Elliptic Curves and Chip-Firing Games on Graphs

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University of Minnesota Colloquium

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Outline

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1. Introduction to Algebraic Curves over Finite Fields
2. Elliptic Curves and a Combinatorial Interpretation of N_k's

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3. Journey into Graph Theory: Spanning Trees

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6. Elliptic Cyclotomic Polynomials and Other Amusements

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6. Elliptic Cyclotomic Polynomials and Other Amusements
7. Further Horizons: Connections to Tropical Geometry

Algebraic Curves over Finite Fields

$\mathbb{F}_q$, a finite field containing q elements, where q is a power of a prime.

$\mathbb{F}_{q^k}$ is a field extension; $\overline{\mathbb{F}_q}$ is an algebraic closure.

Nonsingular Projective Plane Curve (smooth model chosen)

$$C : f(x, y) = 0 \quad \text{plus a single point at infinity.}$$

$$C(\mathbb{F}_q) \subset C(\mathbb{F}_{q^{k_1}}) \subset C(\mathbb{F}_{q^{k_2}}) \subset \cdots \subset C(\overline{\mathbb{F}_q})$$

for any sequence of natural numbers $1|k_1|k_2|\ldots$.

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The **Frobenius** map π acts on curve C over finite field $\mathbb{F}_q$ via

$$\pi(a, b) = (a^q, b^q) \quad \text{and} \quad \pi(P_\infty) = P_\infty.$$

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Fact

For point $P \in C(\overline{\mathbb{F}_q})$,

$$\pi(P) \in C(\overline{\mathbb{F}_q}).$$

Fact

For point $P \in C(\mathbb{F}_{q^k})$,

$$\pi^k(P) = P.$$

Let N_k be the number of points on curve C, over finite field $\mathbb{F}_{q^k}$.

Alternatively, N_k counts the number of points in $C(\overline{\mathbb{F}_q})$ which are fixed by the kth power of the Frobenius map, π^k.

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Using this sequence, we define the **zeta function of an algebraic variety**, which can be written several different ways, including as an exponential generating function.

$$\begin{aligned} Z(C,T) &= \exp\left(\sum_{k=1}^{\infty} N_k \frac{T^k}{k}\right) = 1 + \sum_{k\geq 1} H_k T^k \\ &= \prod_{\mathfrak{p}} \frac{1}{1 - T^{deg\ \mathfrak{p}}} \quad \text{where } \mathfrak{p} \text{ is a prime ideal} \\ \zeta(s) &= \prod_{p \text{ prime integer}} \frac{1}{1-p^{-s}} = \sum_{n\geq 1} \frac{1}{n^s} \end{aligned}$$

Theorem (Rationality - Weil 1948)

$$Z(C,T) = \frac{(1-\alpha_1 T)(1-\alpha_2 T)\cdots(1-\alpha_{2g-1}T)(1-\alpha_{2g}T)}{(1-T)(1-qT)}$$

for complex numbers α_i's, where g is the genus of the curve C. Furthermore, the numerator of $Z(C,T)$ has integer coefficients.

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Theorem (Functional Equation - Weil 1948)

$$Z(C,T) = q^{g-1}T^{2g-2}Z(C,1/qT)$$

$$\begin{aligned} N_k &= p_k[1+q-\alpha_1-\cdots-\alpha_{2g}] \\ &= 1+q^k-\alpha_1^k-\cdots-\alpha_{2g}^k \end{aligned}$$

The Zeta Function of curve C of genus g, hence the entire sequence of $\{N_k\}$'s, only depends on $\{q, N_1, N_2, \ldots, N_g\}$.

Elliptic Curves, and a Combinatorial Interpretation of N_k

Specializing to the case of an elliptic curve E, or a genus one curve, a lot more is known and there is additional structure.

Facts

1. *E can be represented as the zero locus in $\mathbb{P}^2$ of the equation*

$$y^2 = x^3 + Ax + B$$

for $A, B \in \mathbb{F}_q$. (if $p \neq 2,\ 3$)

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2. *E has a group structure where two points on E can be added to yield another point on the curve.*
3. *The Frobenius map is compatible with the group structure:*

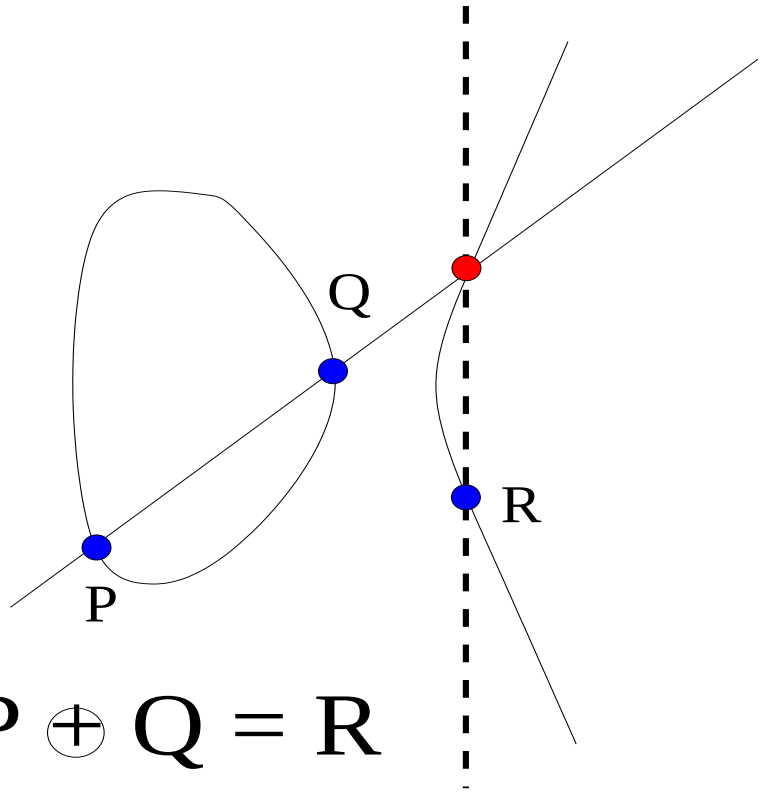
$$\pi(P \oplus Q) = \pi(P) \oplus \pi(Q).$$

