ are the binary digits of $i-1$. This given, note that since, (as we previously observed) the binary digits of $2^k - 1 - i$ are $\overline{\epsilon}_1\overline{\epsilon}_2\cdots\overline{\epsilon}_k$, from 4.19 we derive that

$$A_{2^k+1-i} \;=\; 1/A_i. \qquad 4.20$$

From this it follows that

$$\frac{1 - x_i x_{2^k+1-i}}{(1 - x_i A_i)(1 - x_{2^k+1-i} A_{2^k+1-i})} \;=\; \left(\frac{1}{1 - x_i A_i} + \frac{x_{2^k+1-i}/A_i}{1 - x_{2^k+1-i}/A_i} \right).$$

Thus setting, for convenience, $i' = 2^k + 1 - i$ and combining the factors containing A_i and $A_{i'}$ we may rewrite 4.17 in the form

$$F_k(x_1, x_2, \ldots, x_{2^k}) \;=\; \prod_{i=1}^{2^{k-1}} \frac{1}{1 - x_i x_{i'}} \prod_{i=1}^{2^{k-1}} \left(\frac{1}{(1 - x_i A_i)} + \frac{x_{i'}/A_i}{(1 - x_{i'}/A_i)} \right) \Big|_{a_1^0 a_2^0 \cdots a_k^0}. \qquad 4.21$$

Comparing with 4.17 we derive that the complete generating function of the asymmetric solutions decomposes into the sum

$$F_k^A(x) \;=\; \sum_{S \subseteq [1,2,\ldots,2^{k-1}]} F_S(x). \qquad 4.22$$

with

$$F_S(x) \;=\; \left(\prod_{i \notin S} \frac{1}{(1 - x_i A_i)} \right) \times \left(\prod_{i \in S} \frac{x_{i'}/A_i}{(1 - x_{i'}/A_i)} \right) \Big|_{a_1^0 a_2^0 \cdots a_k^0}. \qquad 4.23$$

Using the notation introduced in Remark 1.3, we can see that $F_S(x)$ is none other than the complete generating function of the reduced system

$$\sum_{i \notin S} p_i V_i \;+\; \sum_{i \in S} p_{i'} V_{i'} \;=\; 0$$

with the added condition that

$$p_{i'} \geq 1 \quad \forall \quad i \in S.$$

Note that for $k = 3$ the summands in 4.22 correspond precisely to the 16 patterns in 4.16 with the added condition that the "$*$" in position $i \geq 5$ should represent $p_i \geq 1$ in the corresponding solution vector. This extra condition is precisely what is needed to eliminate overcounting.

Perhaps all this is best understood with an example. For instance for $k = 3$ the patterns

$$(*, 0, 0, *, 0, *, *, 0) \quad \text{and} \quad (0, *, *, 0, *, 0, 0, *)$$

were the only ones that supported an asymmetric solution represent the two reduced systems

$$\mathcal{S}_{\{14\}} = \left\| \begin{array}{c} p_1 + p_4 - p_6 - p_7 = 0 \\ p_1 - p_4 + p_6 - p_7 = 0 \\ p_1 - p_4 - p_6 + p_7 = 0 \end{array} \right. \qquad \mathcal{S}_{\{23\}} = \left\| \begin{array}{c} p_2 + p_3 - p_5 - p_8 = 0 \\ p_2 - p_3 + p_5 - p_8 = 0 \\ -p_2 + p_3 + p_5 - p_8 = 0 \end{array} \right.$$

and correspond to the following two summands of 4.22 for $k = 3$

$$F_{\{1,4\}}(x) = \frac{1}{1 - x_1 a_1 a_2 a_3} \frac{1}{1 - x_4 a_1/a_2 a_3} \frac{x_6 a_2/a_1 a_3}{1 - x_6 a_2/a_1 a_3} \frac{x_7 a_3/a_1 a_2}{1 - x_7 a_3/a_1 a_2} \bigg|_{a_1^0 a_2^0 a_3^0} = \frac{x_1 x_4 x_6 x_7}{1 - x_1 x_4 x_6 x_7} \qquad 4.24$$

and

$$F_{\{2,3\}}(x) = \frac{1}{1 - x_2 a_1 a_2/a_3} \frac{1}{1 - x_3 a_1 a_3/a_2} \frac{x_5 a_2 a_3/a_1}{1 - x_5 a_2 a_3/a_1} \frac{x_8/a_1 a_2 a_3}{1 - x_8/a_1 a_2 a_3} \bigg|_{a_1^0 a_2^0 a_3^0} = \frac{x_2 x_3 x_5 x_8}{1 - x_2 x_3 x_5 x_8}. \qquad 4.25$$

A close look at these two expressions should reveal the key ingredient that needs to be added to our algorithms that will permit reaching $k = 5$ in the Hdd and Sdd problems. Indeed we see that $F_{\{1,4\}}(x)$ goes onto $F_{\{2,3\}}(x)$ if we act on the vector $(x_1, x_2, \cdots, x_8)$ by the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \end{pmatrix} \qquad 4.26$$

and on the triple (a_1, a_2, a_3) by the operation $a_2 \rightarrow a_2^{-1}$. In fact, σ is none other than an image of the map $(\epsilon_1, \epsilon_2, \epsilon_3) \rightarrow (\epsilon_1, \overline{\epsilon}_2, \epsilon_3)$ on the binary digits of $0, 1, \ldots, 7$, as we can easily see when we replace each i in 4.26 by the binary digits of $i - 1$

$$\sigma = \begin{pmatrix} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 010 & 011 & 000 & 001 & 110 & 111 & 100 & 101 \end{pmatrix}.$$

What goes on is quite simple. Recall that solutions p of our system $\mathcal{S}_k$ can also be viewed as assignments of weights to the vertices of the k-hypercube giving all hyperfaces equal weight. Then clearly any rotation or reflection of the hypercube will carry this assignment onto an assignment with the same property. Thus the Hyperoctahedral group $\mathcal{B}_k$ will act on all the constructs we used to solve $\mathcal{S}_k$.

To make precise the action of $\mathcal{B}_k$ we need some conventions.

(1) We will view the elements of $\mathcal{B}_k$ as pairs (α, η) with $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_k) \in S_k$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_k)$ a binary vector.

(2) Next, for any binary vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ let us set

$$(\alpha, \eta)\epsilon = (\epsilon_{\alpha_1} + \eta_1, \epsilon_{\alpha_2} + \eta_2, \cdots, \epsilon_{\alpha_k} + \eta_k) \qquad 4.27$$

with "*mod 2*" addition.

