A Topological Interpretation of the Cyclotomic Polynomial

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Joint work with Vic Reiner

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Outline

1. Introduction
2. The Main Theorems
3. Reformulation in terms of attaching maps
4. Concordance with known results
5. Open Questions and Final Comments

The Cyclotomic Polynomial

The **Cyclotomic Polynomial** $\Phi_n(x)$ is the minimal polynomial over $\mathbb{Q}$ for any primitive nth root of unity $\zeta \in \mathbb{C}$ (e.g. $\zeta = e^{2\pi i/n}$).

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\Phi_1 &= x - 1 \\
\Phi_2 &= x + 1 \\
\Phi_3 &= x^2 + x + 1 \\
\Phi_4 &= x^2 + 1 \\
\Phi_5 &= x^4 + x^3 + x^2 + x + 1 \\
\Phi_6 &= x^2 - x + 1
\end{aligned}$$

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The polynomial $\Phi_n(x)$ can also be expressed in a number of ways:

1) $\Phi_n(x) = \prod_{(j \in \mathbb{Z}/n\mathbb{Z})^\times} (x - \zeta^j)$; e.g. $\Phi_4(x) = (x - i)(x - i^3)$.

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Example:

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Via Möbius inversion:

$$\Phi_n(x) = \prod_{d|n} (x^{n/d} - 1)^{\mu(d)};$$

$$\mu(d) = \begin{cases} 0 \text{ if } d \text{ is not squarefree,} \\ (-1)^k \text{ if } d = p_1 p_2 \cdots p_k \end{cases}.$$

Example of $\Phi_{15}(x)$

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Our running example will be

$$\begin{aligned} \Phi_{15}(x) &= (x-\zeta)(x-\zeta^2)(x-\zeta^4)(x-\zeta^7)(x-\zeta^8) \\ &\quad \cdot (x-\zeta^{11})(x-\zeta^{13})(x-\zeta^{14}) \\ &= x^8 - x^7 + x^5 - x^4 + x^3 - x + 1 \end{aligned}$$

The complete d-partite simplicial complex $K_{p_1,p_2,\ldots,p_d}$

We focus on the square-free case because if $n = p_1^{e_1} \cdots p_d^{e_d}$, then

$$\Phi_n(x) = \Phi_{p_1 p_2 \cdots p_d}(x^{n/p_1 \cdots p_d}).$$

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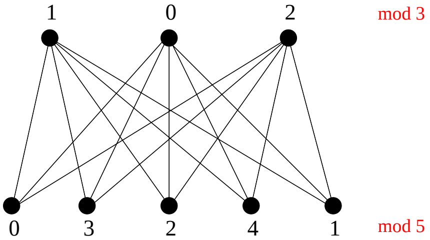
Take the simplicial join of d vertex sets, each with p_i disconnected vertices.

Let $\mathbf{K_{p_1,\ldots,p_d}}$ denote the resulting simplicial complex.

Example: $K_{3,5}$ is the graph (1-complex)



Labeling the facets of $K_{p_1,\ldots,p_d}$

By the Chinese Remainder Theorem, there is a unique $j \in \{0, 1, \ldots, n-1\}$

$$\begin{aligned} j &\equiv j_1 \mod p_1 \\ j &\equiv j_2 \mod p_2 \\ &\cdots \\ j &\equiv j_d \mod p_d \end{aligned}$$

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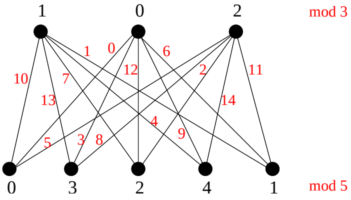
Example: $K_{3,5}$ with the facets (edges) labeled by $0, 1, \ldots, 14$.