Recall that $\pi(x, y) = (x^q, y^q)$ and

$$\pi^k(P) = P \text{ if and only if } P \in E(\mathbb{F}_{q^k}).$$

Elliptic Curve Group Law Geometrically

Draw Chord/Tangent Line and then reflect about horizontal axis



$$P \oplus Q = R$$

Elliptic Curve Group Law Algebraically

If $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, then

$$P_1 \oplus P_2 = P_3 = (x_3, y_3) \quad \text{where}$$

1) If $x_1 \neq x_2$ then

$$x_3 = m^2 - x_1 - x_2 \quad \text{and} \quad y_3 = m(x_1 - x_3) - y_1 \quad \text{with} \quad m = \frac{y_2 - y_1}{x_2 - x_1}.$$

2) If $x_1 = x_2$ but ($y_1 \neq y_2$, or $y_1 = 0 = y_2$) then $P_3 = P_\infty$.
3) If $P_1 = P_2$ and $y_1 \neq 0$, then

$$x_3 = m^2 - 2x_1 \quad \text{and} \quad y_3 = m(x_1 - x_3) - y_1 \quad \text{with} \quad m = \frac{3x_1^2 + A}{2y_1}.$$

4) P_∞ acts as the identity element in this addition.

Rationality (Hasse 1933)

$$Z(E,T) = \frac{(1-\alpha_1 T)(1-\alpha_2 T)}{(1-T)(1-qT)} = \frac{1-(1+q-N_1)T+qT^2}{(1-T)(1-qT)}$$

for complex numbers α_1 and α_2. (In fact $|\alpha_1| = |\alpha_2| = \sqrt{q}$.)

Functional Equation

$$Z(E, 1/qT) = Z(E,T).$$

$$\begin{aligned} N_k &= p_k[1+q-\alpha_1-\alpha_2] \\ &= 1+q^k-\alpha_1^k-\alpha_2^k \end{aligned}$$

and the Functional Equation implies

$$\alpha_1\alpha_2 = q.$$

Thus the entire sequence of N_k's, for elliptic curve E, only depends on q and N_1.

Theorem (Garsia 2004)

For an elliptic curve, we can write N_k as a polynomial in terms of N_1 and q such that

$$N_k = \sum_{i=1}^{k} (-1)^{i-1} P_{k,i}(q) N_1^i$$

where each $P_{k,i}$ is a polynomial in q with positive integer coefficients.

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$$\begin{aligned}
N_2 &= (2+2q)N_1 - N_1^2 \\
N_3 &= (3+3q+3q^2)N_1 - (3+3q)N_1^2 + N_1^3 \\
N_4 &= (4+4q+4q^2+4q^3)N_1 - (6+8q+6q^2)N_1^2 + (4+4q)N_1^3 - N_1^4 \\
N_5 &= (5+5q+5q^2+5q^3+5q^4)N_1 - (10+15q+15q^2+10q^3)N_1^2 \\
&+ (10+15q+10q^2)N_1^3 - (5+5q)N_1^4 + N_1^5
\end{aligned}$$

Question

What is a combinatorial interpretation of these expressions, i.e. of the $P_{k,i}$'s?

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Let $G = (V, E)$ be a finite graph. (We allow multiple edges between vertices, but not loops.)

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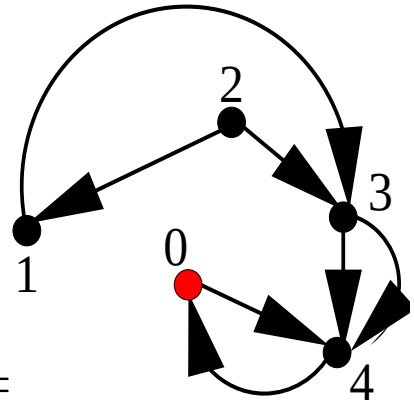
A **rooted oriented spanning tree** of G is a spanning tree of the underlying undirected graph, and orientations of edges along the tree are chosen so that all edges point towards the root.

More Graph Theory Terminology: The Laplacian Matrix

The **Laplacian** matrix of a graph has diagonal entries d_i (outdegree of v_i) and off-diagonal entries $-d_{ij}$ (number of directed edges from v_i to v_j).

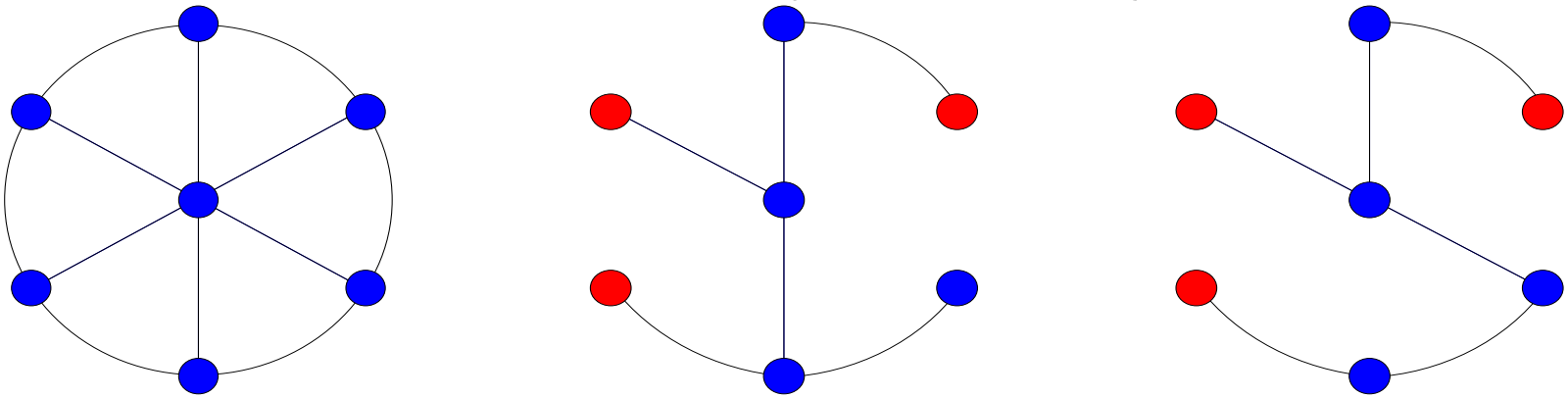
Example: let $G =$ 

with the root vertex v_0 in red. Then

$$L(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}. \text{ (Rows/Columns indexed as } 0, 1, 2, 3, 4)$$