(3) This given, to each element $g = (\alpha, \eta) \in \mathcal{B}_k$ there corresponds a permutation $\sigma(g)$ by setting

$$\sigma(g) = \begin{pmatrix} 1 & 2 & \cdots & 2^k \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2^k} \end{pmatrix}. \qquad 4.28$$

where $\sigma_i = j$ if and only if the k-vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ giving the binary digits of $i-1$ is sent by 4.27 onto the k-vector giving the binary digits of $j-1$. In particular we will set

$$g(x_1, x_2, \ldots, x_{2^k}) \;=\; (x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_{2^k}}). \qquad 4.29$$

(4) In the same vein we will make $\mathcal{B}_k$ act on the k-tuple $(a_1, a_2, \ldots, a_k)$ by setting, again for $g = (\alpha, \eta)$

$$g(a_1, a_2, \ldots, a_k) \;=\; (a_{\alpha_1}^{1-2\eta_1}, a_{\alpha_2}^{1-2\eta_1}, \ldots, a_{\alpha_k}^{1-2\eta_1}). \qquad 4.30$$

With these conventions we can easily derive from 4.19 that 4.29 and 4.30 give

$$g x_i A_i \;=\; x_{\sigma_i} A_{\sigma_i}.$$

thus

$$g \prod_{i=1}^{2^k} \frac{1}{1 - x_i A_i} \Big|_{a_1^0 a_2^0 \cdots a_k^0} \;=\; \prod_{i=1}^{2^k} \frac{1}{1 - x_{\sigma_i} A_{\sigma_i}} \Big|_{a_1^0 a_2^0 \cdots a_k^0} \;=\; \prod_{i=1}^{2^k} \frac{1}{1 - x_i A_i} \Big|_{a_1^0 a_2^0 \cdots a_k^0},$$

from which we again derive the $\mathcal{B}_k$ invariance of the complete generating function

$$F_k(x_1, x_2, \ldots, x_{2^k}).$$

If we let $\mathcal{B}_{k-1}$ not only act on the indices $1, 2, \ldots, 2^{k-1}$, but also on $1', 2', \ldots, {2^{k-1}}'$ by $\sigma_{i'} = \sigma_i'$. Then $\mathcal{B}_{k-1}$ permutes the summands in 4.22 as well as the factors in the product

$$\prod_{i=1}^{2^{k-1}} \frac{1}{1 - x_i x_{i'}}.$$

Note further that if we only want the q-series $G_k(q)$ we can reduce 4.22 to

$$G_k^A(q) \;=\; \sum_{S \subseteq [1,2,\ldots,2^{k-1}]} G_S(q). \qquad 4.31$$

with

$$G_S(q) \;=\; F_S(x)\Big|_{x_i = q} \;=\; \left(\prod_{i \notin S} \frac{1}{(1 - qA_i)} \right) \times \left(\prod_{i \in S} \frac{q/A_i}{(1 - q/A_i)} \right) \Big|_{a_1^0 a_2^0 \cdots a_k^0}.$$

But if for some $g \in \mathcal{B}_{k-1}$ we have

$$F_{S_1}(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_{2^k}}) \;=\; F_{S_2}(x_1, x_2, \ldots, x_{2^k}).$$

Then replacing each x_i by q converts this to the equality

$$G_{S_1}(q) \;=\; G_{S_2}(q).$$

That means that we need only compute the constant terms in 4.31 for orbit representatives, then replace 4.31 by a sum over orbit representatives multiplied by orbit sizes. More precisely we get

$$G_k^A(q) \;=\; \sum_{i=1}^{N_k} m_i G_{S_i}(q). \tag{4.32}$$

where m_i denotes the cardinality of the orbit of $F_{S_i}(x)$. In the computer implementation we obtain orbit representatives as well as orbit sizes, by acting with $\mathcal{B}_{k-1}$ on the monomials

$$M_S \;=\; \prod_{i\in S} x_i.$$

Thus for $k=3$ we found that the 16 summands in 4.22 break up into 6 orbits but only 2 of them do contribute to F_3^A. They corresponds to the monomials 1 and x_1x_4 with respective orbit sizes 1 and 2. The orbit representative that corresponds to 1 is simply the case $S=\phi$ in 4.23 and that corresponding to x_1x_4 is given in 4.24.

Thus from 4.24. 4.32 and 4.21 we derive that

$$G_3(q) \;=\; \frac{1}{(1-q^2)^4}\Big(1+2\frac{q^4}{1-q^4}\Big) \;=\; \frac{1}{(1-q^2)^4}\frac{1+q^4}{1-q^4}.$$

For $k=4$ we have $2^8=256$ summands in 4.22 with 22 orbits but only 11 of these orbits do contribute to F_4^A. The number of denominator factors for each term is 8 which is still a reasonable number for the partial fraction algorithm. The formula for $F_4(x)$ obtained this way can be typed within a page, but we would like to introduce a nicer $F_4(x)$ using the full group $\mathcal{B}_k$ instead of $\mathcal{B}_{k-1}$, as we will do in the next paragraph. For $k=5$ we have 2^{16} summands in 4.22 with 402 orbits but only 341 orbits do contribute to F_5^A. The number of denominator factors for each term is 16 which is out of reach for the partial fraction algorithm to obtain $F_5^A(x)$. Nevertheless, in this manner we can still produce $G_5(q)$ in about 15 minutes.

The decomposition in 4.31 is only $\mathcal{B}_{k-1}$ invariant, and it is natural from the geometry of the hypercube labelings, to ask of a $\mathcal{B}_k$ invariant decomposition. To obtain such a decomposition of $F_k(x)$ we will pair off the factors containing A_i and $A_{i'}$ by means of the more symmetric identity

$$\frac{1-x_ix_{2^k+1-i}}{(1-x_iA_i)(1-x_{2^k+1-i}A_{2^k+1-i})} \;=\; \Big(1+\frac{x_iA_i}{1-x_iA_i}+\frac{x_{i'}A_{i'}}{1-x_{i'}A_{i'}}\Big)$$

and derive that

$$F_k(x) \;=\; \sum_{S+T\subseteq[1,2,\ldots,2^{k-1}]} F_{S,T}(x) \tag{4.33}$$

with

$$F_{S,T}(x) \;=\; \Big(\prod_{i\in S}^{k}\frac{x_iA_i}{1-x_iA_i}\Big)\Big(\prod_{i\in T}\frac{x_i'/A_i}{1-x_i'/A_i}\Big).$$

Note that every pair (S,T) should be identified with the set $S\cup\{i' : i\in T\}\subseteq[1,2,\ldots,2^k]$ when applying the action of $\mathcal{B}_k$.

