Double-dimer configurations and quivers of dP3 (del Pezzo) type.

Gregg Musiker (University of Minnesota)

Online Cluster Algebra Seminar

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http://www-users.math.umn.edu/∼musiker/OCAS20.pdf

Joint work with Helen Jenne (CNRS[†]) and Tri Lai (University of Nebraska),

(†) Institut Denis Poisson, Université de Tours and Université d'Orléans

Brane Tilings and Cluster Algebras

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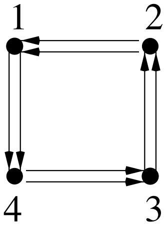

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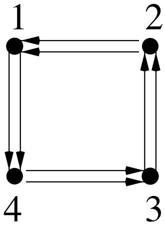

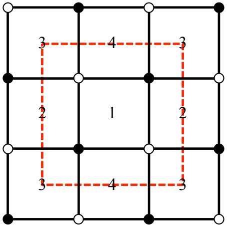

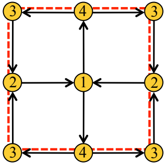

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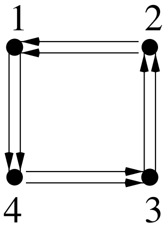

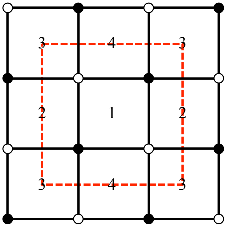

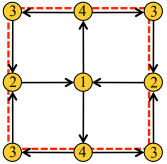

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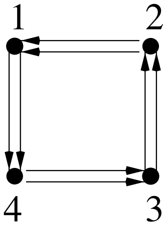

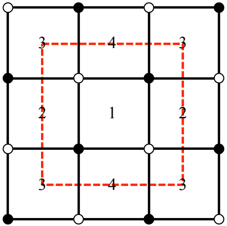

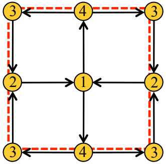

Such cluster algebras are of infinite mutation type, but nonetheless have a finite subset of mutation-equivalent quivers known as **toric phases**,

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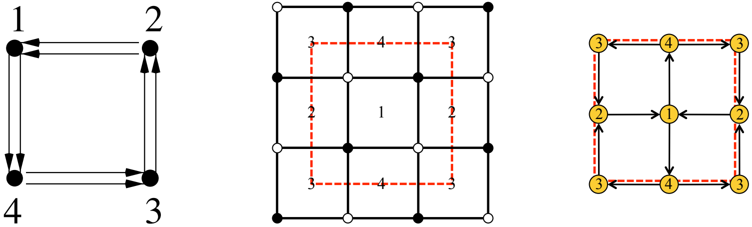
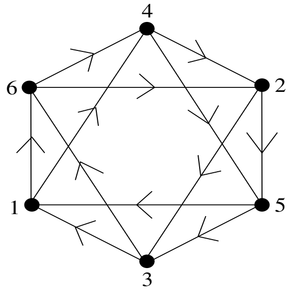


Such cluster algebras are of infinite mutation type, but nonetheless have a finite subset of mutation-equivalent quivers known as **toric phases**, and a well-behaved subset of cluster variables, known as **toric cluster variables** to go along with them.

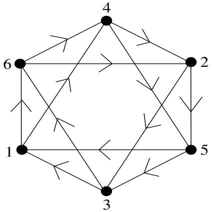
Example (The $dP3$ Quiver): $Q_{dP3} = Q =$  ,

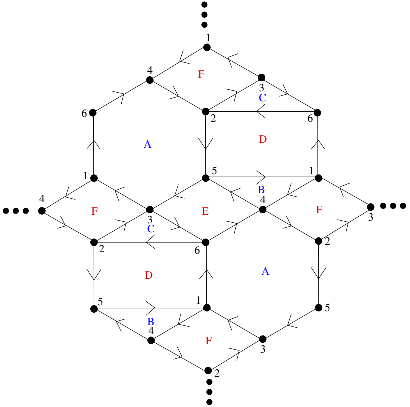
$$
\begin{aligned}
W &= A_{16}A_{64}A_{42}A_{25}A_{53}A_{31} + A_{14}A_{45}A_{51} + A_{23}A_{36}A_{62} \\
&- A_{16}A_{62}A_{25}A_{51} - A_{36}A_{64}A_{45}A_{53} - A_{14}A_{42}A_{23}A_{31}.
\end{aligned}
$$

We unfold Q onto the plane, letting the three positive (resp. negative) terms in W depict clockwise (resp. counter-clockwise) cycles on $\widetilde{Q}$.

Brane Tilings from a Quiver Q with Potential W

Example (continued):

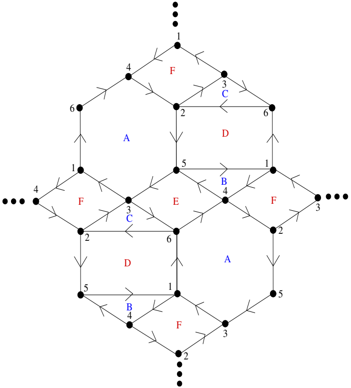
$Q =$ 

unfolds to $\widetilde{Q} =$ 

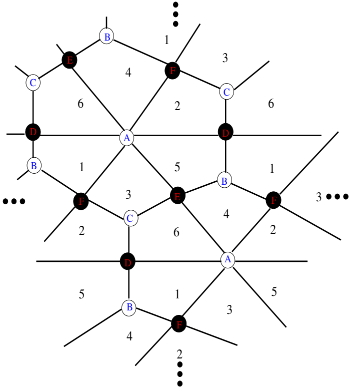
$$W = A_{16}A_{64}A_{42}A_{25}A_{53}A_{31}(A) + A_{14}A_{45}A_{51}\ (B) + A_{23}A_{36}A_{62}\ (C)$$
$$- A_{16}A_{62}A_{25}A_{51}\ (D) - A_{36}A_{64}A_{45}A_{53}\ (E) - A_{14}A_{42}A_{23}A_{31}(F).$$

Brane Tilings from a Quiver Q with Potential W

Taking the planar dual yields a bipartite graph on a torus (**Brane Tiling**):



$$\widetilde{Q} \longrightarrow \mathcal{T}_Q =$$



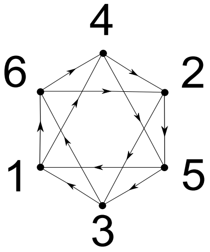
Negative Term in $W \longleftrightarrow$ Counter-Clockwise cycle in $\widetilde{Q} \longleftrightarrow \bullet$ in $\mathcal{T}_Q$

Positive Term in $W \longleftrightarrow$ Clockwise cycle in $\widetilde{Q} \longleftrightarrow \circ$ in $\mathcal{T}_Q$

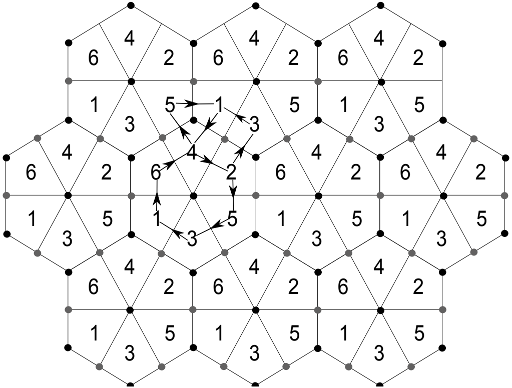
(To obtain $\widetilde{Q}$ from $\mathcal{T}_Q$, we dualize edges so that white is on the right.)

Brane Tilings from a Quiver Q with Potential W

The $dP3$ Example:



Q

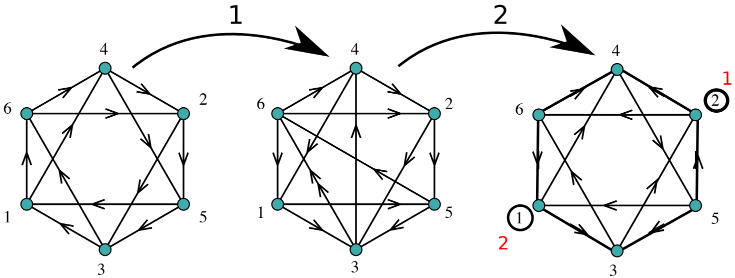


$\mathcal{T}_Q$

Motivational Goal: Study Toric Mutation Sequences of Such Quivers.

We say that a mutation is a toric mutation if it occurs at a vertex with exactly two incoming arrows and two outgoing arrows.