A Family of Examples

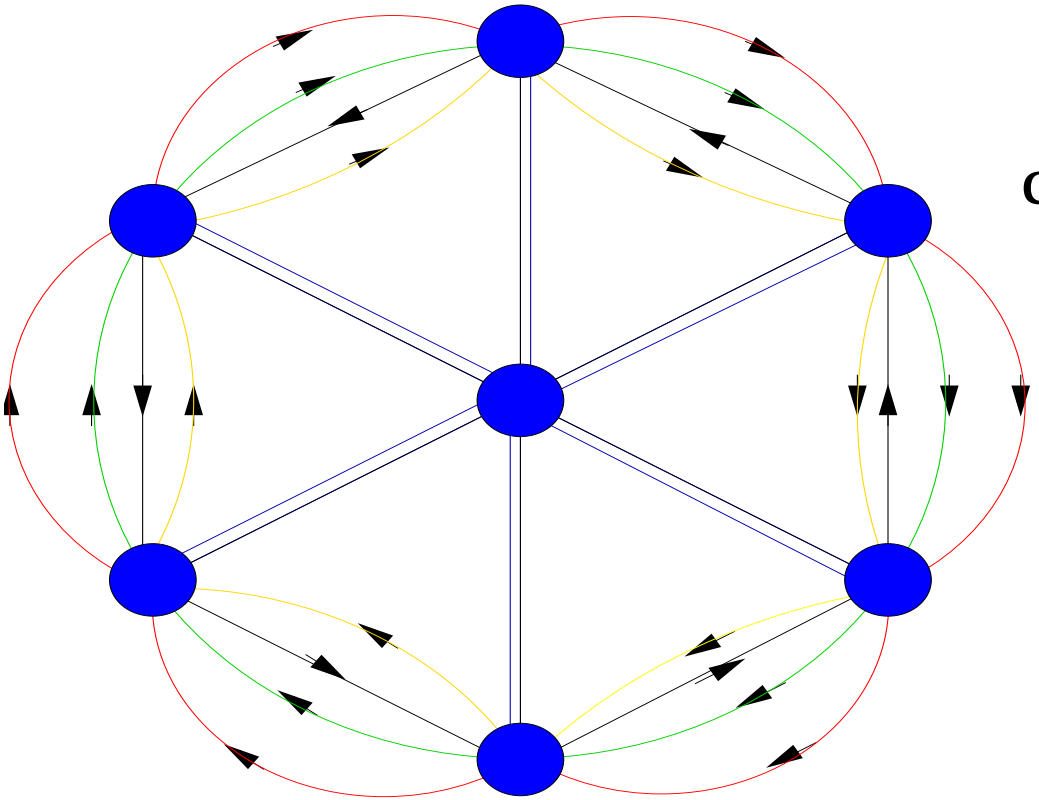
We let W_k denote the wheel graph which consists of k vertices on a circle and a central vertex which is adjacent to every other vertex.



Note that a spanning tree will consist of arcs on the rim and spokes.
We construct a family of digraphs (directed with multiple edges allowed) whose vertex set equal the W_k's.

We replace each rim edge with q clockwise edges and 1 counter-clockwise edge.

We replace each spoke with t spokes pointing towards the root.



$q = 3$
$t = 2$

The (q, t)-wheel graphs $W_k(q, t)$ for $k \geq 1$.

Definition

$\mathcal{W}_k(q,t) =$
The number of rooted oriented spanning trees in graph $W_k(q,t)$.

Theorem (M- 2007)

$\mathcal{W}_k(q,t)$ can be written as a positive bivariate integer polynomial such that the coefficient of t^i in $\mathcal{W}_k(q,t)$ equals $P_{k,i}(q)$ in

$$N_k = \sum_{i=1}^{k} (-1)^{i-1} P_{k,i}(q) N_1^i.$$

In other words, $\mathcal{W}_k(q,-N_1) = N_k$.

The $\mathcal{W}_k(q,t)$'s are integer polynomials

$\mathcal{W}_k(q,t) =$
The number of rooted oriented spanning trees in graph $W_k(q,t)$.

The Laplacian Matrix for $W_k(q,t)$ is

$$L_k = \begin{bmatrix} 1+q+t & -q & 0 & \dots & 0 & -1 & -t \\ -1 & 1+q+t & -q & 0 & \dots & 0 & -t \\ \dots & \dots & \dots & \dots & \dots & \dots & -t \\ 0 & \dots & -1 & 1+q+t & -q & 0 & -t \\ 0 & \dots & 0 & -1 & 1+q+t & -q & -t \\ -q & 0 & \dots & 0 & -1 & 1+q+t & -t \\ -t & -t & -t & \dots & -t & -t & kt \end{bmatrix}.$$

The last row and column correspond to hub vertex, the root.

Proof of Integrality by the Matrix-Tree Theorem

By the Matrix-Tree theorem, the number of directed rooted spanning trees is $\det(L_k)_0$ where $(L_k)_0$ is matrix L_k with the last row and last column deleted.