For $k=3$ we have $3^4=81$ summands with 9 orbits but only 2 orbits do contribute to F_3^A. The two orbits corresponds to the monomials 1 and $x_1x_4x_6x_7$ with respective orbit sizes 1 and 2. The orbit representative that corresponding to 1 is simply the case $F_{\phi,\phi}=1|_{a_1^0a_2^0a_3^0a_4^0}=1$ and that corresponding to $x_1x_4x_6x_7$ is

$$F_{\{1,4\},\{2,3\}}(x)=\frac{x_1A_1}{1-x_1A_1}\frac{x_4A_4}{1-x_4A_4}\frac{x_6A_6}{1-x_6A_6}\frac{x_7A_7}{1-x_7A_7}\Big|_{a_1^0a_2^0a_3^0a_4^0}=\frac{x_1x_4x_6x_7}{1-x_1x_4x_6x_7}.$$

For $k=4$ we have $3^8=6561$ summands with 62 orbits but only 10 orbits do contribute to F_4^A. We obtain the following complete generating functions for the 10 orbit representatives:

$$
\begin{array}{ll}
(1) & 1\\
(24) & \dfrac{x_1x_{15}x_4x_{14}}{1-x_1x_{15}x_4x_{14}}\\
(16) & \dfrac{x_{16}x_7\,(x_9)^2\,x_6x_4}{1-x_{16}x_7{x_9}^2x_6x_4}\\
(96) & \dfrac{x_{15}x_3x_7\,(x_{12})^2\,(x_9)^2\,(x_6)^3}{(1-x_{12}x_7x_9x_6)\,(1-x_{15}x_3x_{12}x_9{x_6}^2)}\\
(96) & \dfrac{x_{16}x_{14}x_5x_7\,(x_{11})^2\,(x_2)^2}{(1-x_2x_7x_{11}x_{14})\,(1-x_{16}x_5x_2x_{11})}\\
(192) & \dfrac{x_9x_{10}x_1\,(x_4)^4\,(x_{15})^3\,(x_5)^2\,(x_{14})^2}{(1-x_1x_{15}x_4x_{14})\,(1-x_{15}x_5x_{10}x_4)\,(1-x_{15}x_5x_9{x_4}^2x_{14})}\\
(64) & \dfrac{x_6x_{16}x_4\,(x_3)^2\,(x_5)^2\,(x_{15})^2\,(x_{10})^3}{(1-x_{15}x_5x_{10}x_4)\,(1-x_{16}x_5x_3x_{10})\,(1-x_{15}x_3x_{10}x_6)}\\
(64) & \dfrac{x_3x_7x_4\,(x_6)^5\,x_1\,(x_9)^3\,(x_{12})^3\,(x_{15})^3}{(1-x_1x_{15}x_{12}x_6)\,(1-x_{12}x_7x_9x_6)\,(1-x_{15}x_9x_6x_4)\,(1-x_{15}x_3x_{12}x_9{x_6}^2)}\\
(32) & \dfrac{(x_{13})^3\,(x_{12})^3\,x_1x_3x_2x_6x_7x_8\,\left(1-x_1x_2x_3x_8{x_{12}}^3x_7{x_{13}}^3x_6\right)}{(1-x_1x_8x_{12}x_{13})\,(1-x_2x_{12}x_7x_{13})\,(1-x_3x_{12}x_{13}x_6)\,(1-x_1{x_{12}}^2x_7x_{13}x_6)\,(1-x_2x_3x_8x_{12}{x_{13}}^2)}\\
(8) & \dfrac{x_4x_5x_3x_6x_9x_{10}x_{15}x_{16}\,\left(1-2\,x_{15}x_{16}x_5x_3x_{10}x_9x_6x_4+{x_{15}}^2{x_{16}}^2{x_5}^2{x_3}^2{x_{10}}^2{x_9}^2{x_6}^2{x_4}^2\right)}{(1-x_{16}x_3x_9x_6)\,(1-x_{16}x_5x_9x_4)\,(1-x_{15}x_9x_6x_4)\,(1-x_{15}x_5x_{10}x_4)\,(1-x_{16}x_5x_3x_{10})\,(1-x_{15}x_3x_{10}x_6)}
\end{array}
\qquad 4.34
$$

Here the numbers in parentheses give the respective multiplicities.

Replacing all the x_i by q and summing as in 4.31, we obtain

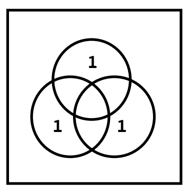
$$F_4(q) \;=\; \frac{1+q^2+21q^4+36q^6+74q^8+86q^{10}+74q^{12}+36q^{14}+21q^{16}+q^{18}+q^{20}}{(1-q^2)^7(1-q^4)^4(1-q^6)}.$$

We should mention that the partial fraction algorithm delivers this rational function in less than a second by directly computing the constant term in I.12 for $k = 4$. We computed the complete generating functions given in 4.32 because we will need them later and also to illustrate an alternate path to $G_4(q)$ and $G_5(q)$.

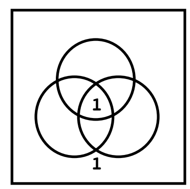
Computing the orbit representatives for $k = 5$ requires the construction of the $2^5 \times 5! = 3840$ elements of $\mathcal{B}_5$ and examining their action on the $3^{16} = 43046721$ symmetric supports. This took a few hours on our computers. We found in this manner that the 43046721 summands in 4.31 break up into 15418 orbits and of these 6341 contribute to the sum. Most of the orbits have denominators of less than 16 factors. It also took about 15 minutes to persuade MAPLE to deliver $G_5(q)$ in the form displayed in the introduction.

Remark 4.1

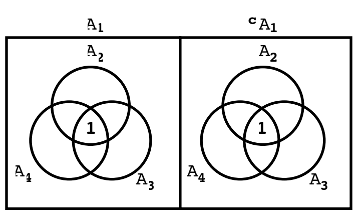
It is interesting to point out that computing complete generating functions for orbit representatives of summands in 4.22 yielded as a byproduct orbit representatives of the extreme rays of our Diophantine cone for $k = 4$ and $k = 5$. Note that for $k = 3$ the representatives can be directly derived from our hand computation, there are only two and the corresponding Venn Diagrams are

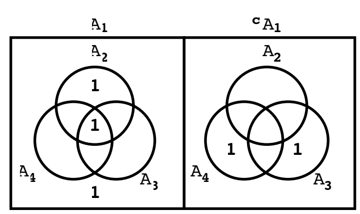


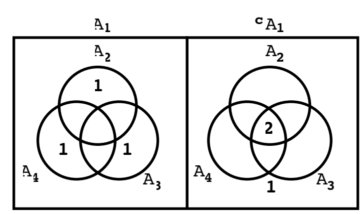
and



here the regions without numbers are empty. The number 1 indicates that the region has only one element. For $k = 4$ we found that there are only three orbits, containing $24, 8$ and 16 elements respectively, the corresponding diagrams are depicted below.