The $dP3$ quiver admits a **periodic** toric mutation sequence beginning as so:



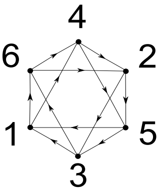
As we mutate Q_{dP3} by $1, 2, 3, 4, 5, 6, 1, 2, \ldots,$ after the first two mutations, we obtain the same quiver back up to relabelling the vertices.

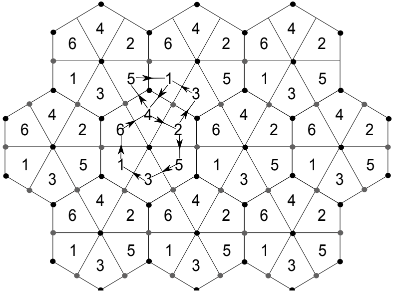
We will discuss other toric mutation sequences momentarily.

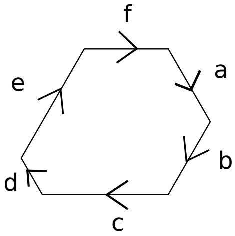
Cluster Variables and Contours on the del Pezzo 3 Lattice

We wish to understand algebraic formulas and combinatorial interpretations for **toric cluster variables**, i.e. those reachable from the initial cluster via a sequence of toric mutations. (We note that in other contexts, such toric mutations are also known as square moves.)

To this end, we cut out subgraphs of the dP3 lattice (Middle) by using six-sided contours

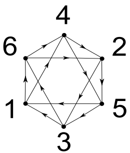


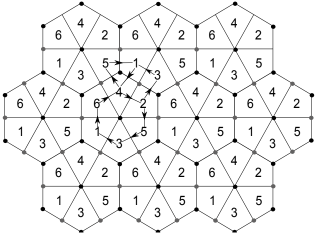




indexed as (a, b, c, d, e, f) with $a, b, c, d, e, f \in \mathbb{Z}$ (Right).

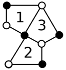
Example from S. Zhang (2012 REU): Periodic mutation $1, 2, 3, 4, 5, 6, 1, 2, \ldots$ yields partition functions for Aztec Dragons (as studied by Ciucu, Cottrell-Young, Propp, and Wieland) under appropriate weighted enumeration of perfect matchings (a.k.a dimers). (Starting from the **initial cluster** $\{x_1, x_2, x_3, x_4, x_5, x_6\}$.)

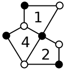


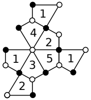


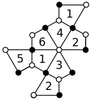

$$\frac{x_3x_5 + x_4x_6}{x_1}$$


$$\frac{x_4x_6 + x_3x_5}{x_2}$$


$$\frac{x_2x_3x_5^2 + x_1x_3x_5x_6 + x_2x_4x_5x_6 + x_1x_4x_6^2}{x_1x_2x_3}$$


$$\frac{x_2x_3x_5^2 + x_1x_3x_5x_6 + x_2x_4x_5x_6 + x_1x_4x_6^2}{x_1x_2x_4}$$


$$\frac{(x_2x_5 + x_1x_6)(x_1x_3 + x_2x_4)(x_3x_5 + x_4x_6)^2}{x_1^2x_2^2x_3x_4x_5}$$

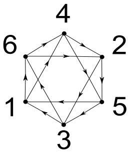

$$\frac{(x_2x_5 + x_1x_6)(x_1x_3 + x_2x_4)(x_3x_5 + x_4x_6)^2}{x_1^2x_2^2x_3x_4x_6}$$

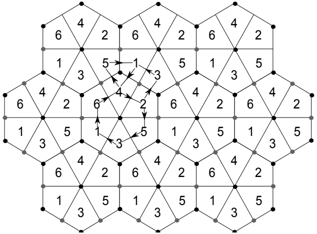
Combinatorial Formula for Toric Cluster Variables parameterized by $\mathbb{Z}^1$

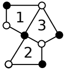
These graphs G admit dimer partition functions $cm(G)\sum_M x(M)$ agreeing with the **Laurent expansion of cluster variables** via the **weighting** $x(M) = \prod_{\text{edge } e \in M} \frac{1}{x_i x_j}$ (for e straddling faces i and j) and $cm(G)$ = the covering monomial recording the face labels contained in G or along its boundary (also see [Speyer] and [Goncharov-Kenyon]).

$$\frac{x_3x_5+x_4x_6}{x_1}$$

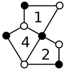
$$\frac{x_4x_6+x_3x_5}{x_2}$$



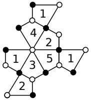




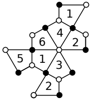
$$\frac{x_2x_3x_5^2+x_1x_3x_5x_6+x_2x_4x_5x_6+x_1x_4x_6^2}{x_1x_2x_3}$$



$$\frac{x_2x_3x_5^2+x_1x_3x_5x_6+x_2x_4x_5x_6+x_1x_4x_6^2}{x_1x_2x_4}$$



$$\frac{(x_2x_5+x_1x_6)(x_1x_3+x_2x_4)(x_3x_5+x_4x_6)^2}{x_1^2x_2^2x_3x_4x_5}$$



$$\frac{(x_2x_5+x_1x_6)(x_1x_3+x_2x_4)(x_3x_5+x_4x_6)^2}{x_1^2x_2^2x_3x_4x_6}$$

All Possible Toric Mutation Sequences

Starting with any of these four models (a.k.a. **toric phases**) of the $dP3$ quiver, any sequence of toric mutations yields a quiver that is graph isomorphic to one of these (up to full reversal).

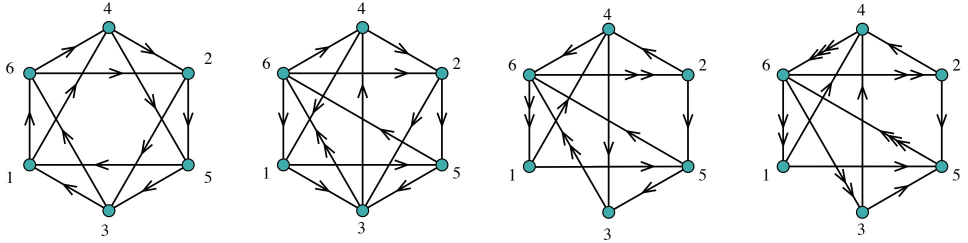
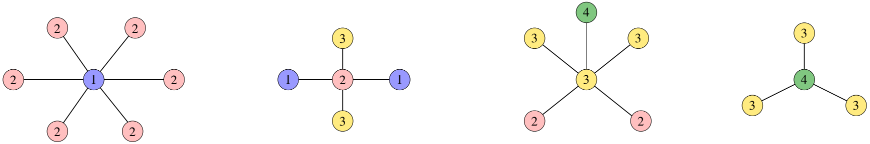
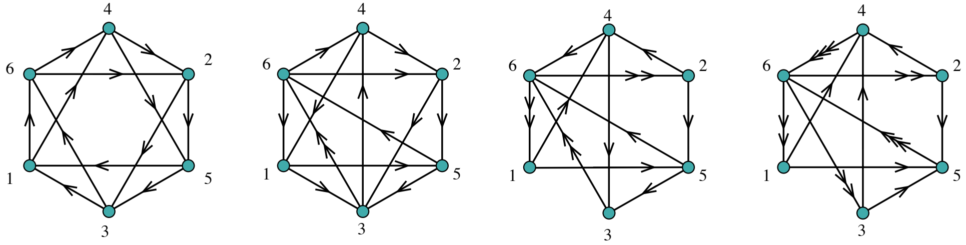


Figure 20 of [Eager-Franco] (Incidences betweeen these Models):



Periodic Examples of Toric Mutation Sequences

The previous example of the periodic sequence $1, 2, 3, 4, 5, 6, 1, 2, \ldots$ corresponds to mutating pairs of antipodal vertices in order, thus alternating between Model 1 and Model 2.



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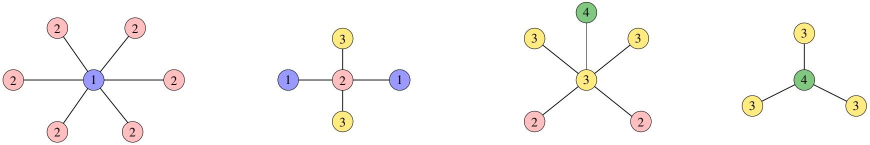
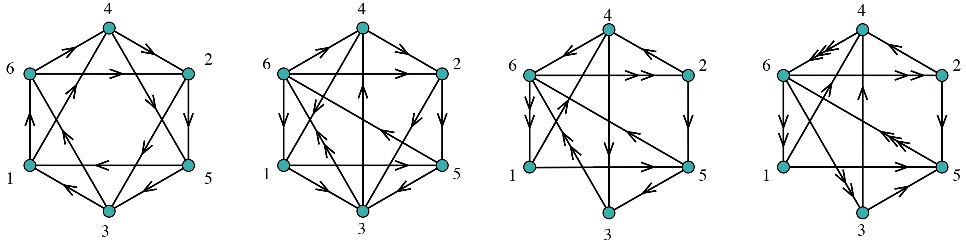


Non-Periodic Examples of Toric Mutation Sequences

We may also mutate at pairs of antipodal vertices in a different order, while still alternating between Model 1 and Model 2. For example, $1, 2, 3, 4, 1, 2, 5, 6$.