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Let $\overline{M_1} = [t]$, $\overline{M_2} = \begin{bmatrix} 1+q+t & -1-q \\ -1-q & 1+q+t \end{bmatrix}$, and for $k \geq 3$, let $\overline{M_k}$ be the k-by-k "three-line" circulant matrix

$$\begin{bmatrix} 1+q+t & -q & 0 & \dots & 0 & -1 \\ -1 & 1+q+t & -q & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & 1+q+t & -q & 0 \\ 0 & \dots & 0 & -1 & 1+q+t & -q \\ -q & 0 & \dots & 0 & -1 & 1+q+t \end{bmatrix}.$$

Theorem (M- 2007)

$$\mathcal{W}_k(q,t) = \det \overline{M_k} \text{ and } N_k(q,t) = -\det \overline{M_k}|_{t=-N_1}$$

The $\mathcal{W}_k$'s also are the cardinalities of a sequence of groups

Consider the **quotient group**

$$K(G, v_0) \cong \mathbb{Z}^{|V(G)|-1} \Big/ Im\ (L_k)_0$$

where $(L_k)_0$ is the Laplacian matrix of graph G with the row and column corresponding to v_0 deleted.

$$|K(G, v_0)| = \#\text{Spanning Trees in Graph } G$$

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Alternative definition with **explicit coset representatives** shortly.

Critical Group of The Complete Graph K_n

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For example, decomposition of $K(W_k)$ involves Fibonacci numbers (Biggs).

Chip-Firing: (Björner, Lovász, Shor 1991)

1. Assign a nonnegative integer value C_i to each vertex v_i (number of chips).
2. Start with vertex v_1.
3. If C_i, the number of chips on v_i, is greater than or equal to the outdegree of v_i, then vertex v_i **fires**. Otherwise move on to v_{i+1}.
4. If vertex v_i fires, then we take d_i chips off of v_i and distribute them to v_i's neighbors.
5. Now $C_i := C_i - d_i$ and $C_j := C_j + d_{ij}$ if v_j is a neighbor of v_i.
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7. We then start over with v_1 and repeat.
8. We continue forever or terminate when all $C_i < d_i$.

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2. To limit extraneous configurations, we presume that the sum $\sum_{i=0}^{\#V-1} C_i = 0$. (Thus in particular, C_0 will be non-positive.)
3. The bank, i.e. vertex v_0, is only allowed to fire if no other vertex can fire. Note that since we now allow C_0 to be negative, v_0 is allowed to fire even when it is smaller than its outdegree.

A configuration is **stable** if v_0 is the only vertex that can fire

A configuration C is **recurrent** if there is firing sequence which will lead back to C.
(Note that this will necessarily require the use of v_0 firing.)

We call a configuration **critical** if it is both stable and recurrent.

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Theorem (Gabrielov 1993)

For any initial configuration C with $\sum_{i=0}^{k} C_i = 0$ and $C_i \geq 0$ for all $1 \leq i \leq k$, there exists a **unique** *critical configuration that can be reached by an allowable firing sequence.*

Coset Representatives for Critical Group

We can define $K(G, v_0)$ to be the set of *critical configurations*, with addition given by $C_1 \oplus C_2 = \overline{C_1 + C_2}$.

Here $+$ signifies the usual pointwise vector addition and $\overline{C}$ represents the unique critical configuration in the same coset as C, modulo the Laplacian.

When v_0 is understood, we will abbreviate this group as the critical group of graph G, and denote it as $K(G)$.

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Here $+$ signifies the usual pointwise vector addition and $\overline{C}$ represents the unique critical configuration in the same coset as C, modulo the Laplacian.

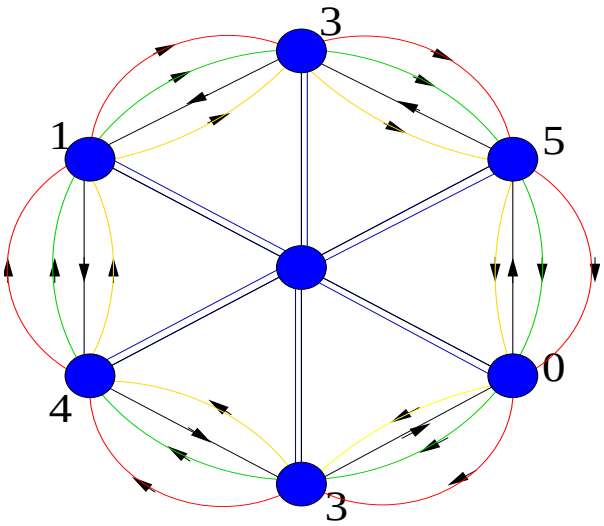
When v_0 is understood, we will abbreviate this group as the critical group of graph G, and denote it as $K(G)$.

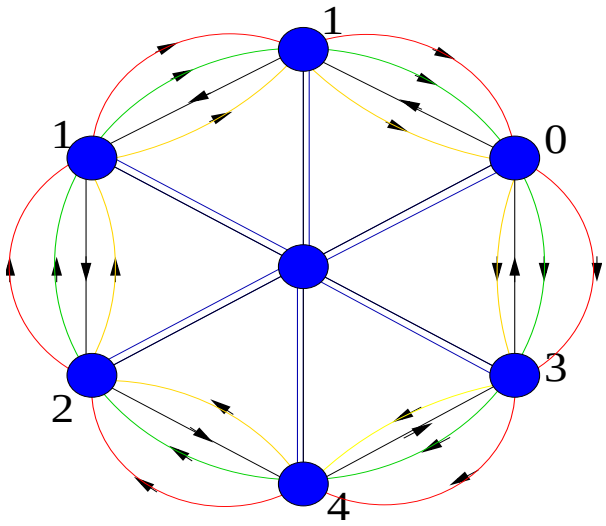
Corollary (Gabrielov 1993)

$K(G)$ is an abelian (associative) group.

