Double Dimer Covers on Snake Graphs from Super Cluster Expansions

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Isaac Newton Cluster Algebra Seminar

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Joint work with Nicholas Ovenhouse and Sylvester Zhang.

http://www-users.math.umn.edu/∼musiker/IsaacNewton21.pdf

https://arxiv.org/pdf/2110.06497.pdf

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001)

A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

Generators:

Specify an initial finite set of them, a Cluster, $\{x_1, x_2, \ldots, x_{n+m}\}$.

Construct the rest via Binomial Exchange Relations:

$$x_\alpha x'_\alpha = \prod x_{\gamma_i}^{d_i^+} + \prod x_{\gamma_i}^{d_i^-}.$$

The set of all such generators are known as Cluster Variables, and the initial pattern B of exchange relations determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Teichmüller and Decorated Teichmüller Spaces

Let $S = S_g^n$ be a smooth oriented surface (possibly with boundary) of genus g equipped with a collection of marked points $p_1, p_2, \ldots, p_n$.

Here $n \geq 0$. The marked points either lie on boundary components, or in the interior of S, in which case they are called punctures.

Roughly speaking, the *Teichmüller space* of such a surface is

$$T(S) = \text{the set of hyperbolic structures on S/isotopy.}$$

Definition

Define the *Teichmüller space* of S to be the quotient space

$$T(S) = \mathrm{Hom}\left(\pi_1(S), \mathrm{PSL}(2, \mathbb{R})\right) / \mathrm{PSL}(2, \mathbb{R}).$$

Definition (Penner)

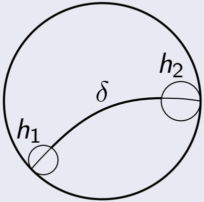
When $n > 0$, any such surface $S = S_g^n$ also admits a *decorated Teichmüller space*, which is a trivial $\mathbb{R}_{>0}^n$-bundle over $T(S)$, denoted $\tilde{T}(S)$.

Decorated Teichmüller Theory

Throughou most of the rest of the talk, let $S = S_0^n$ be a disk with n marked points on its unique boundary (i.e. a polygon). Such surfaces admit the *Poincaré disk* $\mathbb{D}$ model as a hyperbolic structure.

$\mathbb{D} := \{z = x + yi \in \mathbb{C} : |z| < 1\}$, with metric $ds = 2\frac{\sqrt{dx^2+dy^2}}{1-|z|^2}$.

Definition (λ-length via horocycles)



A *horocycle* is a smooth curve in the hyperbolic plane with constant geodesic curvature 1. In $\mathbb{D}$, it is a Euclidean circle tangent to an infinite point, which is the center.

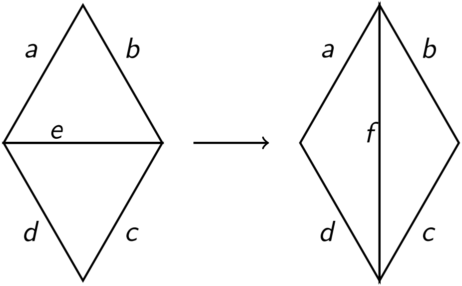
For a pair of horocycles h_1, h_2, the *λ-length* between them is

$$\lambda(h_1, h_2) = e^{\delta/2}$$

where δ is the hyperbolic distance between the two intersections.

Ptolemy Relations

Given a quadruple of horocycles with distinct centers (a decorated ideal quadrilateral), one has the **Ptolemy transformation** induced by flipping the diagonal of the quadrilateral.



At the level of λ-lengths, this induces the identity

$$\lambda(e)\lambda(f) = \lambda(a)\lambda(c) + \lambda(b)\lambda(d).$$

Note that we will often abbreviate this as $ef = ac + bd$.

Structural Theorems for Cluster Algebras

Theorem (Fomin-Zelevinsky 2001, The Laurent Phenomenon)

For any cluster algebra defined by initial seed $(\{x_1, x_2, \dots, x_{n+m}\}, B)$, all cluster variables of $\mathcal{A}(B)$ are Laurent polynomials in $\{x_1, x_2, \dots, x_{n+m}\}$ (with no coefficient $x_{n+1}, \dots, x_{n+m}$ in the denominator).

Because of the Laurent Phenomenon, any cluster variable x_α can be expressed as $\frac{P_\alpha(x_1,\dots,x_{n+m})}{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}$ where $P_\alpha \in \mathbb{Z}[x_1, \dots, x_{n+m}]$ and the α_i's $\in \mathbb{Z}$.

Theorem (Lee-Schiffler 2014, Gross-Hacking-Keel-Kontsevich 2015, Prooof of the Positivity Conjecture)

For any cluster variable x_α and any initial seed (i.e. initial cluster $\{x_1, \dots, x_{n+m}\}$ and initial exchange pattern B), the polynomial $P_\alpha(x_1, \dots, x_n)$ has nonnegative integer coefficients.

Cluster Algebras from Surfaces

Theorem (Fomin-Shapiro-Thurston 2006)

Given a Riemann surface with marked points (S, M)*, one can define a corresponding cluster algebra* $\mathcal{A}(S, M)$*.*

$$\textit{Seed} \leftrightarrow \textit{Triangulation } T = \{\tau_1, \tau_2, \dots, \tau_n\}$$

$$\textit{Cluster Variable} \leftrightarrow \textit{Arc } \gamma \ (x_i \leftrightarrow \tau_i \in T)$$

Cluster Mutation (Binomial Exchange Relations) $\leftrightarrow$ *Flipping Diagonals.*

(Based on earlier work of Gekhtman-Shapiro-Vainshtein and Fock-Goncharov.)

From the perspective of hyperbolic geometry, Laurent expansions of cluster variables may be expressed as λ-lengths of arcs, which can be measured by choosing a point in Penner's decorated Teichmüller space.