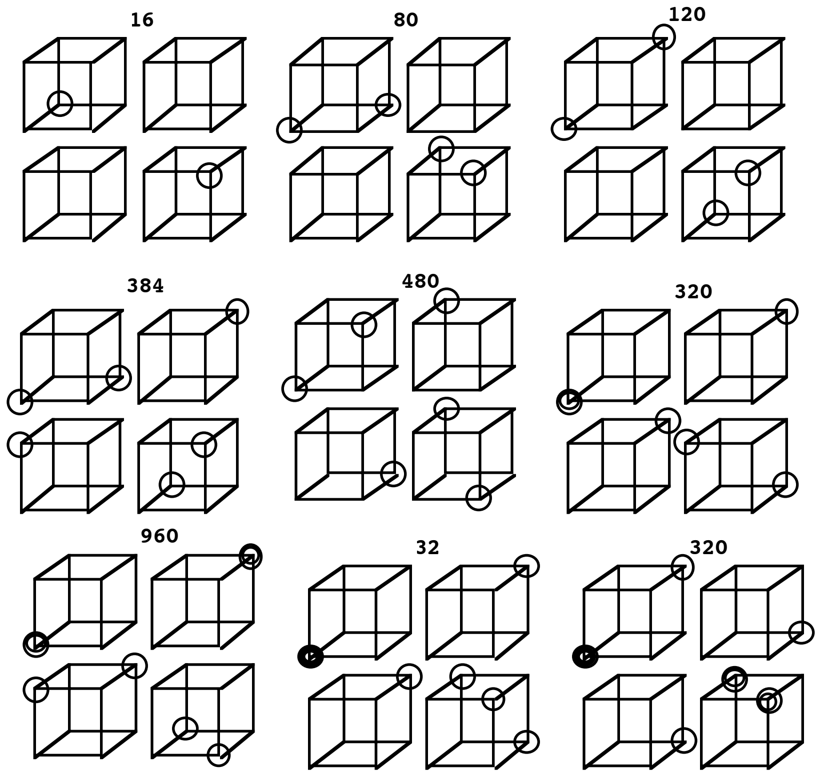




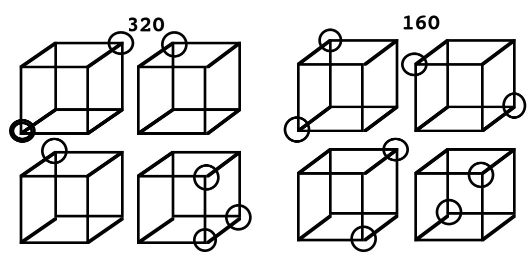


Note, for $k = 4$ each Venn diagram is depicted as a pair of Venn diagrams of $k = 3$. The first member of the pair renders the Venn diagram of $A_1 \cap A_2, A_1 \cap A_3, A_1 \cap A_4$ and the second member renders the Venn diagram of ${}^cA_1 \cap A_2,\ {}^cA_1 \cap A_3,\ {}^cA_1 \cap A_4$.

For $k = 5$ we found that there are 2712 extreme rays which break up into 9 orbits. We give below a set of representatives depicted as assignments of weights to the vertices of the 5 dimensional hypercube. We imagine that the vertices of this hypercube are indexed by the binary digits of $0, 1, 2, \ldots, 31$ with 00000 the vertex at the origin and 11111 giving the coordinates of the opposite vertex. In the following figures each hypercube is represented by two rows of two cubes. The cubes in the first row, from left to right, have the vertices labeled with the binary digits of 1 to 16 (minus 1) and the cubes in the second row rave the vertices labeled with the binary digits of 17 to 32 (minus 1). The possible weights of the vertices here are $0, 1, 2, 3$. Vertices of weight $0, 1, 2, 3$ are respectively surrounded by $0, 1, 2, 3$ concentric circles. The integer on the top of each diagram gives the size of the corresponding orbit.



Each of the corresponding solutions of our system $\mathcal{S}_5$ is "*minimal*" that is it cannot be decomposed into a non-trivial sum of solutions. But we found that there are also 480 minimal solutions that do not come from extreme rays. The latter break up into two orbits. We give below their representatives.



It turns out that the same orbit reduction idea can also be used to compute $W_5(q)$. In fact, we can carry out almost verbatim the same steps that yielded the orbit decomposition of the complete generating function $F_k(x_1, x_2, \ldots, x_{2^k})$ to obtain the complete generating function $W_k(x_1, x_2, \ldots, x_{2^k})$. Recall that the latter was originally defined in 1.18 as the constant term

$$W_k(x_1, x_2, \ldots, x_{2^k}) \;=\; \prod_{j=1}^{k}(1-a_j^2)\prod_{i=1}^{2^k}\frac{1}{1-x_iA_i}\Big|_{a_1^0a_2^0\cdots a_k^0}. \qquad 4.35$$

To carry out its decomposition we need only observe that if we let

$$\widetilde{W}_k(x_1,x_2,\ldots,x_{2^k}) \;=\; \frac{1}{2^k}\prod_{j=1}^{k}(1-a_j^2)(1-a_j^{-2})\prod_{i=1}^{2^k}\frac{1}{1-x_iA_i}\Big|_{a_1^0a_2^0\cdots a_k^0}, \qquad 4.36$$

then

$$W_k(q)=\widetilde{W}_k(q).$$

The reason for this is that using 4.19 we can easily show that, when all the x_i are replaced by q, the constant term in 4.35 is not affected if we replace any a_i by a_i^{-1}. Thus if we average out the right hand side of 4.35 over all these interchanges the result will be simply the right hand side of 4.36 due to the simple relation

$$1-\frac{a_i^2+a_i^{-2}}{2} \;=\; \frac{1}{2}(1-a_i^2)(1-a_i^{-2}).$$

Now 4.36 brings to evidence that $\widetilde{W}_k(x)$ is $\mathcal{B}_k$ invariant while $W_k(x)$ is not. Symmetrizing $W_k(x)$ gives $\widetilde{W}_k(x)$. We can obtain either a $\mathcal{B}_{k-1}$ invariant decomposition or a $\mathcal{B}_k$ invariant decomposition of $\widetilde{W}_k(x)$ just as for $F_k(x)$.