The subcomplexes K_A for a subset A

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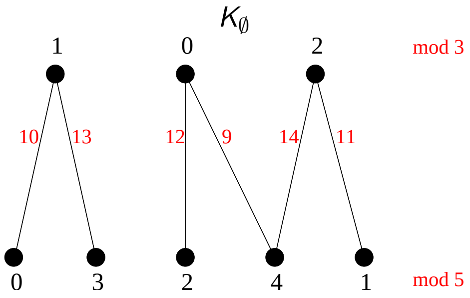
$K_{\{j\}}$ has facets labeled by $\{\varphi(n)+1, \varphi(n)+2, \ldots, n-1\} \cup \{j\}$.

$K_\emptyset$, $K_{\{4\}}$, and $K_{\{6\}}$ for $K_{3,5}$

$$\Phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$$



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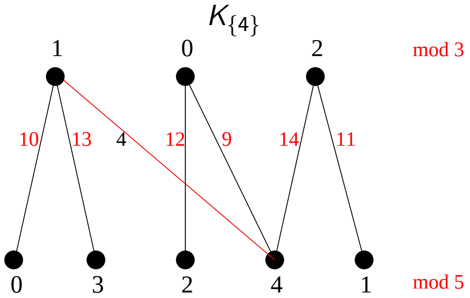

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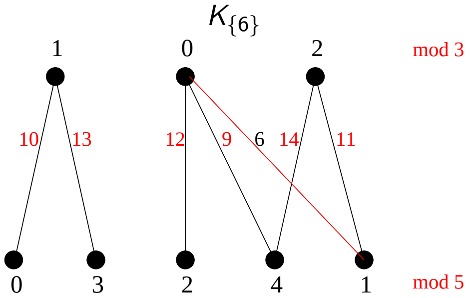

Theorem 1 (M-Reiner)

For a square-free positive integer $n = p_1 p_2 \cdots p_d > 1$ with $\Phi_n(x) = \sum_{j=0}^{\varphi(n)} c_j x^j$, then

$$\widetilde{H}_i(K_{\{j\}}, \mathbb{Z}) = \begin{cases} \mathbb{Z}/c_j\mathbb{Z} \text{ if } i = d-2 \\ \mathbb{Z} \text{ if both } i = d-1 \text{ and } c_j = 0, \text{ and} \\ 0 \text{ otherwise} \end{cases}$$

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$K_{\{j\}}$ is a spanning tree in this case.

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$K_{\{j\}}$ has a 1-cycle and two connected components in this second case.

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Coefficients c_j, $c_{\varphi(n)}$ have the same sign $\longleftrightarrow$ b_j, $b_{\varphi(n)}$ have opposite signs.

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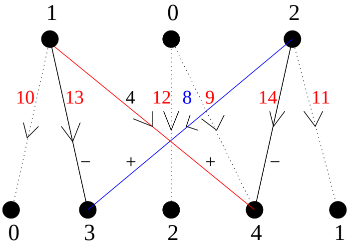

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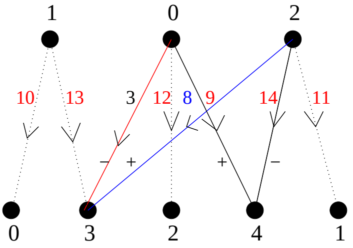

$K_{\{3,8\}} \leftrightarrow c_3 = +1$

Reformulation in terms of attaching maps

(These results are based on discussion with Dmitry Fuchs)

Consider the full $K_{p_1,\ldots,p_d}$ with all the oriented facets $[F_{j \bmod n}]$ for $j \in \{0, 1, \ldots, n-1\}$.

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Example: For $n = 15$, $j = 4$,

$$\begin{aligned} [Z_{4 \mod 15}] &= [1 \mod 3, \quad 4 \widehat{\mod} 5] - [1 \widehat{\mod} 3, \quad 4 \mod 5] \\ &= [1 \mod 3] \qquad\qquad - [4 \mod 5] \end{aligned}$$

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where f_j is a map winding S^{d-2} onto the boundary of the ball B^{d-1} with $\deg(f_j) = c_j$. **Point:** We are gluing one more facet to a homology sphere.

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Question: For $n = p_1 p_2 \ldots p_d$, $d \geq 3$, let b be the $(d-1)$-co-chain with value c_j on $[F_{j \mod n}]$.