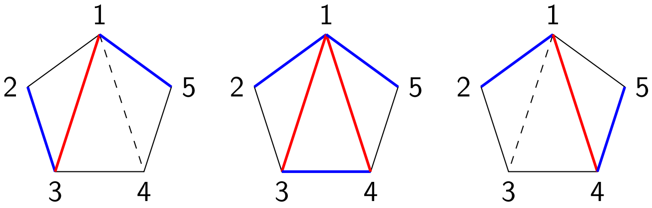
Positivity of Cluster Algebras from Surfaces

Theorem (Schiffler 2006)

Let $\mathcal{A}$ be any cluster algebra of type A_n, i.e. with a seed Σ defined by a triangulation T of an $(n+3)$-gon.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **T-paths**.



$$\lambda_{25} = \frac{x_{23}x_{15}}{x_{13}} + \frac{x_{12}x_{34}x_{15}}{x_{13}x_{14}} + \frac{x_{12}x_{45}}{x_{14}} = \frac{x_{23}x_{14}x_{15} + x_{12}x_{34}x_{15} + x_{12}x_{13}x_{45}}{x_{13}x_{14}}.$$

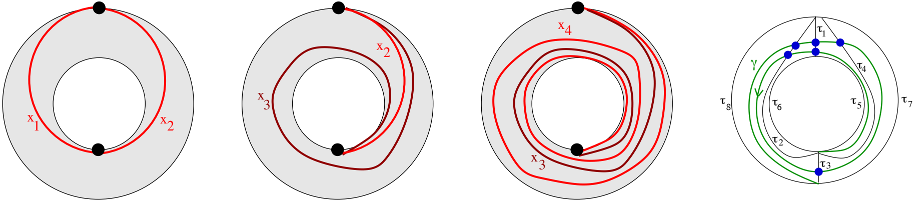
Positivity of Cluster Algebras from Surfaces

Theorem (Schiffler-Thomas 2007, Schiffler 2008)

Let $\mathcal{A}(S, M)$ be any cluster algebra arising from an unpunctured surface S with marked points M, with principal coefficients, and let Σ be any initial seed. Here Σ correponds to a triangulation of S with respect to the marked points M.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **T-paths**.



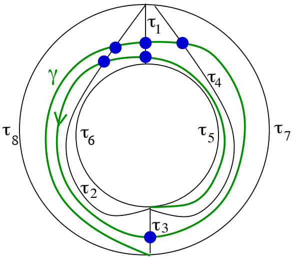
Positivity of Cluster Algebras from Surfaces

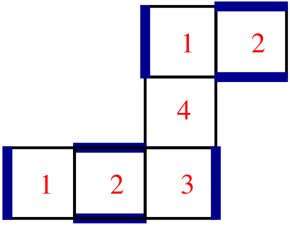
Theorem (M-Schiffler 2008)

Let $\mathcal{A}(S, M)$ be any cluster algebra arising from an unpunctured surface, with principal coefficients, and let Σ be any initial seed.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **snake graphs**.





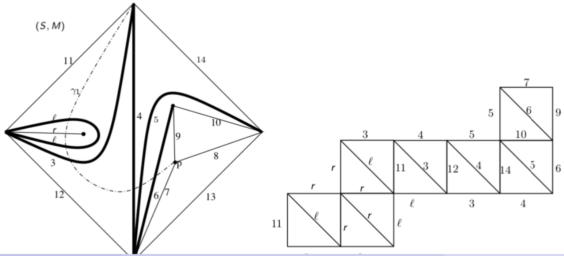
Positivity of Cluster Algebras from Surfaces

Theorem (M-Schiffler-Williams 2009)

Let $\mathcal{A}(S, M)$ be any cluster algebra arising from a surface (with or without punctures), where the coefficient system is of geometric type, and let Σ be any initial seed.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **snake graphs**.



Superalgebras (and towards Superspace)

A super algebra is a $\mathbb{Z}_2$-graded algebra.

i.e. $A = A_0 \oplus A_1$, (the "*even*" and "*odd*" parts) and

$$A_i A_j \subseteq A_{i+j} \text{ for } i, j \in \{0, 1\} \bmod 2$$

The algebra A generated by $x_1, \cdots, x_n, \theta_1, \cdots, \theta_m$, subject to the following relations

$$x_i x_j = x_j x_i \quad x_i \theta_j = \theta_j x_i \quad \theta_i \theta_j = -\theta_j \theta_i$$

is a superalgebra. In particular $\theta_i^2 = 0$.

Here A_0 is spanned by monomials with an even number of θ's and A_1 is spanned by monomials with an odd number of θ's.

E.g. $x_1 x_2 + x_1 \theta_1 \theta_3 + x_2 \theta_1 \theta_2 \theta_3 \theta_4 \in A_0$, $x_1 \theta_1 \theta_2 \theta_3 + x_1 x_4 \theta_2 + \theta_4 \in A_1$

Decorated Super-Teichmüller Spaces [PZ19]

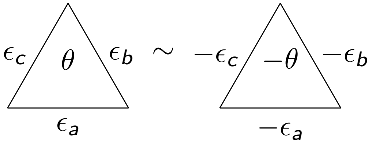
- By replacing $\mathrm{PSL}(2,\mathbb{R})$ with $\mathrm{OSp}(1|2)$, Penner and Zeitlin define the **super-Teichmüller space** of a surface S to be
$$ST(S) = \mathrm{Hom}(\pi_1(S), \mathrm{OSp}(1|2))/\,\mathrm{OSp}(1|2)$$
- Similar to the bosonic case, the decorated space is encoded by a collection of horocycles centered at each ideal point, which leads to the definition of super λ-length.
- But unlike the bosonic case, we need additional invariants to accommodate for the extra degree of freedom coming from the odd dimension.
- They associate an odd variable to each triangle (triple of ideal points), and call them the μ-invariants.

Spin Structures

Components of $ST(S)$ are indexed by the set of **spin structures** on S.

Cimasoni-Reshetikhin formulated the set of spin structures of S in terms of the set of isomorphism classes of Kasteleyn orientations of a fatgraph spine of S.

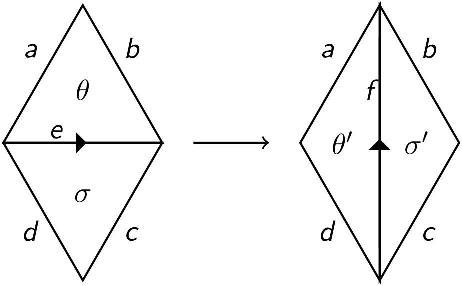
Dual to this formulation, we consider the set of spin structures on S to be the set of equivalence classes of orientations on triangulations of S of the following equivalence relation.



where $\epsilon_a, \epsilon_b, \epsilon_c$ are orientations on the edges, and θ is the μ-invariant associated to the triangle.

Super Ptolemy Relation [PZ19]

The Ptolemy transformation on super λ-length coordinates is given as follows.