The orbit reduction can also be used to considerably speed up steps a_k) and b'_k) in the divided difference algorithm. The idea is that if we are to carry out step b'_k) we do not need the complete generating function $W_{k-1}(x)$. More precisely, if in step $k-1$ we obtain that the orbit representatives in the sum

$$\widetilde{W}_{k-1}(x)=\sum_{S+T\subseteq[1,2,\ldots,2^{k-2}]}\widetilde{W}_{S,T}(x)$$

are the summands

$$\widetilde{W}_{S_1,T_1}(x)\;,\;\widetilde{W}_{S_2,T_2}(x)\;,\;\ldots,W_{S_N,T_N}(x)$$

with respective multiplicities

$$m_1,m_2,\ldots,m_N,$$

then in step a_k) we can replace $\widetilde{W}_{k-1}(x)$ by the sum

$$\widetilde{W}'_{k-1}(x) \;=\; \sum_{i=1}^{N}m_i\widetilde{W}_{S_i,T_i}(x)$$

and obtain

$$\widetilde{WW}'_{k-1}(x) \;=\; \sum_{i=1}^{N}m_i\;\delta_{1,1+2^{k-1}}\cdots\delta_{2^{k-1},2^k}\widetilde{W}_{S_i,T_i}(x).$$

Since the $\mathbf{B}_{k-1}$ invariance of $\widetilde{W}_{k-1}(x)$ yields that

$$\widetilde{WW}_{k-1}(ax_1,\ldots,ax_{2^{k-1}},x_{2^{k-1}+1}/a,\ldots,x_{2^k}/a)\Big|_{x_i=q} \;=\; \widetilde{WW}'_{k-1}(ax_1,\ldots,ax_{2^{k-1}},x_{2^{k-1}+1}/a,\ldots,x_{2^k}/a)\Big|_{x_i=q}$$

we see that replacing $\widetilde{W}_{k-1}(x)$ by $\widetilde{W}'_{k-1}(x)$ does not affect the result of step b'_k).

We can use the same argument to obtain $F_k(x)$. So starting with the orbit representatives in 4.34, applying divided differences to each orbit representatives and computing the constant terms separately, we can obtain $G_5(q)$ in about 12 minutes, which turns out to be the fastest way up to now.

When working with $W_5(q)$, we need an analogue of the collection in 4.34. This idea is best illustrated by the $k=3$ case. We can clearly see the advantage of orbit reduction in producing a compressed version of $\widetilde{W}_k(x)$. For $k=3$, the $\mathcal{B}_3$ decomposition will give 9 orbits with only 7 of them contributing to $\widetilde{W}_3(x)$. We thus get

$$\widetilde{W}_3^A(x) = \frac{1}{|\mathcal{B}_3|}\sum_{g\in\mathcal{B}_3} g\left(9\ monomials + \frac{27 monomials}{1-x_1x_4x_6x_7}\right).$$

The actual formula is a little complicated and its combinatorial meaning is not significant, but it is good enough for us to use the divided difference algorithm to compute $W_4(q)$. ¿From this, by symmetrizing and re-choosing representatives, we obtain a simpler representative. Namely we end up obtaining that

$$\widetilde{W}_3^A(x) = \frac{1}{|\mathcal{B}_3|}\sum_{g\in\mathcal{B}_3} g\left(-1+3\,x_2x_6 - x_1x_2x_6x_4 + \frac{2-6\,x_1x_7 - {x_1}^2 + 6\,x_1{x_4}^2x_7 - {x_1}^2{x_4}^2{x_7}^2}{(1-x_1x_6x_4x_7)}\right).$$

Originally we hoped that this formula would enable us to compute $W_4(q)$ entirely by hand, but we were unable to do so.

For $k=4$, directly using the $\mathcal{B}_4$ decomposition gives us 62 orbits with 27 of them contributing to $\widetilde{W}_4(x)$. The representatives obtained this way are too complex for further computation since several of them have thousands of monomials in their numerators. The similar idea of symmetrizing and re-choosing applies to give us 10 reasonably simple representatives for $\widetilde{W}_4(x)$, but typesetting them will take several pages. Nevertheless we are able to use them in the divided difference algorithm.

Having noticed that for $k=2,3,4$ the divided difference algorithm reduced the computation of $W_k(q)$ to a rather simple constant term evaluation, we tried to see what it gave for $k=5$. Adding the contributions of these 10 representatives, before taking the constant term, yielded a rational function of the form

$$\frac{1}{(1-q^2)\,(1-q^4)^4\,(1-q^6)\left(1-\frac{q^2}{a^2}\right)(1-a^2q^2)\left(1-\frac{q^4}{a^2}\right)^3(1-a^2q^4)^3}\times$$
$$\times\frac{357\ monomials}{\left(1-\frac{q^4}{a^4}\right)^2(1-a^4q^4)^2\left(1-\frac{q^6}{a^2}\right)(1-a^2q^6)\left(1-\frac{q^6}{a^4}\right)(1-a^4q^6)\left(1-\frac{q^6}{a^6}\right)(1-a^6q^6)}.$$

It turns out that this is actually a rational function in q^2 and a^2. Replacing q by $q^{1/2}$ and a by $a^{1/2}$ and then taking constant term in a, we can obtain $W_5(q^{1/2})$. Using this approach Maple can deliver $W_5(q)$ in only about 5 minutes in total which is the shortest time we have been able to compute this series.

Before closing it will be worthwhile to include a description of the first algorithm that was used to obtain $G_5(q)$ and $W_5(q)$ since it contains another trick that clearly shows the flexibility afforded by the partial fraction algorithm in the computation of constant terms.