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3) $K_{\{j\}} \simeq S^{d-2} \cup_{f_j} B^{d-1}$ with $\deg(f_j) = c_j$.

Question: For $n = p_1 p_2 \dots p_d$, $d \geq 3$, let b be the $(d-1)$-co-chain with value c_j on $[F_{j \mod n}]$.

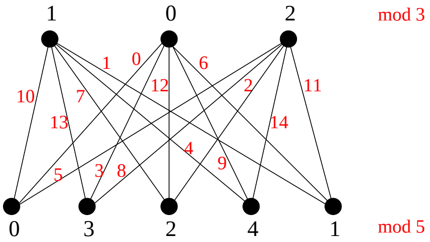
Is there a natural way to write a $(d-2)$-chain with a co-boundary b?

Corollary to this approach (suggested by Fuchs)

Example: For $n = pq$, $p < q$,

$$\begin{aligned} b &= \delta([0 \mod p] + [q \mod p] + \cdots + [d_1 q \mod p] \\ &+ [1 \mod q] + [p+1 \mod q] + \cdots + [d_2 p + 1 \mod q] \end{aligned}$$

where $(d_1 + 1)q \equiv 1 \mod p$ and $(d_2 + 1)p + 1 \equiv 0 \mod q$.



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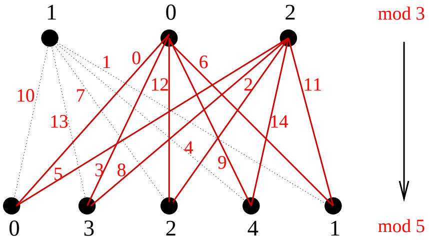

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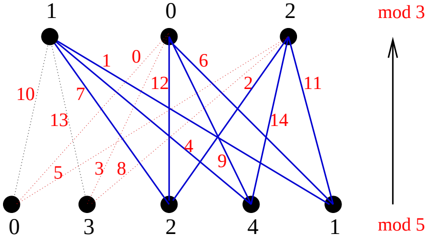

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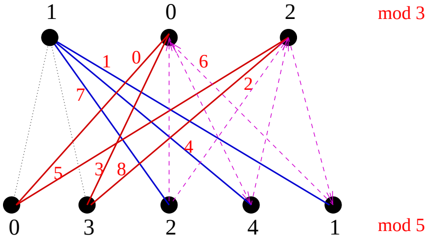

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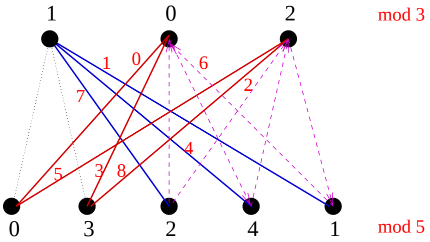

Agrees with *pq* case elsewhere in literature, e.g. Sam Elder.

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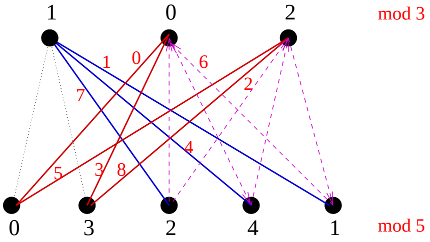

Ricky Liu also has analyzed co-boundaries related to $\Phi_{pqr}(x)$.

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This symmetry can be seen by simplicial automorphisms.

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(Nathan Kaplan showed that if $r \equiv \pm 1 \mod pq$, then $\Phi_{pqr}(x)$ has is flat.)

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Revised Beiter Conjecture (although recently solved by other means): If p, q, r are distinct primes, then the absolute values of coefficients of $\Phi_{pqr}(x)$ can only be so big. (e.g. bound for $\Phi_{3qr}(x)$ is 2).

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Thanks for Listening!

The Cyclotomic Polynomial Topologically (with Vic Reiner),
`http://arxiv.org/pdf/1012.1844.pdf`