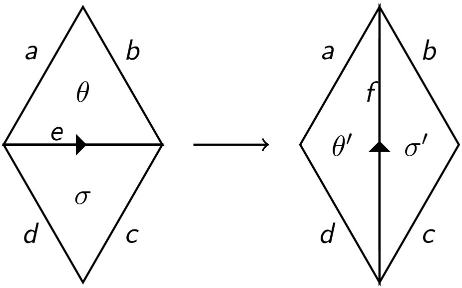


$$ef = (ac + bd)\left(1 + \frac{\sigma\theta\sqrt{\chi}}{1+\chi}\right), \quad \chi = \frac{ac}{bd}$$

$$\sigma' = \frac{\sigma - \sqrt{\chi}\theta}{\sqrt{1+\chi}} \quad \text{and} \quad \theta' = \frac{\theta + \sqrt{\chi}\sigma}{\sqrt{1+\chi}}$$

Super Ptolemy Relation [PZ19]

The Ptolemy transformation on super λ-length coordinates is given as follows.



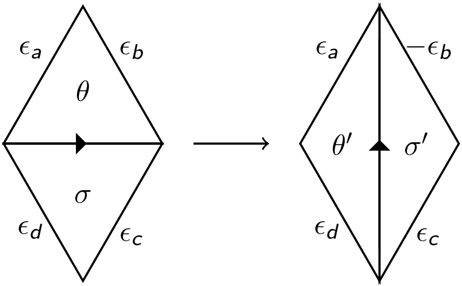
$$ef = ac + bd + \sqrt{abcd}\,\sigma\theta$$

$$\sigma' = \frac{\sigma\sqrt{bd} - \theta\sqrt{ac}}{\sqrt{ac+bd}} \quad \text{and} \quad \theta' = \frac{\theta\sqrt{bd} + \sigma\sqrt{ac}}{\sqrt{ac+bd}}$$

$$\sigma\theta = \sigma'\theta'$$

Super Ptolemy Relation [PZ19]

Super-flip also reverses the orientation of the edge b.

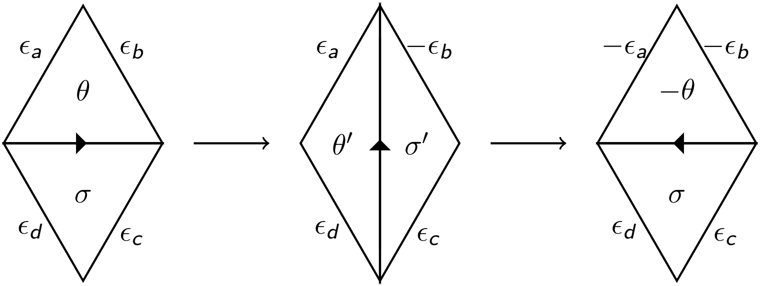


Remark

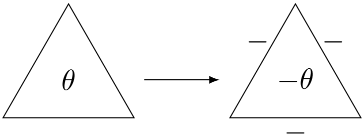
- Super Ptolemy moves are not involutions: $\mu_i^8 = I$.
- The even-degree-0 terms of a super λ-length are exactly the (ordinary) λ-length in the bosonic decorated space.

Super Ptolemy Relation [PZ19]

If we flip a diagonal twice

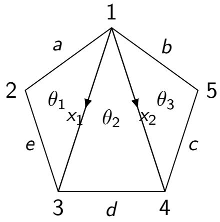


the orientations of the triangle θ are reversed and θ is changed to $-\theta$.



This orientation is equivalent to the original one, i.e. both the first and third pictures represent the same spin structure.

Super Ptolemy Relation - Example



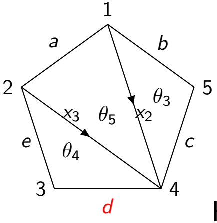
Start with a Pentagon with given orientation.

The boundary orientations are ignored, because they are irrelevant in the calculations.

What are λ_{24}, λ_{25}, and λ_{35}?

We first flip the edge x_1.

Super Ptolemy Relation - Example



After flipping x_1 to x_3, we get:

$$x_3 = \frac{ad + ex_2}{x_1} + \frac{\sqrt{adex_2}}{x_1}\theta_1\theta_2$$

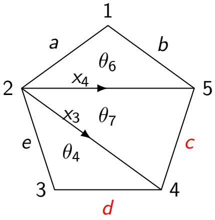
$$\theta_4 = \frac{\sqrt{ad}\,\theta_1 - \sqrt{ex_2}\,\theta_2}{\sqrt{x_1 x_3}}$$

$$\theta_5 = \frac{\sqrt{ad}\,\theta_2 + \sqrt{ex_2}\,\theta_1}{\sqrt{x_1 x_3}}$$

Here the red color indicates that the orientation on the boundary edge has been reversed.

Next we flip x_2.

Super Ptolemy Relation - Example



After flipping x_2 to x_4, we have:

$$x_4 = \frac{ac + bx_3}{x_2} + \frac{\sqrt{acbx_3}}{x_2}\,\theta_5\theta_3$$

$$= \frac{acx_1 + abd + bex_2}{x_1x_2} + \frac{b\sqrt{adex_2}}{x_1x_2}\theta_1\theta_2 + \frac{\sqrt{acb\left(\frac{ad+ex_2}{x_1} + \frac{\sqrt{adex_2}}{x_1}\theta_1\theta_2\right)}}{x_2}\left(\frac{\sqrt{ad}\,\theta_2 + \sqrt{ex_2}\,\theta_1}{\sqrt{x_1x_3}}\right)\theta_3$$

$$= \frac{acx_1}{x_1x_2} + \frac{abd}{x_1x_2} + \frac{bex_2}{x_1x_2} + \frac{b\sqrt{ade}}{x_1\sqrt{x_2}}\theta_1\theta_2 + \frac{a\sqrt{bcd}}{\sqrt{x_1}x_2}\theta_2\theta_3 + \frac{\sqrt{abce}}{\sqrt{x_1x_2}}\theta_1\theta_3$$

Question: If we now flip x_3 to x_5, what do we expect x_5 to look like?

Main Question

In a cluster algebra A, any cluster variable can be expressed as a positive Laurent polynomial in the initial cluster, i.e.

$$A \subset \mathbb{R}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}].$$

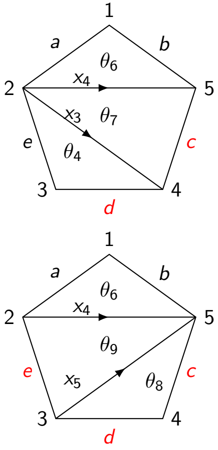
Questions

- Does the super λ-length satisfy some Laurent phenomenon?
- Is there a "positivity" for terms with anti-commuting variables?