In this approach we begin by replacing our system $\mathcal{S}_k$ by a system $\mathcal{S}'_k$ which has the same cone of solutions. To describe the new system we will use the k-tuple of sets model. The idea is that originally we got $\mathcal{S}_k$ by equating the cardinality of each set to the cardinality of its complement obtaining

$$\mathcal{S}_k \;=\; \left\| \begin{array}{c} |A_1| = |\,{}^cA_1| \\ |A_2| = |\,{}^cA_2| \\ \cdots \\ |A_k| = |\,{}^cA_k| \end{array} \right. .$$

Now it is quite clear that this is equivalent to set

$$\mathcal{S}'_k \;=\; \left\| \begin{array}{c} |A_1| = d \\ |A_2| = d \\ \cdots \\ |A_k| = d \\ |\,{}^cA_1| = d \end{array} \right. . \qquad 4.37$$

For instance, using the binary digit indexing of the variables, for $k = 3$ this results in the following system of 4 equations in 9 unknowns

$$\begin{array}{ccccccccccccccccc} p_{000} & + & p_{001} & + & p_{010} & + & p_{011} & & & & & & & & -d & = & 0 \\ p_{000} & + & p_{001} & + & & & & + & p_{100} & + & p_{101} & & & & -d & = & 0 \\ p_{000} & & & + & p_{010} & & & + & p_{100} & & & + & p_{110} & & -d & = & 0 \\ & & & & & & & & p_{100} & + & p_{101} & + & p_{110} & + \; p_{111} & -d & = & 0 \end{array} . \qquad 4.38$$

This given, our rational function $G_3(q) = G_3(q,1)$ may be also obtained by taking the following constant term

$$G_3(q,t) \;=\; \frac{1}{1-qa_1a_2a_3}\frac{1}{1-qa_1a_2}\frac{1}{1-qa_1a_3}\frac{1}{1-qa_1} \frac{1}{1-qa_2a_3a_4}\frac{1}{1-qa_2a_4}\frac{1}{1-qa_3a_4}\frac{1}{1-qa_4}\frac{1}{1-t/a_1a_2a_3a_4}\Big|_{a_1^0a_2^0a_3^0a_4^0} . \qquad 4.39$$

Here we choose the order $q < t < a_1 < a_2 < \cdots$ and we can not set $t = 1$ as this moment yet.

Now it turns out to be expedient to start by eliminating a_4. This can simply be done by omitting the factor $1/(1 - t/a_1a_2a_3a_4)$ and making the substitution $a_4 \to t/a_1a_2a_3$, obtaining

$$G_3(q,t) \;=\; \frac{1}{1-qa_1a_2a_3}\frac{1}{1-qa_1a_2}\frac{1}{1-qa_1a_3}\frac{1}{1-qa_1} \frac{1}{1-qt/a_1}\frac{1}{1-qt/a_1a_3}\frac{1}{1-qt/a_1a_2}\frac{1}{1-qt/a_1a_2a_3}\Big|_{a_1^0a_2^0a_3^0} . \qquad 4.40$$

Setting $t = 1$ is valid here. Grouping terms containing the same subset of the variables a_1, a_2, a_3 gives

$$G_3(q) = \frac{1}{1-qa_1}\frac{1}{1-q/a_1} \frac{1}{1-qa_1a_2}\frac{1}{1-q/a_1a_2} \frac{1}{1-qa_1a_3}\frac{1}{1-q/a_1a_3} \frac{1}{1-qa_1a_2a_3}\frac{1}{1-q/a_1a_2a_3}\Big|_{a_1^0a_2^0a_3^0} . \qquad 4.41$$

Likewise, we can easily see that the general form of 4.39 is

$$G_k(q,t) \;=\; \Big(\prod_{S\subseteq[2,\dots,k]}\frac{1}{1-qa_1A(S)}\Big)\Big(\prod_{S\subseteq[2,\dots,k]}\frac{1}{1-qA(S)a_{k+1}}\Big)\frac{1}{1-t/a_1a_2\cdots a_ka_{k+1}}\Big|_{a_1^0a_2^0\cdots a_k^0a_{k+1}^0}$$

with

$$A(S) \;=\; \prod_{i\in S}a_i.$$

Removing the last factor and setting $a_{k+1}=t/a_1a_2\cdots a_k$ gives

$$G_k(q,t) \;=\; \Big(\prod_{S\subseteq[2,\dots,k]}\frac{1}{1-qa_1A(S)}\Big)\Big(\prod_{S\subseteq[2,\dots,k]}\frac{1}{1-qtA(S)/a_1a_2\cdots a_k}\Big)\Big|_{a_1^0a_2^0\cdots a_k^0}$$

and by setting $t=1$ this can be rewritten as

$$G_k(q) \;=\; \Big(\prod_{S\subseteq[2,\dots,k]}\frac{1}{1-qa_1A(S)}\frac{1}{1-q/a_1A(S)}\Big)\Big|_{a_1^0a_2^0\cdots a_k^0}.$$

Now comes the next trick: grouping terms according as $A(S)$ contains a_2 or not. This gives

$$G_k(q)=\Big(\prod_{S\subseteq[3,\dots,k]}\frac{1}{1-qa_1A(S)}\frac{1}{1-q/a_1A(S)}\Big)\Big(\prod_{S\subseteq[3,\dots,k]}\frac{1}{1-qa_1a_2A(S)}\frac{1}{1-q/a_1a_2A(S)}\Big)\Big|_{a_1^0a_2^0\cdots a_k^0}. \quad 4.42$$

To appreciate the significance of this step let us see what this gives for $k=3$. Grouping terms in 4.41 as was done in 4.42 gives

$$\begin{aligned}G_3(q)=&\frac{1}{1-qa_1}\frac{1}{1-q/a_1}\frac{1}{1-qa_1a_3}\frac{1}{1-q/a_1a_3}\\ &\frac{1}{1-qa_1a_2}\frac{1}{1-q/a_1a_2}\frac{1}{1-qa_1a_2a_3}\frac{1}{1-q/a_1a_2a_3}\Big|_{a_1^0a_2^0a_3^0}.\end{aligned} \quad 4.43$$

Let us now see what the partial fraction algorithm gives if we first eliminate a_2. This entails computing the constant term

$$Q \;=\; \frac{1}{1-qa_1a_2}\frac{1}{1-q/a_1a_2}\frac{1}{1-qa_1a_2a_3}\frac{1}{1-q/a_1a_2a_3}\Big|_{a_2^0}.$$