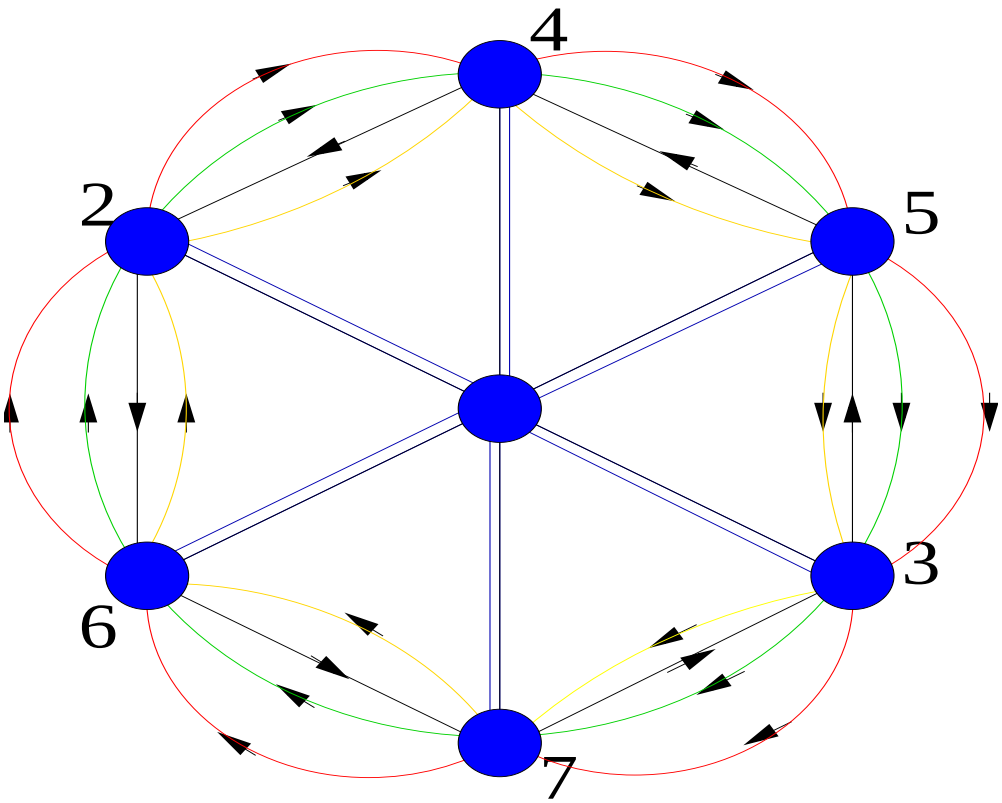
For example, consider the following two wheels with chip distributions as given. These are both critical configurations.

We do not label the number of chips on the hub vertex since forced.

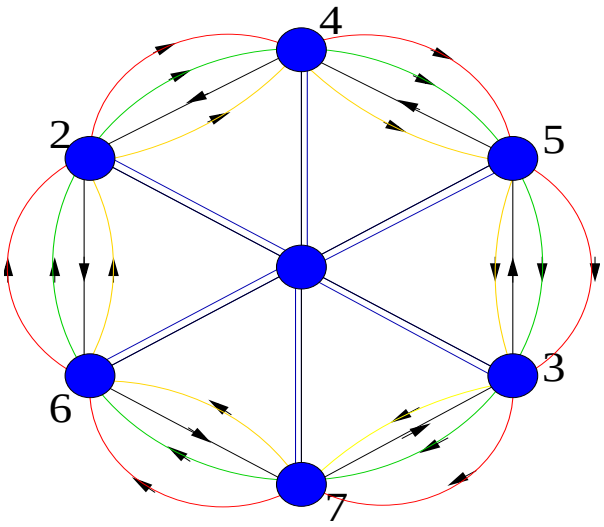




If we add these together pointwise we obtain



This is not a critical configuration, but by the theorem, reduces to a unique critical configuration.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2
5
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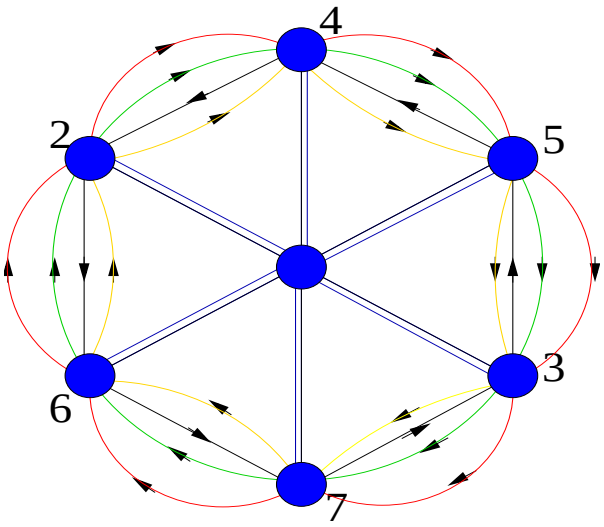
3
7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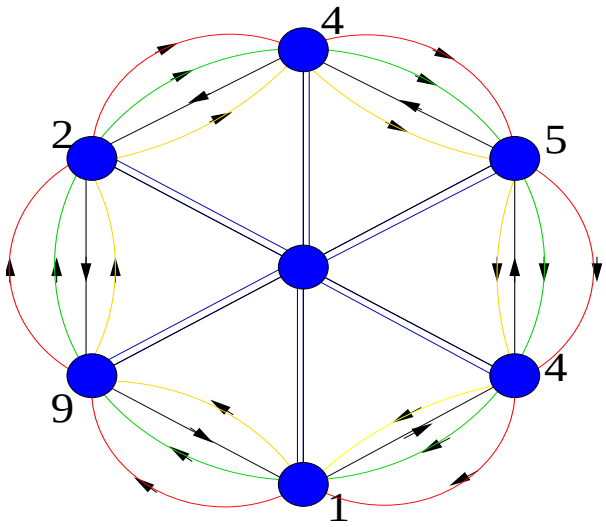
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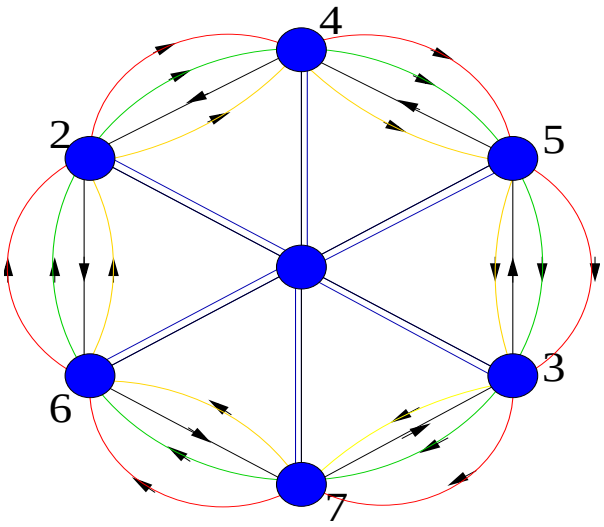