Answers (Spoiler Alert)

- Super λ-lengths live in $\mathbb{R}[x_1^{\pm\frac{1}{2}}, \cdots, x_1^{\pm\frac{1}{2}} | \theta_1, \cdots, \theta_{n+1}]$.
- There exists an ordering on the odd variables, called *positive ordering*, such that if we multiply θ's in the positive ordering then the coefficients are positive.

Super Ptolemy Relation - Example Continued



Before giving the general answer, we illustrate the result of flipping x_3 to x_5: We first recall

that $\theta_4 = \dfrac{\sqrt{ad}\,\theta_1 - \sqrt{ex_2}\,\theta_2}{\sqrt{x_1x_3}}$ and note that

$$\theta_7 = \frac{\sqrt{ac}\,\theta_5 - \sqrt{bx_3}\,\theta_3}{\sqrt{x_2x_4}} =$$

$$\frac{1}{\sqrt{cx_3x_4}}\left(c\sqrt{\frac{ae}{x_1}}\theta_1 + ac\sqrt{\frac{d}{x_1x_2}}\theta_2 - x_3\sqrt{\frac{bc}{x_2}}\theta_3\right).$$

We then proceed to obtain

$$x_5 = \frac{ce + dx_4}{x_3} + \frac{\sqrt{cdex_4}}{x_3}\theta_4\theta_7 = \cdots =$$

$$\frac{bd + cx_1}{x_2} + \frac{\sqrt{bcdx_1}}{x_2}\theta_2\theta_3.$$

Continuing with super-flips of x_4 and x_5, in order, yields x_1 and x_2, respectively.

Schiffler's T-paths [Sch08]

Let T be a triangulation of a polygon, thought of as a graph of vertices and edges.

A *T-path* from i to j is a path in T starting at vertex i, ending at j, such that

(T1) the path does not use any edge twice

(T2) the path has an odd number of edges

(T3) the even-numbered edges cross the diagonal (i, j)

(T4) The intersections of the path and (i, j) move from progressively i to j.

Let T_{ij} denote the set of T-paths from i to j.

For a T-path $\gamma = (x_1, x_2, \cdots)$, define it's weight to be

$$\text{wt}(\gamma) = \prod_{i \text{ odd}} \lambda(x_i) \prod_{i \text{ even}} \lambda(x_i)^{-1}$$

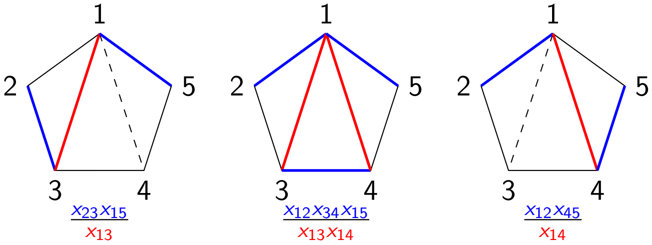
where $\lambda(x_i)$ denote the λ-length of the edge x_i.

Schiffler's T-paths [Sch08]

Theorem (Schiffler)

$$\lambda(x_{i,j}) = \sum_{t \in T_{i,j}} \mathrm{wt}(t)$$

Here are the T-paths in T_{25}. (odd steps are blue and even steps are red)



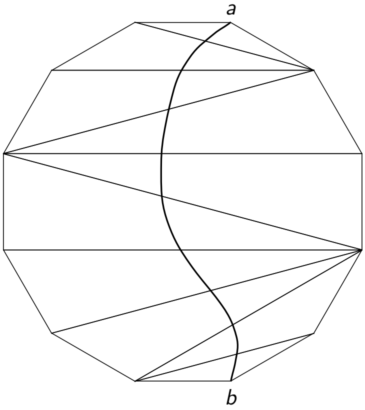
$$\lambda(x_{2,5}) = \sum_{t \in T_{25}} \mathrm{wt}(t) = \frac{x_{23}x_{15}}{x_{13}} + \frac{x_{12}x_{34}x_{15}}{x_{13}x_{14}} + \frac{x_{12}x_{45}}{x_{14}}$$

First Result: Super T-paths and Twisted Super T-paths

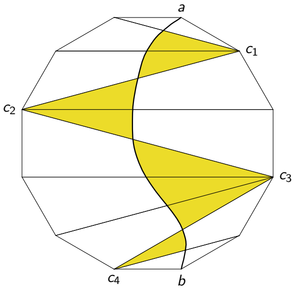
https://arxiv.org/pdf/2102.09143.pdf and https://arxiv.org/pdf/2110.06497.pdf

From now on we only consider triangulations with a longest arc crossing all internal diagonals.

In other words, every triangle has a boundary edge. Call the end points of the longest arc a and b.



Fan Decomposition



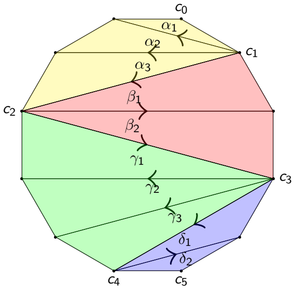
For a triangulation T, we will define a canonical fan decomposition.

The arc (a, b) intersect with internal diagonals, and create smaller triangles (colored yellow).

Vertices of these yellow triangles are called fan centers, denoted $c_1, \cdots, c_n$, ordered by their distance from a. And we further denote $a = c_0$ and $b = c_{n+1}$.

The sub-triangulation bounded by c_{i-1}, c_i, c_{i+1} is called the i-th fan segment of T.

Default Orientation and Positive Ordering



We define a default orientation on the interior diagonals.

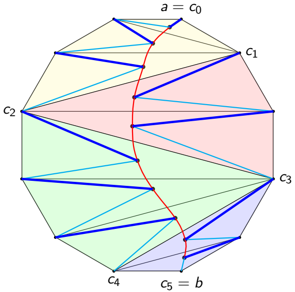
- Edges inside each fan segment are directed away from the center.
- Others are oriented as $c_1 \to c_2 \to \cdots \to c_n$.

We define a positive ordering on μ-invariants.

- μ-invariants in a fan are ordered counterclockwise around the center.
- "Alternate" across the fans.

$$\alpha_1 > \alpha_2 > \alpha_3 > \gamma_1 > \gamma_2 > \gamma_3 > \delta_2 > \delta_1 > \beta_2 > \beta_1$$

The Twisted Auxiliary Graph



For each triangle in T, we place an internal vertex.

The internal vertices are connected to the complement of the nearest fan centers by two σ-edges, with one closer to the starting vertex a while the other is closer to the ending vertex b.

We color the former in **thick blue** and call it a σ^A-edge and color the latter in cyan and call it a σ^B-edge.