Using the terminology of [2] we note that the first and third factors are contributing and the other two are dually contributing. Thus,

$$Q \;=\; \frac{A_1}{1-qa_1a_2}+\frac{A_3}{1-qa_1a_2a_3}\Big|_{a_2^0} \;=\; A_1+A_3 \quad 4.44$$

with

$$\begin{aligned}A_1 \;&=\; \frac{a_1^2a_2^2a_3}{(a_1a_2-q)(1-qa_1a_2a_3)(a_1a_2a_3-q)}\Big|_{a_2=1/qa_1}\\ &=\; \frac{a_3/q^2}{(1/q-q)(1-a_3)(a_3/q-q)}\Big|_{a_2=1/qa_1} \;=\; \frac{1}{(1-q^2)(1-a_3)(1-q^2/a_3)}\end{aligned} \quad 4.45$$

and

$$\begin{aligned} A_3 &= \frac{a_1^2a_2^2a_3}{(1-qa_1a_2)(a_1a_2-q)(a_1a_2a_3-q)}\Big|_{a_2=1/qa_1a_3} \\ &= \frac{1/q^2a_3}{(1-1/a_3))(1/qa_3-q)(1/q-q)} = \frac{a_3}{(a_3-1)(1-q^2a_3)(1-q^2)} \end{aligned}. \qquad 4.46$$

Using 4.44 in 4.43 gives

$$\begin{aligned} G_3(q) &= \frac{1}{1-qa_1}\frac{1}{1-q/a_1}\frac{1}{1-qa_1a_3}\frac{1}{1-q/a_1a_3}\Big(A_1 + A_3\Big)\Big|_{a_1^0a_3^0} \\ &= \frac{1}{1-qa_1}\frac{1}{1-q/a_1}\frac{1}{1-qa_1a_3}\frac{1}{1-q/a_1a_3}\Big|_{a_1^0}\Big(A_1 + A_3\Big)\Big|_{a_3^0} \end{aligned}. \qquad 4.47$$

The last equality is due to the fact that A_1 and A_3 do not contain a_1. Next we will compute the constant term

$$Q' = \frac{1}{1-qa_1}\frac{1}{1-q/a_1}\frac{1}{1-qa_1a_3}\frac{1}{1-q/a_1a_3}\Big|_{a_1^0}.$$

The surprise, which is the whole point of the factorization in 4.42, is that this leads to the same partial fraction decomposition! More precisely we see that

$$Q' = \frac{B_1}{1-qa_1} + \frac{B_3}{1-qa_1a_3}\Big|_{a_1^0} = B_1 + B_3$$

with

$$\begin{aligned} B_1 &= \frac{a_1^2a_3}{(a_1-q)(1-qa_1a_3)(a_1a_3-q)}\Big|_{a_1=1/q} \\ &= \frac{a_3/q^2}{(1/q-q)(1-a_3)(a_3/q-q)}\Big|_{a_2=1/qa_1} = \frac{1}{(1-q^2)(1-a_3)(1-q^2/a_3)} = A_1 \end{aligned}$$

and

$$\begin{aligned} B_3 &= \frac{a_1^2a_3}{(1-qa_1)(a_1-q)(a_1a_2a_3-q)}\Big|_{a_2=1/qa_1a_3} \\ &= \frac{1/q^2a_3}{(1-1/a_3))(1/qa_3-q)(1/q-q)} = \frac{a_3}{(a_3-1)(1-q^2a_3)(1-q^2)} = A_3 \end{aligned}.$$

Thus 4.47 becomes

$$G_3(q) = (A_1+A_3)^2\Big|_{a_3^0} = A_1^2\Big|_{a_3^0} + A_3^2\Big|_{a_3^0} + 2A_1A_3\Big|_{a_3^0}.$$

It is easy to see that the same collapse of terms occurs in the general case. Indeed we can rewrite 4.42 in the form

$$\begin{aligned} G_k(q) = &\Bigg(\prod_{S\subseteq[3,\dots,k]}\frac{1}{1-qa_1a_2A(S)}\frac{1}{1-q/a_1a_2A(S)}\Big|_{a_2^0}\Bigg)\times \\ &\times\Bigg(\prod_{S\subseteq[3,\dots,k]}\frac{1}{1-qa_1A(S)}\frac{1}{1-q/a_1A(S)}\Big|_{a_1^0}\Bigg)\Big|_{a_3^0\cdots a_k^0} \end{aligned}. \qquad 4.48$$

We can see that, in both constant terms with respect to a_1 and a_2, the first member of each pair of factors contributes and the second dually contributes, and the partial fraction algorithm yields

$$\prod_{S\subseteq[3,\ldots,k]}\frac{1}{1-qa_1a_2A(S)}\frac{1}{1-q/a_1a_2A(S)}\Big|_{a_2^0} \;=\; \sum_{T\subseteq[3,\ldots,k]}\frac{C_T}{1-qa_1a_2A(T)}\Big|_{a_2^0} \;=\; \sum_{T\subseteq[3,\ldots,k]} C_T$$

with

$$\begin{aligned} C_T &= \Big(1-qa_1a_2A(T)\Big)\prod_{S\subseteq[3,\ldots,k]}\frac{1}{1-qa_1a_2A(S)}\frac{1}{1-q/a_1a_2A(S)}\Big|_{a_2=1/qa_1A(T)} \\ &= \frac{1}{(1-q/a_1a_2A(T))}\prod_{\substack{S\subseteq[3,\ldots,k]\\ S\neq T}}\frac{1}{1-qa_1a_2A(S)}\frac{1}{1-q/a_1a_2A(S)}\Big|_{a_2=1/qa_1A(T)} \\ &= \frac{1}{(1-q^2)}\prod_{\substack{S\subseteq[3,\ldots,k]\\ S\neq T}}\frac{1}{1-A(S)/A(T)}\frac{1}{1-q^2A(T)/A(S)} \end{aligned}$$

and we see that, as in the case $k=3$, all of these coefficients are independent of a_1. Moreover we can also easily see that

$$\Big(1-qa_1A(T)\Big)\prod_{S\subseteq[3,\ldots,k]}\frac{1}{1-qa_1A(S)}\frac{1}{1-q/a_1A(S)}\Big|_{a_1=1/qA(T)} \;=\; C_T.$$

This reduces the computation of $G_k(q)$ to the sum of $2^{k-2}+\binom{2^{k-2}}{2}$ constant terms of the form

$$G_k(q) \;=\; \sum_{i=1}^{2^{k-2}} A_i^2\Big|_{a_3^0\cdots a_k^0} + 2\sum_{1\leq i<j\leq 2^{k-2}} A_iA_j\Big|_{a_3^0\cdots a_k^0}$$

Note that for $k=5$ we are reduced to the calculation of $2^3+\binom{2^3}{2}=36$ constant terms. Most importantly in each of these constant terms the denominators have at most 14 factors. The latest version of the partial fraction algorithm (motivated by the computation of $G_5(q)$) posted in the web site

http://www.combinatorics.net.cn/homepage/xin/maple/ell2.rar"

computed these 36 constant terms on a Pentium 4 Windows system computer with a 3G Hz processor in about 22 minutes which is a considerable time reduction from the 2 hours and 15 minutes that took previous versions of the algorithm to compute these constant terms.