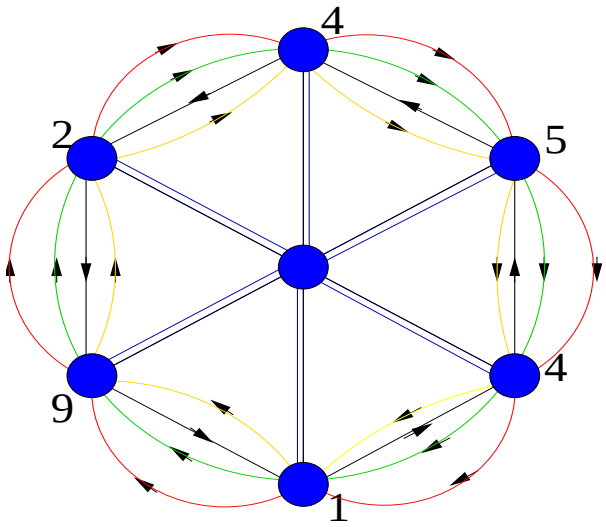

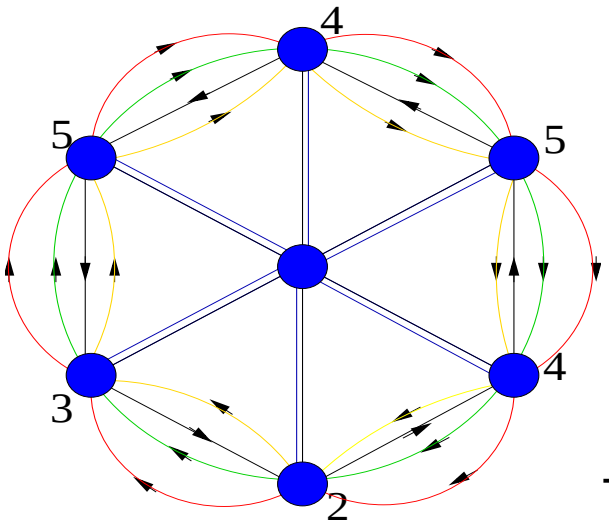

This last one is critical.

Critical Groups of (q, t)-Wheel Graphs

We want to analogize theory of elliptic curves: For example, there is a tower of groups

$$E(\mathbb{F}_q) \subset E(\mathbb{F}_{q^{k_1}}) \subset E(\mathbb{F}_{q^{k_2}}) \subset \cdots \subset E(\overline{\mathbb{F}_q})$$

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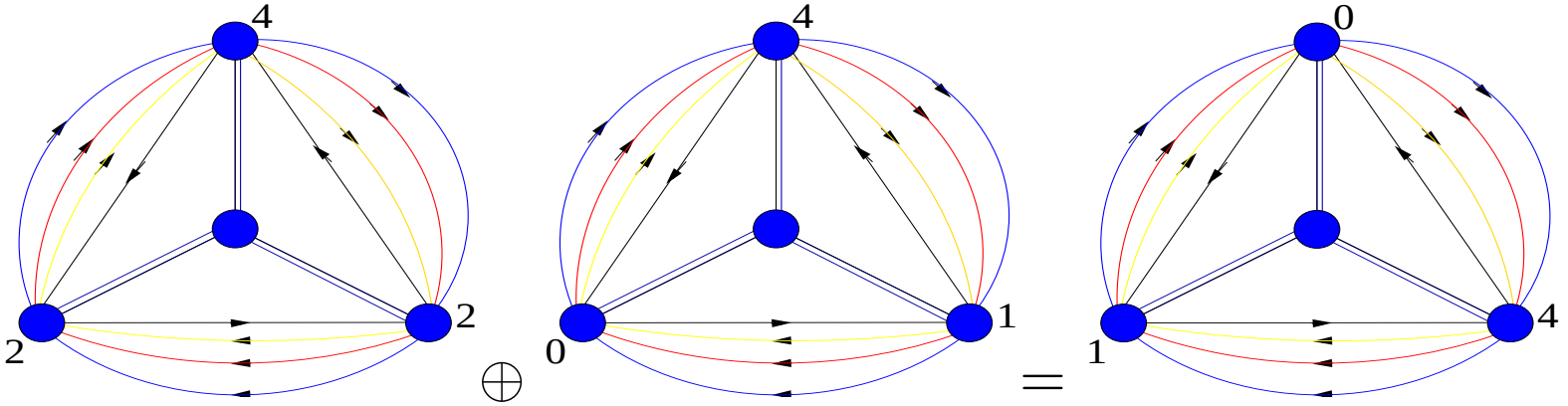
Understanding the sequence of Critical Groups:

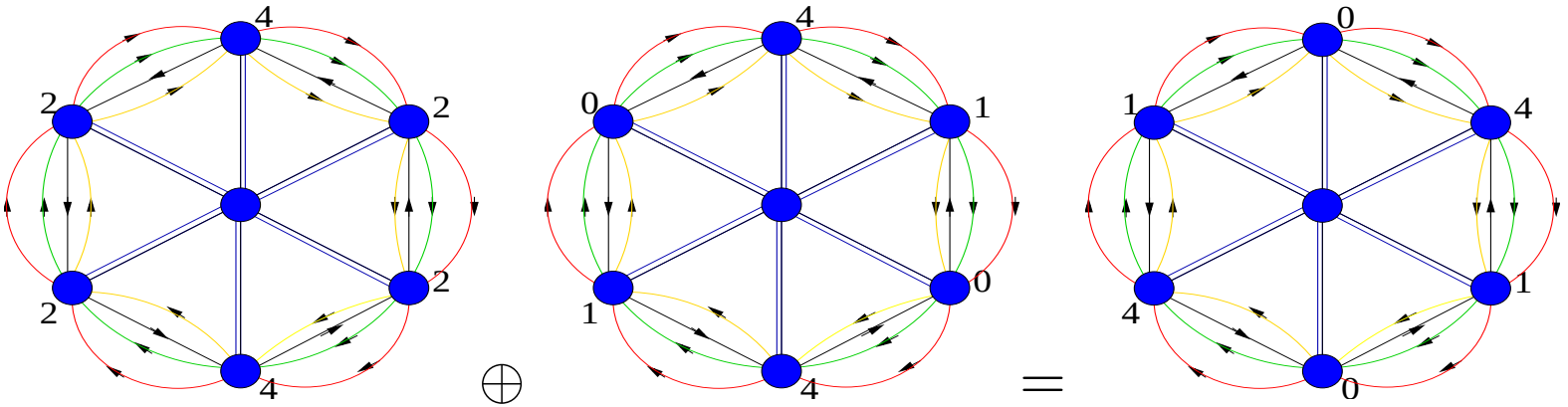
$$K(W_1(q,t)),\ K(W_2(q,t)),\ K(W_3(q,t)),\ \ldots$$

The set $\left\{\text{Elements of the critical group } K(W_k(q,t))\right\}$ is a subset of the set of length k words in alphabet $\{0, 1, 2, \ldots, q+t\}$.

Example: $[2,4,2] \oplus [0,4,1] \equiv [1,0,4]$ in $W_3(q=3,t=2)$ versus



$[2,4,2,2,4,2] \oplus [0,4,1,0,4,1] \equiv [1,0,4,1,0,4]$ in $W_6(q=3,t=2)$



Chip-firing is a local process.

Proposition

The map $\psi : w \longrightarrow www\ldots w$ is an injective group homomorphism between $K(W_{k_1}(q,t))$ and $K(W_{k_2}(q,t))$ whenever $k_1|k_2$. Here map ψ replaces w with k_2/k_1 copies of w.

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Define ρ to be the counter-clockwise rotation map on $K(W_k(q,t))$.

$$\rho([C_1, C_2, \ldots, C_k]) = [C_2, C_3, \ldots, C_k, C_1].$$

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We therefore can define a direct limit

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In particular we obtain

$$K(W_k(q,t)) \cong Ker(1-\rho^k) : K(\overline{W}(q,t)) \to K(\overline{W}(q,t)).$$

Shift map ρ is the wheel graph-analogue of the Frobenius map π on elliptic curves.

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4 One last surprising connection ...

Behavior of Torsion Subgroups of $K(\overline{W}(q,t))$

4 The Group $K(\overline{W}(q,t))$ (the direct limit of the $K(W_k(q,t))$'s) contains the subgroup $\mathbb{Z}/n\mathbb{Z}$ for all $n \geq 1$, and

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Given an integer $n \geq 1$, does there exist a $k \geq 1$ such that n divides the kth Fibonacci number?

Answer provided by a result of D.D Wall from 1960.

Lemma (Wall 1960)

The sequence $\{F_k \mod n : k \in \mathbb{Z}\}$ *is periodic, and* $F_k \equiv 0 \mod n$ *for some* $k \geq 1$.

Proof. Finite number (n^2) of possibilities for a window of length two, and an infinite number of k. Thus there will be two identical windows.

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Letting $F_1 = F_2 = 1$ and running recurrence backwards, $F_0 = 0$. Thus $F_{k_0} \equiv 0 \mod n$ for some $k_0 \geq 1$ too.

Application to Torison Groups

Theorem (M- 2009)

For $k \geq 3$, the Smith normal form of $(L_k)_0$ is equivalent to a direct sum of the identity matrix and

$$\begin{bmatrix} q\hat{F}_{2k-4}+1 & q\hat{F}_{2k-2} \\ \hat{F}_{2k-2} & \hat{F}_{2k}-1 \end{bmatrix} \equiv \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad d_1 | d_2$$

where $\hat{F}_k$ denotes a bivariate analogue of the Fibonacci numbers:

We let S range over all subsets $\{1, 2, \ldots, 2k\}$ with no two consecutive elements, and define

$$\hat{F}_{2k}(q,t) = \sum_S q^{\#\text{ even elements in } S}\ t^{k-\#S}.$$

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The $\hat{F}_k$'s satisfy the reccurence $\hat{F}_{2k+2} = (1+q+t)\hat{F}_{2k} - q\hat{F}_{2k-2}$.

Factorizations of N_k and Elliptic Cyclotomic Polynomials

$$\mathcal{W}_k(q,t) = -N_k\big|_{N_1=-t} = \sum_{i=1}^{k} P_{k,i}(q)\ t^i \quad \text{for all } k \geq 1.$$

$\overline{M_k}$ be the k-by-k "three-line" circulant matrix

$$\begin{bmatrix} 1+q+t & -q & 0 & \dots & 0 & -1 \\ -1 & 1+q+t & -q & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & 1+q+t & -q & 0 \\ 0 & \dots & 0 & -1 & 1+q+t & -q \\ -q & 0 & \dots & 0 & -1 & 1+q+t \end{bmatrix}.$$

Let $M_k = \overline{M_k}|_{t=-N_1}$.