Every pair of internal vertices are connected by a teleportation, called a τ-edge. (Note that the τ-edges are drawn to be overlapping.)

The resulting graph $\Gamma_T^{a,b}$ is the **twisted auxiliary graph** associated to $\{T, a, b\}$.

Twisted Super T-paths

Finally, we define **twisted super T-paths** to be paths on the twisted auxiliary graph such that:

(T1) $a = a_0, a_1, \cdots, a_{\ell(t)} = b$ are vertices on $\Gamma_T^{a,b}$.

(T2) For each $1 \leq i \leq \ell(t)$, t_i is an edge in $\Gamma_T^{a,b}$ connecting a_{i-1} and a_i.

(T3) $t_i \neq t_j$ if $i \neq j$.

(T4) $\ell(t)$ is odd.

(T5') t_i crosses (a,b) if i is even. The τ-edges (teleporation) are considered to cross (a,b), and any step along a τ-edge must end further from endpoint a and closer to endpoint b.

(T6') $t_i \in \sigma$ only if i is odd, $t_i \in \tau$ only if i is even.

(T7) If $i < j$ and both t_i and t_j cross the arc (a,b), then the intersection $t_i \cap (a,b)$ is closer to the vertex a than the intersection $t_j \cap (a,b)$.

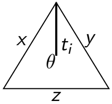
Let $\tilde{T}_{a,b}$ denote the set of **twisted super T-paths** on $\Gamma_T^{a,b}$.

Every ordinary T-path is also a twisted super T-path: $T_{a,b} \subset \tilde{T}_{a,b}$

Weights of Twisted Super T-paths

If a super T-path uses edges $t_1, t_2, \ldots,$ we define its weight as follows.

- If t_i is a diagonal in the triangulation, then:
  $\mathrm{wt}(t_i) = \lambda(t_i)$ if i odd, and
  $\mathrm{wt}(t_i) = \lambda(t_i)^{-1}$ if t is even.
- If t_i is a τ-edge, then $\mathrm{wt}(t_i) = 1$ (teleportation)
- If t_i is a σ-edge, then $\mathrm{wt}(t_i) = \sqrt{\frac{xy}{z}}\,\theta$. Here x, y, z are λ-lengths and θ is the μ-invariant.



If t is a twisted super T-path with edges $t_1, t_2, \ldots,$ set $\mathrm{wt}(t) = \prod_i \mathrm{wt}(t_i)$. Here the product is taken under the positive ordering.

First Theorem: Formula for Super λ-lengths

Theorem (M-Ovenhouse-Zhang 2021)

Under default orientation, the super λ-length of the arc (a, b) (assuming to be the longest arc in T) is given by (twisted) super T-paths:

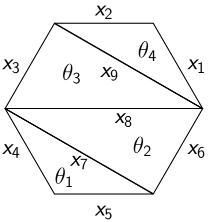
$$\lambda(a, b) = \sum_{t \in \tilde{T}_{a,b}} \mathrm{wt}(t)$$

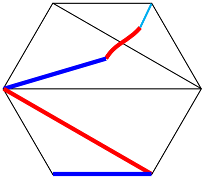
With the following lemma, we can apply the main theorem for triangulations with arbitrary orientation.

Lemma ([MOZ21])

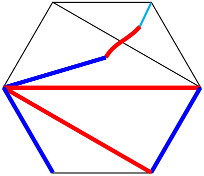
In the equivalence class of any spin structure, there exists (at least) one default orientation. (In other words, up to possibly negating boundary edges, or negating a μ-invariant and its three incident edges, we can transform any orientation on T into the default orientation.)

Twisted Super T-paths and their Weights: Examples

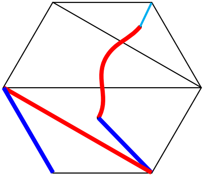




$$\frac{x_5\sqrt{x_1x_2x_3x_8}}{x_7x_9}\,\theta_4\theta_3$$

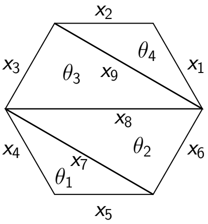


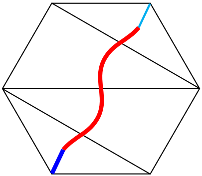
$$\frac{x_4x_6\sqrt{x_1x_2x_3}}{x_7x_9\sqrt{x_8}}\,\theta_4\theta_3$$



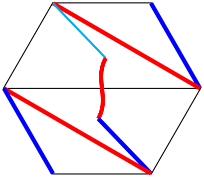
$$\frac{x_4\sqrt{x_1x_2x_6}}{\sqrt{x_7x_8x_9}}\,\theta_2\theta_4$$

Twisted Super T-paths and their Weights: Examples II

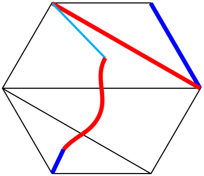




$$\frac{\sqrt{x_1x_2x_4x_5}}{\sqrt{x_7x_9}}\,\theta_1\theta_4$$

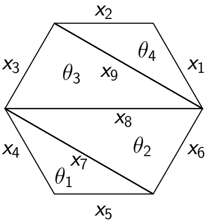


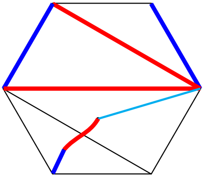
$$\frac{x_1x_4\sqrt{x_3x_6}}{x_8\sqrt{x_7x_9}}\,\theta_2\theta_3$$



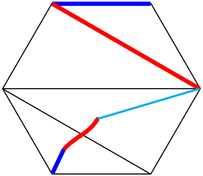
$$\frac{x_1\sqrt{x_3x_4x_5}}{\sqrt{x_7x_8x_9}}\,\theta_1\theta_3$$

Twisted Super T-paths and their Weights: Examples III

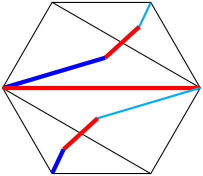




$$\frac{x_1 x_3 \sqrt{x_4 x_5 x_6}}{x_7 x_9 \sqrt{x_8}} \theta_1 \theta_2$$



$$\frac{x_2 \sqrt{x_4 x_5 x_6 x_8}}{x_7 x_9} \theta_1 \theta_2$$

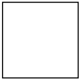


$$\frac{\sqrt{x_1 x_2 x_3 x_4 x_5 x_6}}{x_7 x_9} \theta_1 \theta_2 \theta_4 \theta_3$$

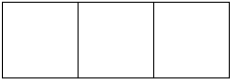
Recent Work: A Second Combinatorial Interpretation

A **snake graph** is a planar graph consisting of a sequence of square tiles, each connected to either the top or right side of the previous tile.