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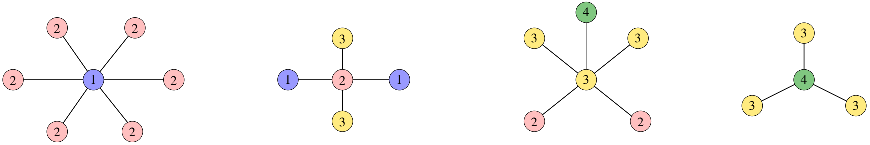

Example from M. Leoni, S. Neel, and P. Turner (2013 REU): We refer to sequences made out of mutations at antipodal vertices of the dP3 quiver as τ-mutation sequences.

e.g. $1, 2, 3, 4, 1, 2, 5, 6$ yields a cluster variable (which is **not** an Aztec Dragon)

$$
\begin{aligned}
& (x_1x_2^2x_3^3x_5^4 + x_2^3x_3^2x_4x_5^4 + 2x_1^2x_2x_3^3x_5^3x_6 + 4x_1x_2^2x_3^2x_4x_5^3x_6 + 2x_2^3x_3x_4^2x_5^3x_6 + x_1^3x_3^3x_5^2x_6^2 \\
+ \quad & 5x_1^2x_2x_3^2x_4x_5^2x_6^2 + 5x_1x_2^2x_3x_4^2x_5^2x_6^2 + x_2^3x_4^3x_5^2x_6^2 + 2x_1^3x_3^2x_4x_5x_6^3 + 4x_1^2x_2x_3x_4^2x_5x_6^3 \\
+ \quad & 2x_1x_2^2x_4^3x_5x_6^3 + x_1^3x_3x_4^2x_6^4 + x_1^2x_2x_4^3x_6^4)/x_1^2x_2^2x_3^2x_4^2x_6 = \frac{(x_1x_3 + x_2x_4)(x_4x_6 + x_3x_5)^2(x_1x_6 + x_2x_5)^2}{x_1^2x_2^2x_3^2x_4^2x_6}
\end{aligned}
$$

Combinatorial Formula for Toric Cluster Variables parameterized by $\mathbb{Z}^2$

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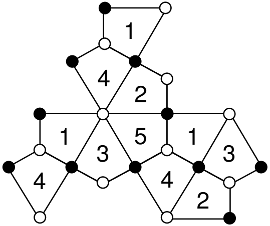
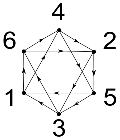
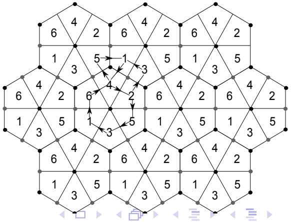
e.g. $1, 2, 3, 4, 1, 2, 5, 6$ yields a cluster variable (which is **not** an Aztec Dragon)

$$\begin{aligned} & (x_1x_2^2x_3^3x_5^4 + x_2^3x_3^2x_4x_5^4 + 2x_1^2x_2x_3^3x_5^3x_6 + 4x_1x_2^2x_3^2x_4x_5^3x_6 + 2x_2^3x_3x_4^2x_5^3x_6 + x_1^3x_3^3x_5^2x_6^2 \\ + \quad & 5x_1^2x_2x_3^2x_4x_5^2x_6^2 + 5x_1x_2^2x_3x_4^2x_5^2x_6^2 + x_2^3x_4^3x_5^2x_6^2 + 2x_1^3x_3^2x_4x_5x_6^3 + 4x_1^2x_2x_3x_4^2x_5x_6^3 \\ + \quad & 2x_1x_2^2x_4^3x_5x_6^3 + x_1^3x_3x_4^2x_6^4 + x_1^2x_2x_4^3x_6^4)/x_1^2x_2^2x_3^2x_4^2x_6 = \frac{(x_1x_3 + x_2x_4)(x_4x_6 + x_3x_5)^2(x_1x_6 + x_2x_5)^2}{x_1^2x_2^2x_3^2x_4^2x_6} \end{aligned}$$

Resulting Laurent polynomials correspond to **Aztec Castles** under appropriate weighted enumeration of dimers.







Combinatorial Formula for Toric Cluster Variables parameterized by $\mathbb{Z}^2$

τ-mutation sequences (i.e. toric mutation sequences stemming from mutating antipodal pairs of vertices) can be built out of $\tau_1 = \mu_1\mu_2$, $\tau_2 = \mu_3\mu_4$, and $\tau_3 = \mu_5\mu_6$.

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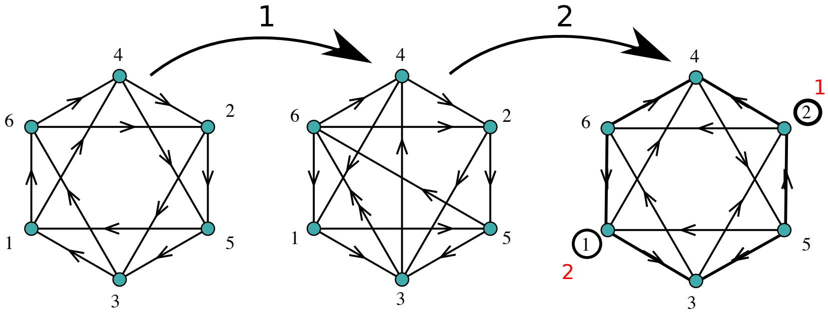

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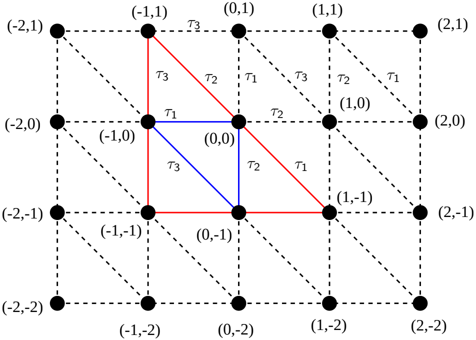
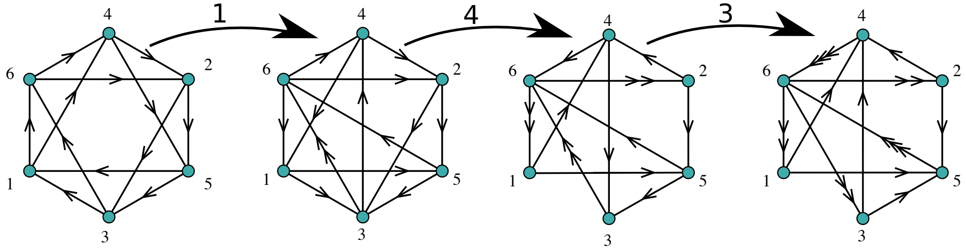
These relations are provably the only equivalences, and thus we can describe clusters reachable by such toric mutation sequences in terms of $\mathbb{Z}^2$ (each cluster is a triangle in this lattice):

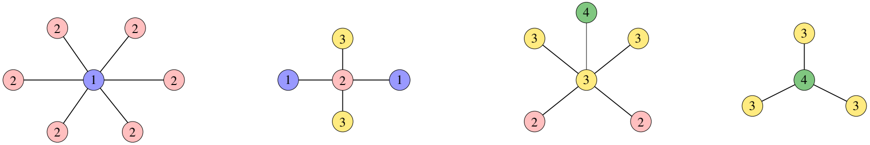


Even More Examples of Non-Periodic Toric Mutation Sequences

More complicated mutation sequences lead us from Model 1 to Model 3 and/or Model 4 and back. For example, consider the sequences $1, 4, 1, 5, 1$ or $1, 4, 3, 2, 4, 1$.



Incidences from Figure 20 of [Eager-Franco]: Models $1, 2, 3, 3, 2, 1$ and $1, 2, 3, 4, 3, 2, 1$, resp.



$\mathbb{Z}^3$ Parameterization for Toric Cluster Variables of the Model 1 $dP3$ Quiver

In [Lai-M 2017], we showed that the set of toric cluster variables is parameterized by $\mathbb{Z}^3$.