Corollary (M- 2007)

The sequence of integers $N_k = \#E(\mathbb{F}_{q^k})$ satisfies the relation

$$N_k = -\det M_k \text{ for all } k \geq 1.$$

Elliptic Cyclotomic Polynomials

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Combinatorial interpretations for the summands when we write N_k as an alternating sum in powers of N_1

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Combinatorial interpretations for the summands when we write N_k as an alternating sum in powers of N_1

We now look at factorizations of N_k into $\mathbb{Z}[q, N_1]$ polynomials.

$$\text{e.g.} \quad N_2 = N_1\Big(2 + 2q - N_1\Big)$$

Motivates a combinatorial interpretation of $E(\mathbb{F}_{q^k})$ as Cartesian Product of smaller subsets.

$$
\begin{aligned}
N_2 &= N_1\Big(2+2q-N_1\Big)\\
N_3 &= N_1\Big((3+3q+3q^2)-(3+3q)N_1+N_1^2\Big)\\
N_4 &= N_1\Big(2+2q-N_1\Big)\Big((2q^2+2)-(2q+2)N_1+N_1^2\Big)\\
N_5 &= N_1\Big((5+5q+5q^2+5q^3+5q^4)-(10+15q+15q^2+10q^3)N_1\\
&+ (10+15q+10q^2)N_1^2-(5+5q)N_1^3+N_1^4\Big)\\
N_6 &= N_1\Big(2+2q-N_1\Big)\Big((3+3q+3q^2)-(3+3q)N_1+N_1^2\Big)\\
&\times \Big((q^2-q+1)-(q+1)N_1+N_1^2\Big)
\end{aligned}
$$

Factoring N_k in general:

Theorem (M- 2007)

There exists integral polynomials, which we will denote as $ECyc_d$, in N_1 and q, only depending on d such that

$$N_k(N_1, q) = \prod_{d|k} ECyc_d.$$

Compare with $1 - x^k = \prod_{d|k} Cyc_d(x)$.

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We call these **Elliptic Cyclotomic Polynomials**.

Definition

$ECyc_d(q, N_1) = Cyc_d(\alpha_1)Cyc_d(\alpha_2)$ where α_1 and α_2 are the two complex roots of quadratic $T^2 - (1 + q - N_1)T + q$, and

$$Cyc_d(x) = \prod_{e|d} (1 - x^e)^{\mu(d/e)}.$$

$$
\begin{aligned}
ECyc_1 &= N_1 \\
ECyc_2 &= 2 + 2q - N_1 \\
ECyc_3 &= (3 + 3q + 3q^2) - (3 + 3q)N_1 + N_1^2 \\
ECyc_4 &= (2q^2 + 2) - (2q + 2)N_1 + N_1^2 \\
ECyc_5 &= (5 + 5q + 5q^2 + 5q^3 + 5q^4) - (10 + 15q + 15q^2 + 10q^3)N_1 \\
&+ (10 + 15q + 10q^2)N_1^2 - (5 + 5q)N_1^3 + N_1^4 \\
ECyc_6 &= (q^2 - q + 1) - (q + 1)N_1 + N_1^2
\end{aligned}
$$

Proposition (M- 2007)

$$ECyc_d\Big|_{N_1=0} = Cyc_d(1) \cdot Cyc_d(q)$$

where $Cyc_1(1) = 0$, $Cyc_d(1) = p$ *if* $d = p^k$ *and* $Cyc_d(1)$ *equals* 1 *otherwise.*

Conjecture

$$\text{For } d \geq 2,\ ECyc_d(q, N_1) = Cyc_d(1) \cdot Cyc_d(q) + \sum_{i=1}^{\phi(d)} (-1)^i Q_{i,d}(q) N_1^i$$

where $Q_{i,d}$ is a univariate polynomial with positive integer coefficients.

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However, Conjecture fails for $d = 105$.

Nonetheless, we can give a geometric interpretation of the values $ECyc_d(q, N_1)$ for a given q and $N_1 = |E(\mathbb{F}_q)|$.

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Theorem (M- 2007)

$$ECyc_d(q, N_1) = \left| Ker\ Cyc_d(\pi) : E(\overline{\mathbb{F}_q}) \circlearrowleft \right|$$

where $Cyc_d(\pi)$ denotes the isogeny obtained from the dth Cyclotomic polynomial of the Frobenius map.

Ker $M = \{P \in E(\overline{\mathbb{F}_q}) : M(P) = P_\infty\}$

From Chip-Firing to Tropical Geometry

Variant of earlier discussion: Let $G = (V, E)$ be any **undirected** graph.

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Definition

The **degree** *of a divisor* $D = \sum_{i=1}^{n} C_i v_i$ *is* $\sum_{i=1}^{n} C_i$.

D *is* **effective** *if* $C_i \geq 0$ *for all* i.

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Equivalently, $D_1 - D_2$ is a $\mathbb{Z}$-sum of columns of the Laplacian matrix $L(G)$.

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The **Linear System of** D, *denoted as* $|D|$, *is the set* $\{D' : D' \sim D \text{ and } D' \text{ is effective.}\}$.

The following definitions are from Baker-Norine.

1. Let $K(G) = [(deg\ v_1) - 2, (deg\ v_2) - 2, \ldots, (deg\ v_n) - 2]$, the **canonical divisor** of G.

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3. The **rank** of D, $r(D)$, is the biggest $k \geq 0$ such that for all effective E of degree k, $|D - E| \neq \emptyset$ if such a k exists.
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Theorem (Baker-Norine 2006 - Riemann-Roch Theorem for Graphs)

$$r(D) - r(K - D) = \deg D - g + 1.$$

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With Christian Haase and Josephine Yu:

1. We explictly describe cell structures of $|D|$ as a polyhedral cell complex
2. Show how to embed $|D|$ into tropical projective space.
3. Also get generalization of chip-firing to metric graphs, called weighted chip-firing games.

`http://arxiv.org/pdf/0909.3685.pdf`

Thanks For Coming

http://math.mit.edu/∼ musiker/CGs.pdf

G. Musiker, *Combinatorial aspects of elliptic curves*, Seminaire Lotharingien de Combinatoire 56 (2007), Article B56f, 1-31

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