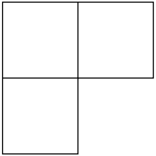
Given a snake graph G, the *word* of G, denoted $W(G)$, is a string in the alphabet $\{$R,U$\}$ (standing for "*right*" and "*up*") indicating how each tile is connected to the previous.



$W(G) = \varnothing$



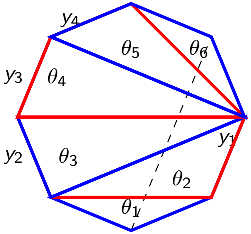
$W(G) = \text{RR}$



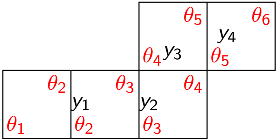
$W(G) = \text{UR}$

Recent Work: A Second Combinatorial Interpretation

Every square tile in a snake graph represents two triangles in the triangulation. We label tiles with the odd variables of those triangles.



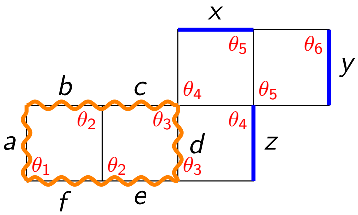
$$W(G) = RRUR$$



We built the snake graph from this triangulation traversing the dashed line from bottom-to-top, gluing tiles together based on boundary edges shared by adjacent quadrilaterals.

Recent Work: A Second Combinatorial Interpretation

Every square tile in a snake graph represents two triangles in the triangulation. We label tiles with the odd variables of those triangles.



$$\text{weight} = xyz\sqrt{abcdef}\,\theta_1\theta_3$$

A **double dimer cover** of a graph is the union of two dimer covers. It is composed of cycles and doubled edges.

Dimers will be drawn as wavy orange lines, and double dimers will be drawn as straight blue lines.

The **weight** of a double dimer cover is the product of the square roots of the edge weights, times the odd variables at the beginning and end of cycles.

Recent Work: A Second Combinatorial Interpretation

Theorem (M-Ovenhouse-Zhang 2021)

Consider a triangulation where f is the longest edge, we follow the construction of [MSW11] to build the snake graph G corresponding to the arc f. Then the super λ-length for f is given as follows:

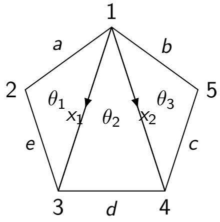
$\frac{1}{\mathrm{cross}(f)} \sum_{M \in DD(G)} \mathrm{wt}(M)$ *where $DD(G)$ is the set of double-dimers on G.*

Here, $\mathrm{cross}(f)$ denotes the monomial given by the product of the edges crossed by the arc f, and wt decomposes into an even and odd part, $\mathrm{wt} = \mathrm{wt}_x \mathrm{wt}_\theta$.

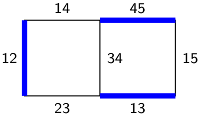
The value of wt_x is the product of the weights of the edges in M with multiplicity, but the weight of each individual edge is given by a square-root.

Additionally each cycle around tiles appearing in M contributes a weight of $\theta_i \theta_j$ to wt_θ, where θ_i and θ_j label the first and last triangles of that cycle in G, respectively.

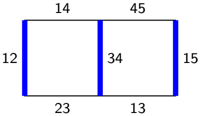
Recent Work: A Second Combinatorial Interpretation



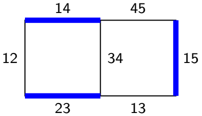
$$\textit{Recall } \lambda_{2,5} = \frac{acx_1}{x_1x_2} + \frac{abd}{x_1x_2} + \frac{bex_2}{x_1x_2} + \frac{b\sqrt{ade}}{x_1\sqrt{x_2}}\theta_1\theta_2 + \frac{a\sqrt{bcd}}{\sqrt{x_1}x_2}\theta_2\theta_3 + \frac{\sqrt{abce}}{\sqrt{x_1x_2}}\theta_1\theta_3$$



$\frac{acx_1}{x_1x_2}$

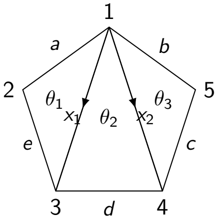


$\frac{abd}{x_1x_2}$

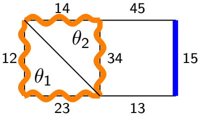


$\frac{bex_2}{x_1x_2}$

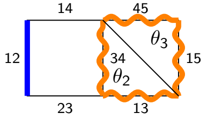
Recent Work: A Second Combinatorial Interpretation



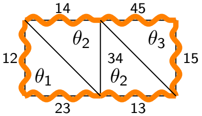
$$Recall\ \lambda_{2,5} = \frac{acx_1}{x_1x_2} + \frac{abd}{x_1x_2} + \frac{bex_2}{x_1x_2} +$$
$$\frac{b\sqrt{ade}}{x_1\sqrt{x_2}}\theta_1\theta_2 + \frac{a\sqrt{bcd}}{\sqrt{x_1}x_2}\theta_2\theta_3 + \frac{\sqrt{abce}}{\sqrt{x_1x_2}}\theta_1\theta_3$$



$$\frac{b\sqrt{adex_2}}{x_1x_2}\theta_1\theta_2$$



$$\frac{a\sqrt{bcdx_1}}{x_1x_2}\theta_2\theta_3$$



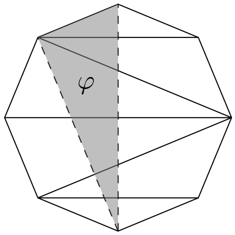
$$\frac{\sqrt{abcex_1x_2}}{x_1x_2}\theta_1\theta_3$$

What about odd variables?

Consider an arc γ as before.

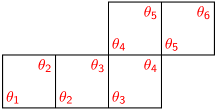
Let φ be a triangle with γ as a side, and also a boundary side.

Can we express the μ-invariant θ_φ in terms of the initial triangulation?



The Toggle Involution

Recall that snake graphs are labelled with odd variables.

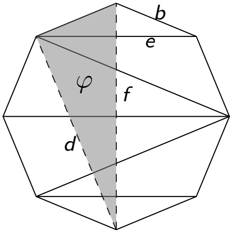


If θ_n is the label on the upper-right of the last tile, define an involution $x \mapsto x^*$ on monomials which adds/removes θ_n.