In particular, we quotient by toric mutation sequences that result in the same dP3 quiver up to vertex relabelling σ, and the permuted initial cluster $\{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)}\}$. Accounting for this, we deduced there are exactly three degrees of freedom and no torsion.

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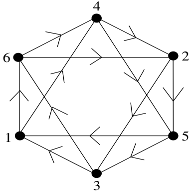
Initializing the initial cluster $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ as corresponding to the triangular prism

$$\{(0,-1,1),(0,-1,0),(-1,0,0),(-1,0,0),(-1,0,1),(0,0,1),(0,0,0)\} \subset \mathbb{Z}^3,$$

we let $z_{i,j,k}$ be the toric cluster variable corresponding to $(i,j,k) \in \mathbb{Z}^3$.



Note: We will sometimes denote $z_{i,j,k}$ as $z(i,j,k)$ instead.

Various moves in the $\mathbb{Z}^3$ lattice correspond to mutation of Q_{dP3}.

$\mathbb{Z}^3$ Parameterization for Toric Cluster Variables and an Algebraic Formula

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We obtained **algebraic expressions** for all such corresponding Laurent polynomials.

$$\text{Let} \qquad A = \frac{x_3x_5 + x_4x_6}{x_1x_2}, \quad B = \frac{x_1x_6 + x_2x_5}{x_3x_4}, \quad C = \frac{x_1x_3 + x_2x_4}{x_5x_6},$$

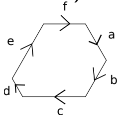
$$D = \frac{x_1x_3x_6 + x_2x_3x_5 + x_2x_4x_6}{x_1x_4x_5}, \text{ and } E = \frac{x_2x_4x_5 + x_1x_3x_5 + x_1x_4x_6}{x_2x_3x_6}.$$

$$z_{i,j,k} = x_r \; A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} D^{\lfloor \frac{(k-1)^2}{4} \rfloor} E^{\lfloor \frac{k^2}{4} \rfloor}$$

Note: x_r is an initial cluster variable with r depending on $(i-j)$ modulo 3 and k modulo 2.

Map from $\mathbb{Z}^3$ to $\mathbb{Z}^6$:

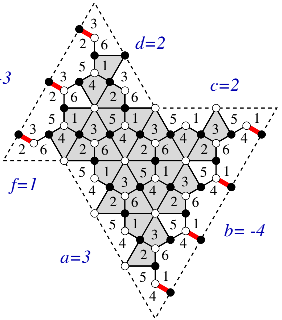
$$(i,j,k) \to (a,b,c,d,e,f) = (j+k, -i-j-k, i+k, j-k+1, -i-j+k-1, i-k+1)$$

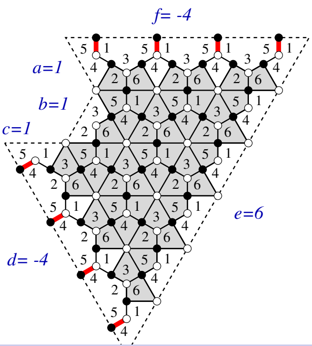


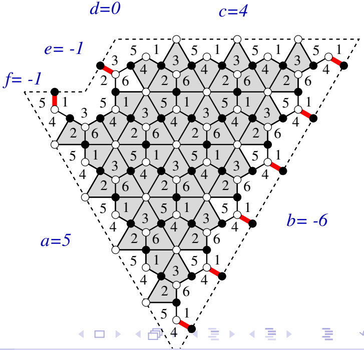
Magnitude determines Length and Sign determines Direction.

$+/-$ Sign also determines white/black vertices on the contour boundary.

Examples: $(1,2,1) \to (3,-4,2,2,-3,1)$, $(-2,-2,3) \to (1,1,1,-4,6,-4)$, and $(1,2,3) \to (5,-6,4,0,-1,-1)$

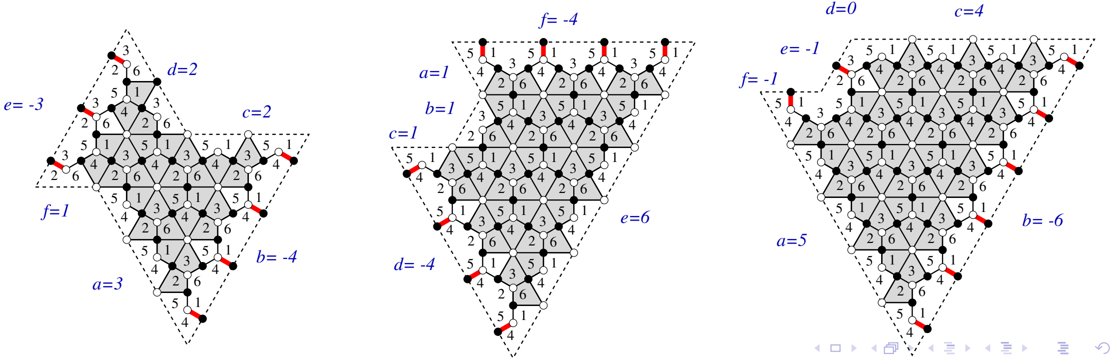




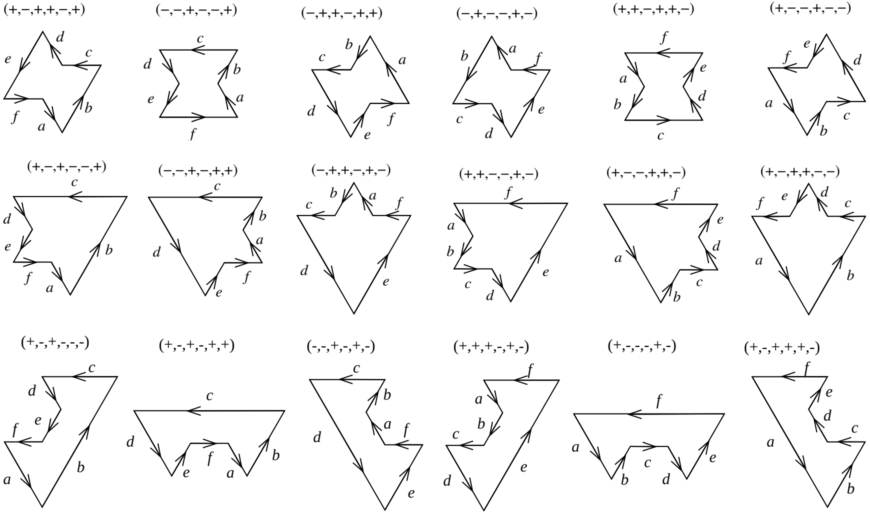


Theorem [Lai-M 2017]: For **most** toric cluster variables $z_{i,j,k}$, we let G be the subgraph cut out by the contour $(a, b, c, d, e, f) = (j + k, -i - j - k, i + k, j - k + 1, -i - j + k - 1, i - k + 1)$. Then the Laurent expansion of $z_{i,j,k}$ agrees with the partition function of weighted enumeration of dimers on G.

Examples: $(1, 2, 1) \to (3, -4, 2, 2, -3, 1)$, $(-2, -2, 3) \to (1, 1, 1, -4, 6, -4)$, and $(1, 2, 3) \to (5, -6, 4, 0, -1, -1)$



Possible Shapes of Aztec Castles



Sketch of Proof in [Lai-M 2017]

Our proof of this combinatorial interpretation for most toric cluster variables in the cluster algebra associated to the dP3 quiver relies on comparing cluster mutations of $z_{i,j,k}$'s to decomposing superpositions of dimers on graphs via Kuo's Graphical Condensation.

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Kuo Graphical Condensation Review: First used by Eric H. Kuo to (re)prove the Aztec diamond theorem by Elkies, Kuperberg, Larsen and Propp. Kuo condensation can be considered as a combinatorial interpretation of Dodgson condensation (or the Jacobi-Desnanot identity) on determinants of matrices.