The same approach can be used to calculate $W_5(q)$, but in a much simpler way. The constant terms have to be appropriately modified. Again we will start with the case $k=3$.

The k-tuple of sets interpretation of the constant term in 2.3 given in section 2, yields that to obtain $W_k(q)$ we must compute the constant terms corresponding to the 2^k systems obtained by requiring each A_i to have 2 or 0 more elements than its complement in all possible ways and then carry out an inclusion exclusion type alternating sum of the results.

A moments reflection should reveal that to get $W_3(q) = W_3(q,1)$ we need only modify 4.39 to

$$\begin{aligned} W_3(q,t) \;&=\; \Big(\big(1-a_4/a_1\big)\big(1-1/a_2\big)\big(1-1/a_3\big)\Big)\times \\ &\quad\times \frac{1}{1-qa_1a_2a_3}\frac{1}{1-qa_1a_2}\frac{1}{1-qa_1a_3}\frac{1}{1-qa_1} \\ &\quad\quad\times \frac{1}{1-qa_2a_3a_4}\frac{1}{1-qa_2a_4}\frac{1}{1-qa_3a_4}\frac{1}{1-qa_4}\frac{1}{1-t/a_1a_2a_3a_4}\Big|_{a_1^0a_2^0a_3^0a_4^0} \end{aligned} \qquad 4.49$$

In fact expanding the first factor gives the 8 terms

$$1-1/a_2-1/a_3-a_4/a_1+a_4/a_1a_2+a_4/a_1a_3+1/a_2a_3-a_4/a_1a_2a_3.$$

And we see that the 8 constant terms obtained by expanding this factor in 4.49 correspond in order to the following 8 modified versions of $\mathcal{S}_3'$

$$\left\|\begin{array}{rcl}|A_1| &=& d\\ |A_2| &=& d\\ |A_3| &=& d\\ |{}^cA_1| &=& d\end{array}\right., \quad \left\|\begin{array}{rcl}|A_1| &=& d+1\\ |A_2| &=& d\\ |A_3| &=& d\\ |{}^cA_1| &=& d-1\end{array}\right., \quad \left\|\begin{array}{rcl}|A_1| &=& d\\ |A_2| &=& d+1\\ |A_3| &=& d\\ |{}^cA_1| &=& d\end{array}\right., \quad \left\|\begin{array}{rcl}|A_1| &=& d\\ |A_2| &=& d\\ |A_3| &=& d+1\\ |{}^cA_1| &=& d\end{array}\right.$$

$$\left\|\begin{array}{rcl}|A_1| &=& d+1\\ |A_2| &=& d+1\\ |A_3| &=& d\\ |{}^cA_1| &=& d-1\end{array}\right., \quad \left\|\begin{array}{rcl}|A_1| &=& d+1\\ |A_2| &=& d\\ |A_3| &=& d+1\\ |{}^cA_1| &=& d-1\end{array}\right., \quad \left\|\begin{array}{rcl}|A_1| &=& d\\ |A_2| &=& d+1\\ |A_3| &=& d+1\\ |{}^cA_1| &=& d\end{array}\right., \quad \left\|\begin{array}{rcl}|A_1| &=& d+1\\ |A_2| &=& d+1\\ |A_3| &=& d+1\\ |{}^cA_1| &=& d-1\end{array}\right.$$

Now the elimination of a_4 in 4.49 and then setting $t=1$ (as for $G_3(q)$) gives

$$\begin{aligned} W_3(q) \;&=\; \Big(\big(1-1/a_1^2a_2a_3\big)\big(1-1/a_2\big)\big(1-1/a_3\big)\Big)\times \\ &\quad\times \frac{1}{1-qa_1a_2a_3}\frac{1}{1-qa_1a_2}\frac{1}{1-qa_1a_3}\frac{1}{1-qa_1} \\ &\quad\quad\times \frac{1}{1-q/a_1}\frac{1}{1-q/a_1a_3}\frac{1}{1-q/a_1a_2}\frac{1}{1-q/a_1a_2a_3}\Big|_{a_1^0a_2^0a_3^0}. \end{aligned}$$

For general k, we are left to compute the constant term

$$W_k(q) = \big(1-1/a_1^2a_2\cdots a_k\big)\prod_{i=2}^{k}\big(1-1/a_i\big)\Bigg(\prod_{S\subseteq[2,\ldots,k]}\frac{1}{1-qa_1A(S)}\frac{1}{1-q/a_1A(S)}\Bigg)\Big|_{a_1^0a_2^0\cdots a_k^0}.$$

Using this formula, the updated package will directly deliver $W_5(q)$ in about 17 minutes. This is because the factors in the numerator nicely cancel some of the denominators of the intermediate rational functions.

REFERENCES

[1] G.E. Andrews, *MacMahon's partition analysis. I. The lecture hall partition theorem*, Mathematical Essays in Honor of Gian-Carlo Rota (Cambridge MA 1996), 1–22, Progr. Math., 161, Birkhäuser Boston, Boston, MA, 1998.

[2] A. M. Garsia, N. Wallach, G. Xin, M. Zabrocki, *Kronecker coefficients via Symmetric Functions and Constant Term Identities*, J. Combin. Theory A., to appear.

[3] A. M. Garsia , N. Wallach, G. Xin, M. Zabrocki, *Hilbert Series of Invariants, Constant terms and Kostka-Foulkes Polynomials* , Discrete Math., to appear.

[4] J-G. Luque, J. Y. Thibon, *Polynomial Invariants of four cubits*. Physical Review A **67**, 042303 (2003).

[5] J-G. Luque, J. Y. Thibon, *Algebraic Invariants of five cubits*, J. Phys. A: Math. Gen. 39 (2006) 371-377.

[6] R. P. Stanley, *Enumerative Combinatorics* , Volume I, Cambridge Studies in Advanced Mathematics, 49. Cambridge University Press, Cambridge, 1997.

[7] N. Wallach, *Quantum Computing and entaglement for mathematicians* , CIME proceedings of the Venice Summer School June 2006, to appear.

[8] N. Wallach, *The Hilbert series of measures of entaglement for 4 qubits* , Acta Appl. Math. **86** (2005), no 1-2 pp. 203-220.

[9] G. Xin, *A fast algorithm for MacMahon's partition analysis*, Electron. J. Combin., 11 (2004), R53. arXiv: math.CO/0408377