Examples:

$$(\theta_1\theta_2)^* = \theta_1\theta_2\theta_6, \qquad (\theta_4\theta_6)^* = \theta_4$$

Formula for Odd Variables

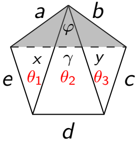


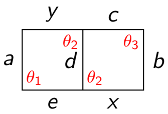
Theorem [M-Ovenhouse-Zhang 2021]

$$\sqrt{df}\,\theta_\varphi = \frac{1}{\mathrm{cross}(f)}\frac{\sqrt{e}}{\sqrt{b}}\sum_{M\in D_t(G_f)} \mathrm{wt}(M)^*$$

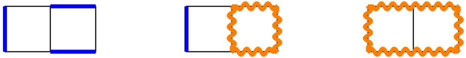
where D_t is the set of double dimer covers using the **top** edge of the last tile (as long as the polygon has an odd number of triangles; otherwise use the **right** edge on the last tile instead).

Example of μ-invariant Formula





$D_t(G)$:



$$\sum_{M\in D_t(G)} \mathrm{wt}(M) = acx + a\sqrt{bcdx}\,\theta_2\theta_3 + \sqrt{abcexy}\,\theta_1\theta_3$$

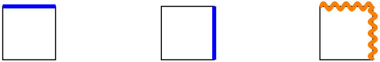
$$\sum_{M\in D_t(G)} \mathrm{wt}(M)^* = acx\,\theta_3 + a\sqrt{bcdx}\,\theta_2 + \sqrt{abcexy}\,\theta_1$$

$$\frac{\sqrt{y}}{\sqrt{c}} \sum_{M\in D_t(G)} \mathrm{wt}(M)^* = ax\sqrt{cy}\,\theta_3 + a\sqrt{bdxy}\,\theta_2 + y\sqrt{abex}\,\theta_1$$

$$\sqrt{a\gamma}\,\theta_\varphi = \frac{1}{xy}\left(ax\sqrt{cy}\,\theta_3 + a\sqrt{bdxy}\,\theta_2 + y\sqrt{abex}\,\theta_1\right)$$

The Proof

Looking at the top-right corner of the last tile of $G = G_f$, there are 3 cases:



So we have $DD(G) = D_T(G) \cup D_R(G) \cup D_{tr}(G)$.
The super Ptolemy relation also has 3 terms:

$$f = \frac{1}{e}\Big(ac + bd + \sqrt{abcd}\,\sigma\theta\Big)$$

The strategy of the proof is to show that

$$\frac{ac}{e} = \sum_{M \in D_T(G)} \mathrm{wt}(M)$$

$$\frac{bd}{e} = \sum_{M \in D_R(G)} \mathrm{wt}(M)$$

$$\frac{\sqrt{abcd}}{e}\,\sigma\theta = \sum_{M \in D_{tr}(G)} \mathrm{wt}(M)$$

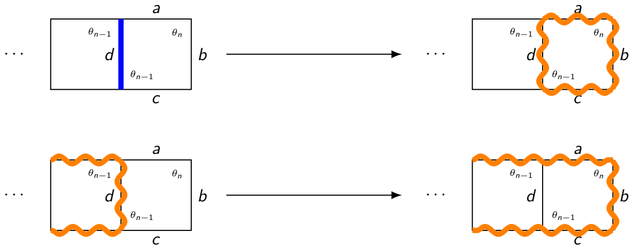
The Proof

The details involve induction on the number of tiles in the snake graph (equivalently, the number of triangles in the polygon).

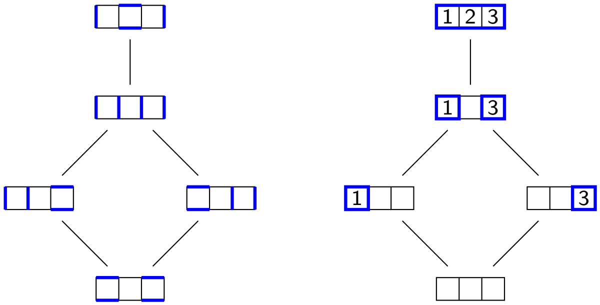
However, since the formula for super λ-lengths involves μ-invariants, we must inductively prove the super λ-length formula and μ-invariant formula simultaneously.

The induction steps themselves are proven using recursion formulas that are combinatorially satisfied by the set of double dimers.

Example of Mapping $D_r(G^{(-1)})$ into $D_{tr}(G)$:

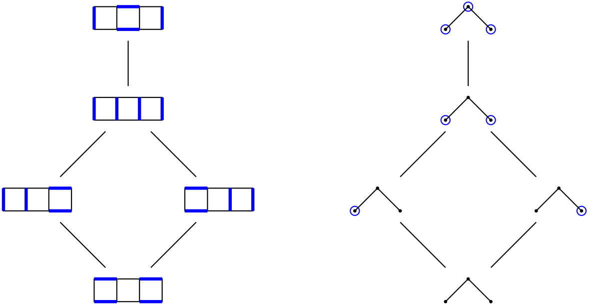


Lattice Structure in Dimer Case



Superimpose the minimal dimer cover (but do not draw doubled edges) to see this is isomorphic to a lattice of subsets ordered under inclusion.

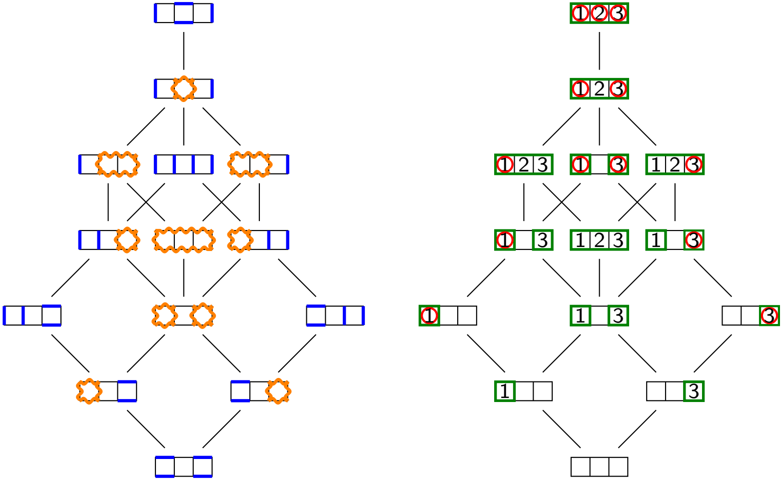
Lattice Structure in Dimer Case



Lattice isomorphism

There is a poset isomorphism $L(G) \cong J(P(G))$, between the set of dimer covers on G and the lattice of lower order ideals in $P(G)$, the fence poset corresponding to the snake graph G.