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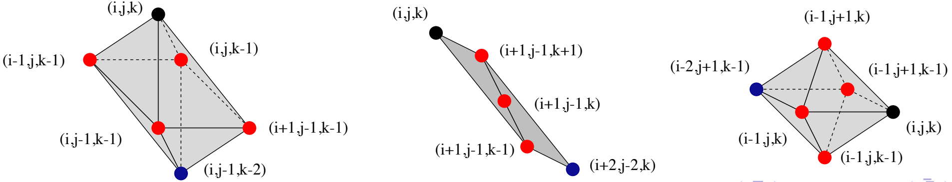

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$$\begin{aligned} z_{i,j-1,k-2}\ z_{i,j,k} &= z_{i+1,j-1,k-1}\ z_{i-1,j,k-1} + z_{i,j-1,k-1}\ z_{i,j,k-1} \\ z_{i+2,j-2,k}\ z_{i,j,k} &= z_{i+1,j-1,k+1}\ z_{i+1,j-1,k-1} + (z_{i+1,j-1,k})^2 \\ z_{i-2,j+1,k-1}\ z_{i,j,k} &= z_{i-1,j+1,k}\ z_{i-1,j,k-1} + z_{i-1,j,k}\ z_{i-1,j+1,k-1} \end{aligned}$$

Crash Course on Kuo Condensation

Let $G = (V_1, V_2, E)$ be a (weighted) planar bipartite graph and let p_1, p_2, p_3, p_4 be four vertices appearing in cyclic order on a face of G.

Theorem (Balanced Kuo Condensation) [Theorem 5.1 in [Kuo]]
Let $|V_1| = |V_2|$ with $p_1, p_3 \in V_1$ and $p_2, p_4 \in V_2$. Then

$$\begin{aligned} w(G)w(G - \{p_1, p_2, p_3, p_4\}) &= w(G - \{p_1, p_2\})w(G - \{p_3, p_4\}) \\ &+ w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}). \end{aligned}$$

Theorem (Non-alternating Balanced) [Theorem 5.2 in [Kuo]]
Let $|V_1| = |V_2|$ with $p_1, p_2 \in V_1$ and $p_3, p_4 \in V_2$. Then

$$\begin{aligned} w(G)w(G - \{p_1, p_2, p_3, p_4\}) &= w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}) \\ &- w(G - \{p_1, p_3\})w(G - \{p_2, p_4\}). \end{aligned}$$

Let $G = (V_1, V_2, E)$ be a (weighted) planar bipartite graph and let p_1, p_2, p_3, p_4 be four vertices appearing in cyclic order on a face of G.

Theorem (Unbalanced Kuo Condensation) [Theorem 5.3 in [Kuo]]
Let $|V_1| = |V_2| + 1$ with $p_1, p_2, p_3 \in V_1$ and $p_4 \in V_2$. Then

$$\begin{aligned} w(G - \{p_2\})w(G - \{p_1, p_3, p_4\}) &= w(G - \{p_1\})w(G - \{p_2, p_3, p_4\}) \\ &+ w(G - \{p_3\})w(G - \{p_1, p_2, p_4\}). \end{aligned}$$

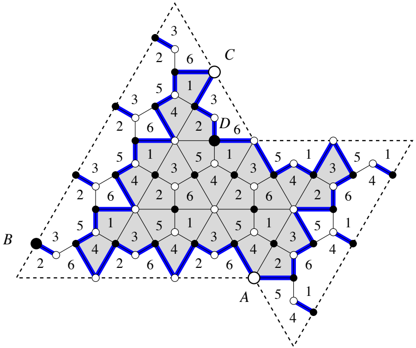
Theorem (Monochromatic Condensation) [Theorem 5.4 in [Kuo]] Let $|V_1| = |V_2| + 2$ with $p_1, p_2, p_3, p_4 \in V_1$. Then

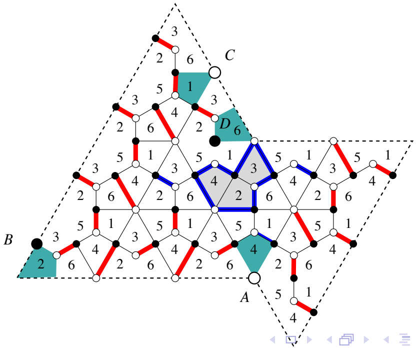
$$\begin{aligned} w(G - \{p_1, p_3\})w(G - \{p_2, p_4\}) &= w(G - \{p_1, p_2\})w(G - \{p_3, p_4\}) \\ &+ w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}). \end{aligned}$$

Example of Kuo Balanced Graphical Condensation

Let $G = (V_1, V_2, E)$ be a (weighted) planar bipartite graph and let p_1, p_2, p_3, p_4 be four vertices appearing in cyclic order on a face of G. Assume $|V_1| = |V_2|$ with $p_1, p_3 \in V_1$ and $p_2, p_4 \in V_2$.

$$\mathbf{w(G) \cdot w(G - \{A, B, C, D\})} = w(G - \{A, B\}) \cdot w(G - \{C, D\}) + w(G - \{A, D\}) \cdot w(G - \{B, C\})$$





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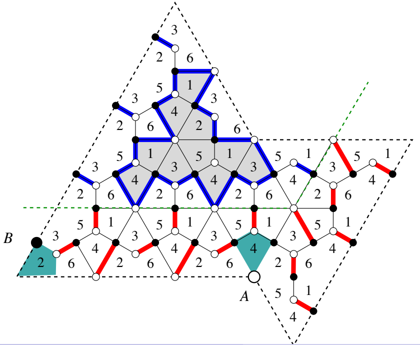

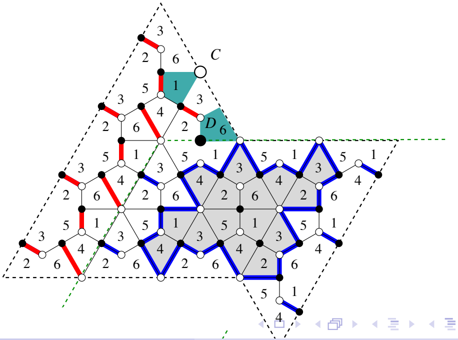

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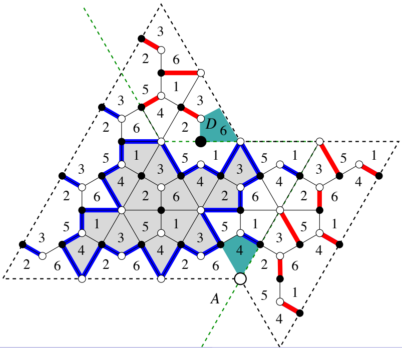
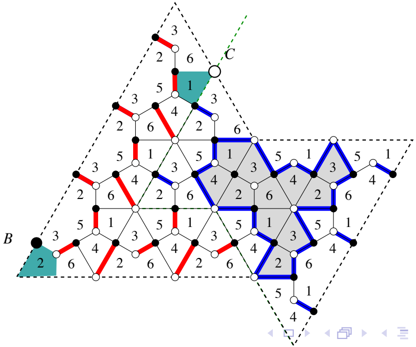
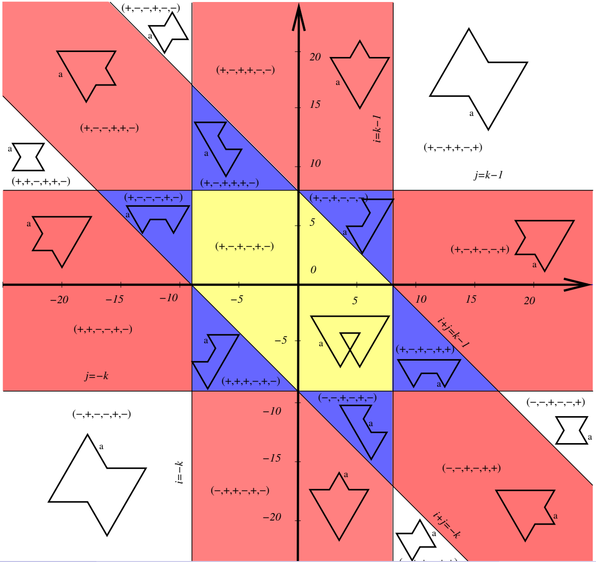
$$w(G)\cdot w(G-\{A,B,C,D\}) = w(G-\{A,B\})\cdot w(G-\{C,D\}) + \mathbf{w(G-\{A,D\})\cdot w(G-\{B,C\})}$$

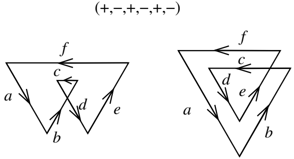




Cross-section when k is positive



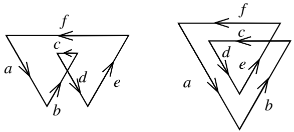
Self-intersecting Contours



For (i, j, k) associated to a self-intersecting contour, our Algebraic formula still works:

$$z_{i,j,k} = x_r \;\; A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} \; B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} \; C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} \; D^{\lfloor \frac{(k-1)^2}{4} \rfloor} \; E^{\lfloor \frac{k^2}{4} \rfloor}$$

(+,−,+,−,+,−)



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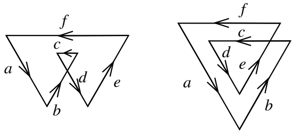
However, when the contour (a, b, c, d, e, f) alternates in sign, what is a combinatorial formula?