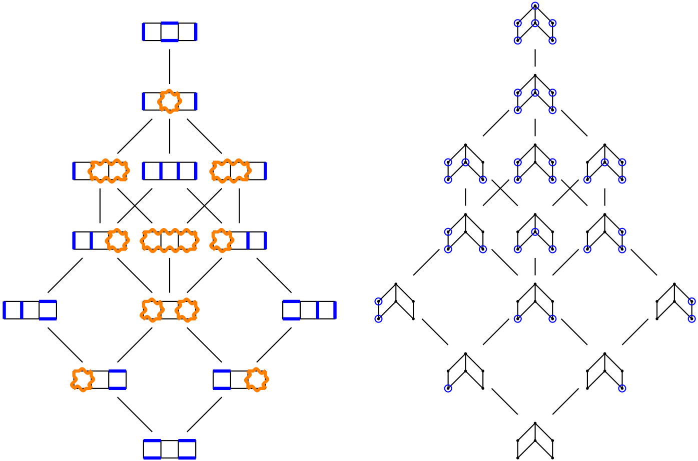
Application: Lattice Structure in Double Dimer Case



Application: Lattice Structure in Double Dimer Case

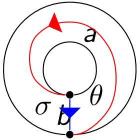
Theorem

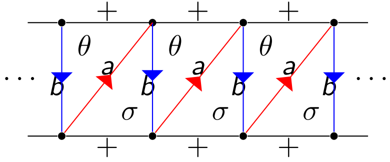
There is a poset isomorphism $L(G) \cong J(\mathbb{P}(G))$, between double dimer covers on G and lower order ideals in $\mathbb{P}(G) := P(G) \times \{0, 1\}$.

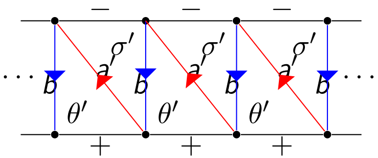


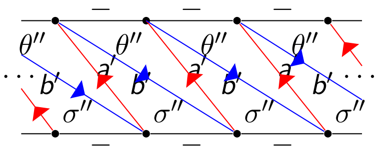
Application: Super Fibonacci Numbers

Given a triangulation of an annulus, we consider the periodic mutation sequence $a, b, a, b, a, b, \ldots$ in the universal cover.





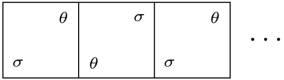




Since $\sigma\theta = \sigma'\theta' = \sigma''\theta'' = \ldots$, if we let $\epsilon = \sigma\theta$, the Super Ptolemy Relation will always have the form $ef = a^2 + b^2 + ab\epsilon$. Thus letting $Z_1 = a$, $Z_2 = b$, we get the recurrence $Z_m Z_{m-2} = Z_{m-1}^2 + Z_{m-1}\epsilon + 1$ for the resulting infinite sequence of super λ-lengths.

Application: Super Fibonacci Numbers

Letting G_m denote the snake graph for the word $W(G) = RR\ldots R$, i.e. with m tiles in a horizontal row, where all edges have weight 1, and all tiles alternate between the same two μ-invariants σ and θ



we obtain that the Z_m's are the double dimer partition functions for the snake graphs G_{2m-5}.

Further, when we initialize $Z_1 = a = 1$ and $Z_2 = b = 1$, we get for $m \geq 3$

$$Z_m = F_{2m-3} + \left(\sum_{k=0}^{m-3} (2k+1)\binom{m+k-1}{2k+2}\right)\epsilon,$$

where F_k is the kth Fibonacci number such that $F_1 = F_2 = 1$.

Application: Super Fibonacci Numbers

Further, when we initialize $Z_1 = a = 1$ and $Z_2 = b = 1$, we get for $m \geq 3$

$$Z_m = F_{2m-3} + \left(\sum_{k=0}^{m-3} (2k+1) \binom{m+k-1}{2k+2} \right) \epsilon,$$

where F_k is the kth Fibonacci number such that $F_1 = F_2 = 1$.
We also can let $W_m = F_{2m-2} + \left(\sum_{k=0}^{m-3} (2k) \binom{m+k-2}{2k+1} \right) \epsilon$, which is the double dimer partition function for G_{2m-4}.

Examples: `https://oeis.org/A054454`

$$Z_3 = 2 + \epsilon$$
$$W_3 = 3 + 2\epsilon$$
$$Z_4 = 5 + 6\epsilon$$
$$W_4 = 8 + 12\epsilon$$
$$Z_5 = 13 + 26\epsilon$$
$$W_5 = 21 + 50\epsilon$$
$$Z_6 = 34 + 97\epsilon$$

Open Problems

Conjecture

If we let $W_1 = W_2 = 1$ (or if we let $W_1 = a$ and $W_2 = b$), and set W_m to be the double dimer partition function of G_{2m-4}, then W_m corresponds to the super λ-lengths of a peripheral arc in an annulus, except in the context of the decorated super-Teichmüller space.

Question

Begin with an oriented triangulation of the once-punctured torus, and allow flips in all three directions. The resulting super λ-lengths of such arcs correspond to super analogues of the Markoff numbers satisfying

$$x^2 + y^2 + z^2 + (xy + yz + xz)\epsilon = 3(1+\epsilon)xyz.$$

Do they have combinatorial interpretations using double dimer covers of the snake graphs appearing in Section 7 of [Propp 2005] in the presence of appropriately specialized μ-invariants?

Open Problems

Question

Does using double dimer covers on snake graphs rather than (twisted) super T-paths allow us to combinatorially calculate super λ-lengths more easily for other surfaces? Do we recover super analogues of skein relations (rather than only when applying diagonal flips in a quadrilateral?

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http://www-users.math.umn.edu/∼musiker/IsaacNewton21.pdf

Thank You for Listening! https://arxiv.org/pdf/2110.06497.pdf

Gregg Musiker, Nicholas Ovenhouse, and Sylvester W. Zhang.
An expansion formula for decorated super-Teichmüller spaces.
SIGMA. Symmetry, Integrability and Geometry: Methods and Applications, 17:080, September 2021.

Musiker, Schiffler, and Williams.
Positivity for cluster algebras from surfaces.
Advances in Mathematics, 227(6):2241–2308, 2011.

Robert C Penner and Anton M Zeitlin.
Decorated super-Teichmüller space.
Journal of Differential Geometry, 111(3):527–566, 2019.

Ralf Schiffler.
A cluster expansion formula (an case).
the electronic journal of combinatorics, 15(R64):1, 2008.