Self-intersecting Contours

(+,−,+,−,+,−)



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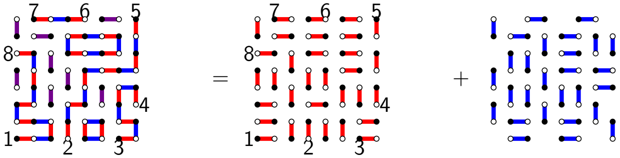
However, when the contour (a, b, c, d, e, f) alternates in sign, what is a combinatorial formula?

Speculation: Instead of dimer partition functions, what if we use double dimers instead?



Jenne Condensation

Theorem 1.0.2 in [Jenne]: Let G be a bipartite graph with $|V_1| = |V_2|$ with $p_1, p_3 \in V_1$ and $p_2, p_4 \in V_2$ as before.

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$$\begin{aligned} Z_{\sigma}^{DD}(G,N)Z_{\sigma_5}^{DD}(G,N-\{p_1,p_2,p_3,p_4\}) &= Z_{\sigma_1}^{DD}(G,N-\{p_1,p_2\})Z_{\sigma_2}^{DD}(G,N-\{p_3,p_4\}) \\ &+ Z_{\sigma_3}^{DD}(G,N-\{p_1,p_4\})Z_{\sigma_4}^{DD}(G,N-\{p_2,p_3\}) \end{aligned}$$

where $Z_{\sigma_i}^{DD}$ counts (*) the number of **double dimer configurations (with nodes)** $\mathcal{M}$ on (G, N)

Theorem 1.0.2 in [Jenne]: Let G be a bipartite graph with $|V_1| = |V_2|$ with $p_1, p_3 \in V_1$ and $p_2, p_4 \in V_2$ as before. Furthermore, choose a subset of nodes N **on the boundary of** graph G, and divide N into three circularly continguous sets R, G, and B. (We also assume that the sizes $|R|$, $|G|$ and $|B|$ satisfy the triangle inequality.) Then

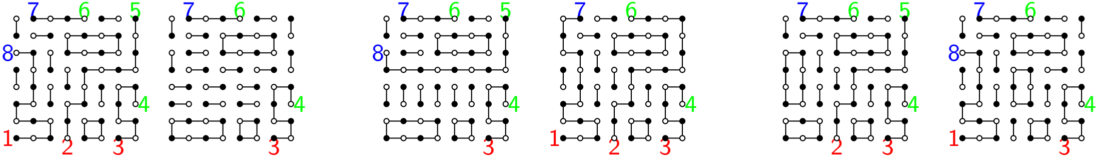
$$\begin{aligned} Z_{\sigma}^{DD}(G,N) Z_{\sigma_5}^{DD}(G, N - \{p_1,p_2,p_3,p_4\}) &= Z_{\sigma_1}^{DD}(G, N - \{p_1,p_2\}) Z_{\sigma_2}^{DD}(G, N - \{p_3,p_4\}) \\ &+ Z_{\sigma_3}^{DD}(G, N - \{p_1,p_4\}) Z_{\sigma_4}^{DD}(G, N - \{p_2,p_3\}) \end{aligned}$$

where $Z_{\sigma_i}^{DD}$ counts (*) the number of **double dimer configurations (with nodes)** $\mathcal{M}$ on (G,N) such that 1) every vertex in $G - N$ is incident to exactly two edges (or a doubled-edge) of $\mathcal{M}$, 2) every node in N is incident to exactly one edge of $\mathcal{M}$, and 3) each path from N to N included in $\mathcal{M}$ has endpoints of different colors.

Note (*) : When calculating $Z_{\sigma_i}^{DD}$, each contribution from $\mathcal{M}$ is multiplied by $2^{\#\text{ cycles in }\mathcal{M}}$.

Example:

$$Z^{DD}_{\sigma}(\mathbf{N})Z^{DD}_{\sigma_{1258}}(\mathbf{N}-1,2,5,8) = Z^{DD}_{\sigma_{12}}(\mathbf{N}-1,2)Z^{DD}_{\sigma_{58}}(\mathbf{N}-5,8) + Z^{DD}_{\sigma_{18}}(\mathbf{N}-1,8)Z^{DD}_{\sigma_{25}}(\mathbf{N}-2,5)$$



Remark: Because of the fact that N (and $N-S$ for each subset S) is divided into three circularly continguous sets, there is a unique non-monochromatic non-crossing pairing σ_i.

Remark: By attaching a leaf vertex to (and subsituting for) a node, it is possible to turn any of these factors $Z^{DD}_{\sigma}(G, N-S)$ into $Z^{DD}_{\sigma}(G-S, N-S)$ instead; i.e. we can turn a $1-valent$ node into a non-node by either 1) making it 2-valent or 2) making it 0-valent (deleting it).

Jenne Condensation Example

Consider the case of $(i, j, k) = (-1, -1, 3)$.

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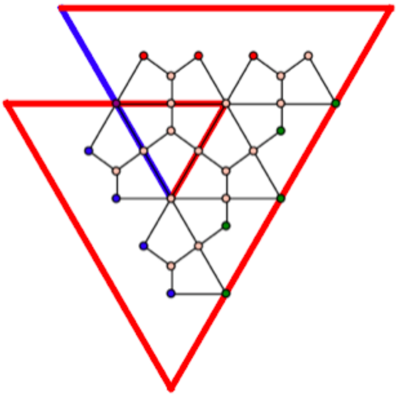

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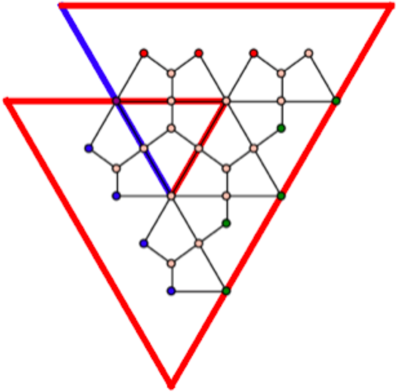


$(-1, -1, 3) \to (2, -1, 2, -3, 4, -3)$

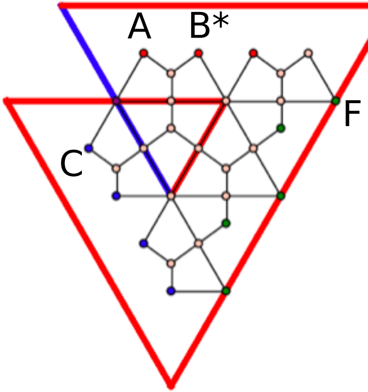
To prove this, we note that a certain octahedron in the $\mathbb{Z}^3$ lattice should correspond to a cluster mutation (see below).

Here, $z(-1, -2, 1)$, as well as the four cluster variables on the right-hand-side correspond to ordinary dimers.

$$z(-1, -1, 3) \cdot z(-1, -2, 1) = z(0, -2, 2) \cdot z(-2, -1, 2) + z(-1, -1, 2) \cdot z(-1, -2, 2)$$

It thus suffices to illustrate this recurrence as an example of Jenne Condensation.

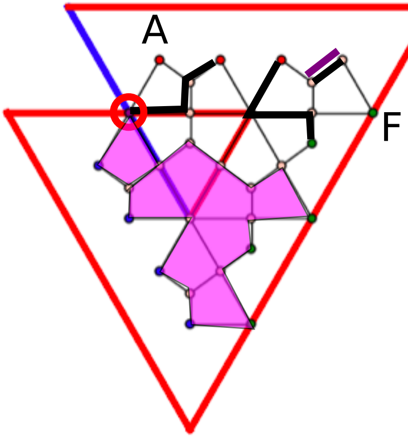
We label four of the nodes as A, B, C, and F as indicated. (Here B plays a special role that we will discuss shortly.) We begin by illustatrating the deletion of vertices $\{A, F\}$.

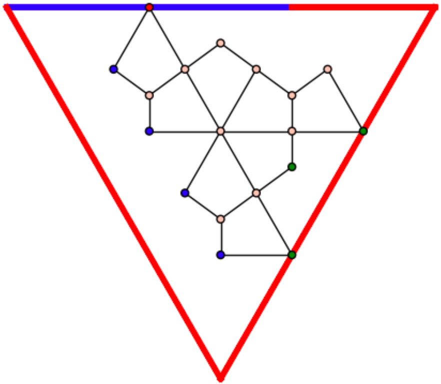


$$\mathbf{z(-1,-1,3)} \cdot z(-1,-2,1) = z(0,-2,2) \cdot z(-2,-1,2) + z(-1,-1,2) \cdot z(-1,-2,2)$$

Jenne Condensation Example

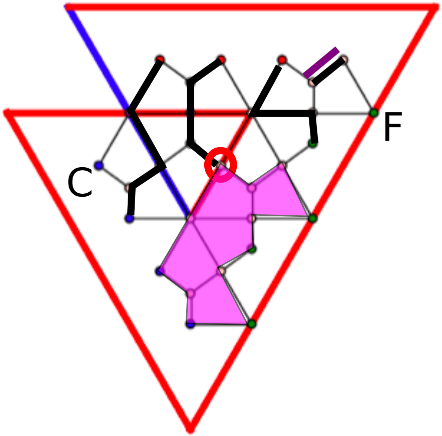
If we delete vertices A and F, we obtain $z(0,-2,2)$ as desired. $(0,-2,2) \to (0,0,2,-3,3,-1)$

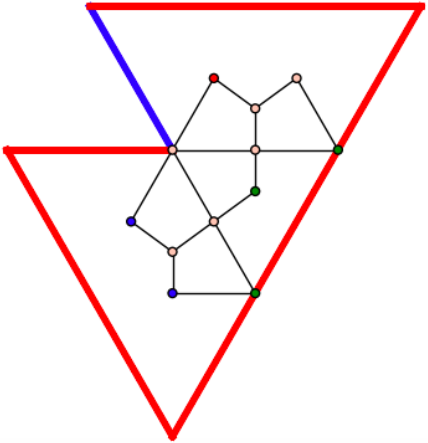




$$z(-1,-1,3)\cdot z(-1,-2,1) = \mathbf{z(0,-2,2)}\cdot z(-2,-1,2) + z(-1,-1,2)\cdot z(-1,-2,2)$$

If we instead delete vertices C and F, we get $z(-1,-1,2)$. $(-1,-1,2) \to (1,0,1,-2,3,-2)$





$$z(-1,-1,3)\cdot z(-1,-2,1) = z(0,-2,2)\cdot z(-2,-1,2) + \mathbf{z(-1,-1,2)}\cdot z(-1,-2,2)$$

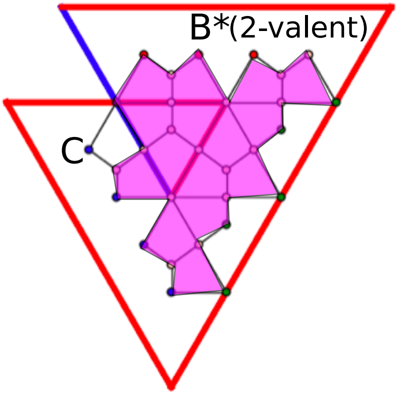
Jenne Condensation Example

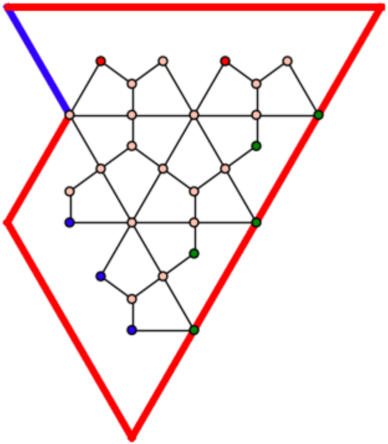
We next "delete" nodes B and C, but observe that while vertex C is deleted from the graph, we turn off node B (making it valence two) instead.

Jenne Condensation Example

We next "delete" nodes B and C, but observe that while vertex C is deleted from the graph, we turn off node B (making it valence two) instead. We then obtain $z(-2,-1,2)$ as desired.

$(-2,-1,2) \to (1,1,0,-2,4,-3)$



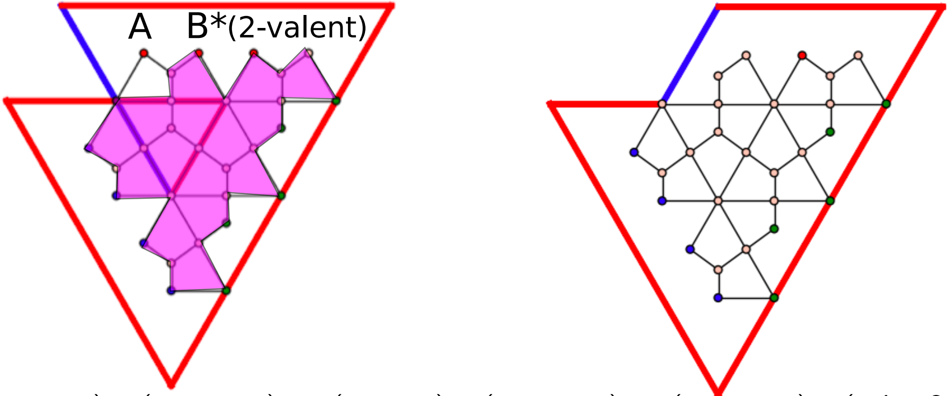


$$z(-1,-1,3)\cdot z(-1,-2,1) = z(0,-2,2)\cdot \mathbf{z(-2,-1,2)} + z(-1,-1,2)\cdot z(-1,-2,2)$$

Jenne Condensation Example

The case of "deleting" nodes A and B is analogous: vertex A is still deleted from the graph, but we turn off node B (making it valence two). We then obtain $z(-1,-2,2)$ as desired.

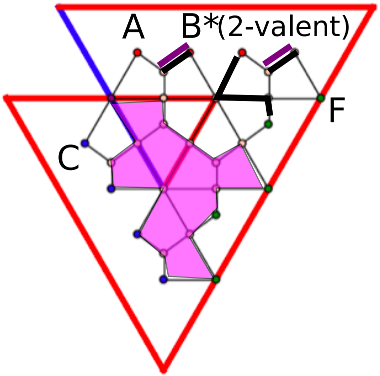
$(-1,-2,2) \to (0,1,1,-3,4,-2)$

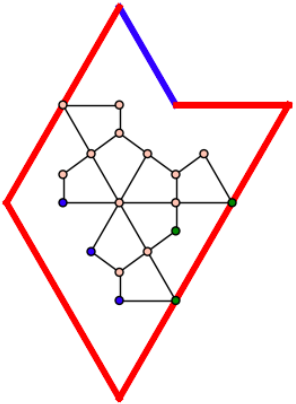


$$z(-1,-1,3)\cdot z(-1,-2,1) = z(0,-2,2)\cdot z(-2,-1,2) + z(-1,-1,2)\cdot \mathbf{z(-1,-2,2)}$$

Jenne Condensation Example

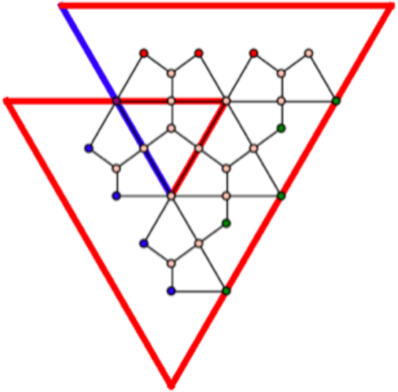
Finally, "deleting" nodes A, B, C, and F sees the deletion of vertices A, C, and F, while we turn off node B (making it valence two). We obtain the cluster variable $z(-1,-2,1)$.
$(-1,-2,1) \to (-1,2,0,-2,3,-1)$

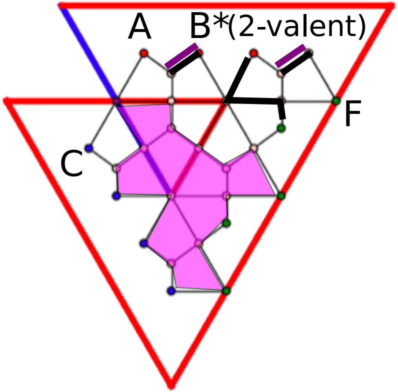




$$z(-1,-1,3)\cdot \mathbf{z(-1,-2,1)} = z(0,-2,2)\cdot z(-2,-1,2) + z(-1,-1,2)\cdot z(-1,-2,2)$$

Summary:





$$\mathbf{z}(-\mathbf{1},-\mathbf{1},\mathbf{3})\cdot\mathbf{z}(-\mathbf{1},-\mathbf{2},\mathbf{1}) = z(0,-2,2)\cdot z(-2,-1,2)+z(-1,-1,2)\cdot z(-1,-2,2)$$

$$54\ (G)\cdot 16\ (G-ABCF) = 12\ (G-AF)\cdot 48\ (G-BC)+6\ (G-CF)\cdot 48\ (G-AB)$$

Jenne Condensation Example

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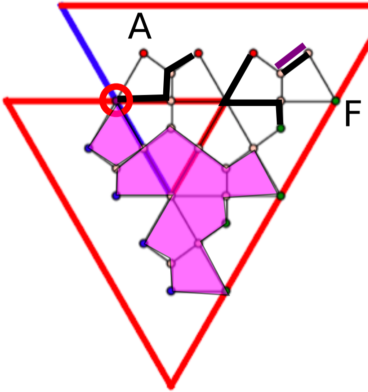

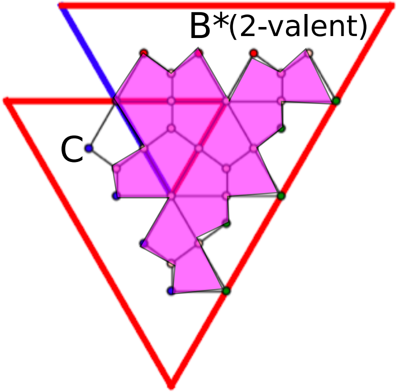

$$z(-1,-1,3)\cdot z(-1,-2,1) \;=\; \mathbf{z(0,-2,2)}\cdot \mathbf{z(-2,-1,2)} + z(-1,-1,2)\cdot z(-1,-2,2)$$

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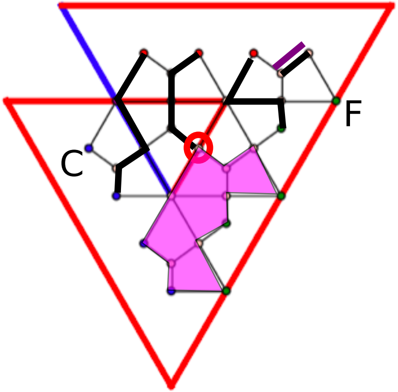

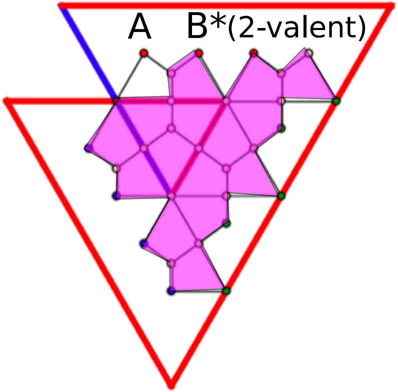

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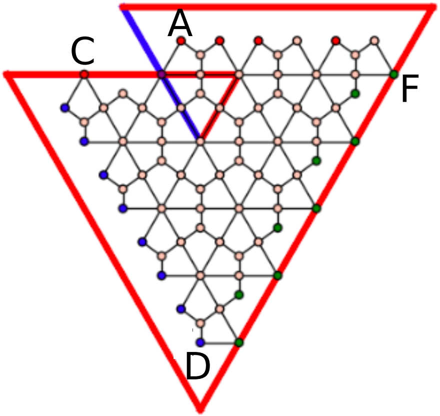
Jenne Condensation Example 2



$$\mathbf{z}(-1,-2,4)\cdot\mathbf{z}(0,-2,2) = \mathbf{z}(-1,2,3)\cdot\mathbf{z}(0,-2,3)+\mathbf{z}(-1,-1,3)\cdot\mathbf{z}(0,-3,3)$$

$$11664 \cdot 12 = 432 \cdot 108 + 54 \cdot 1728$$

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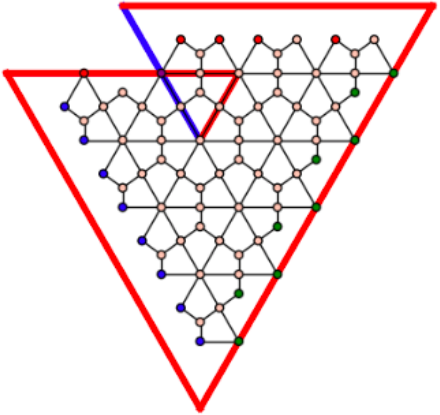

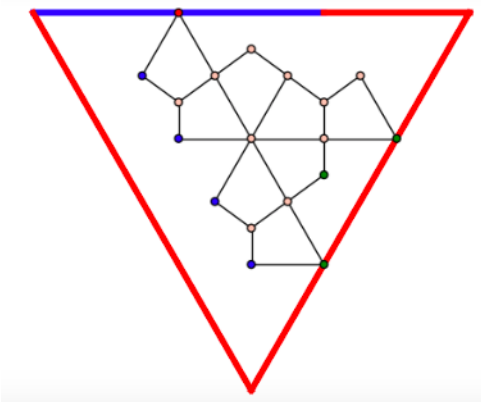

$$\mathbf{z}(-\mathbf{1},-\mathbf{2},\mathbf{4})\cdot\mathbf{z}(\mathbf{0},-\mathbf{2},\mathbf{2}) \;=\; z(-1,2,3)\cdot z(0,-2,3)+z(-1,-1,3)\cdot z(0,-3,3)$$

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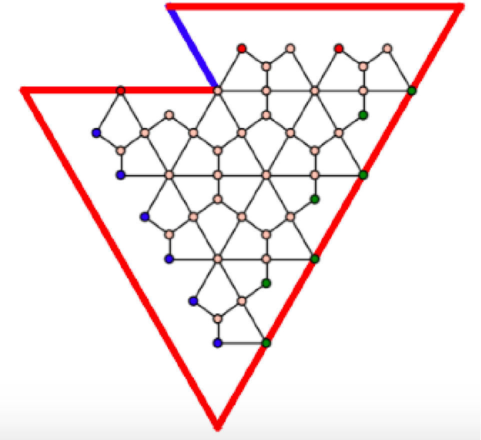

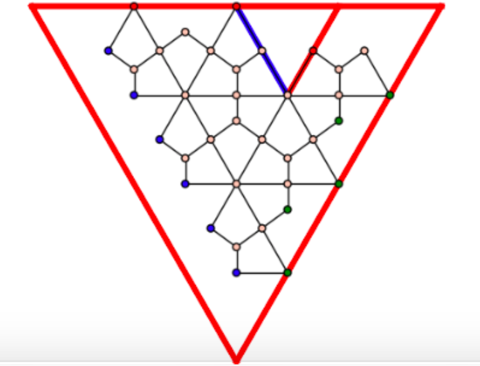

$$z(-1,-2,4)\cdot z(0,-2,2) \;=\; \mathbf{z}(-\mathbf{1},\mathbf{2},\mathbf{3})\cdot \mathbf{z}(\mathbf{0},-\mathbf{2},\mathbf{3}) + z(-1,-1,3)\cdot z(0,-3,3)$$

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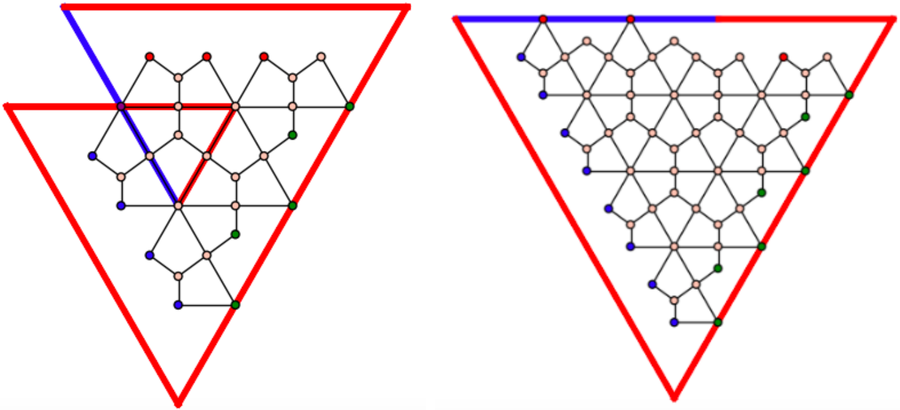

$$z(-1,-2,4)\cdot z(0,-2,2) \quad = \quad z(-1,2,3)\cdot z(0,-2,3)+\mathbf{z(-1,-1,3)}\cdot\mathbf{z(0,-3,3)}$$

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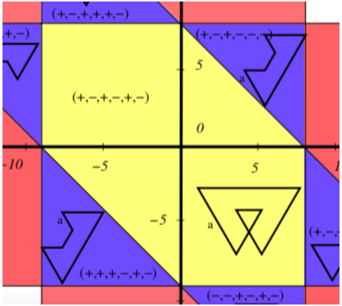
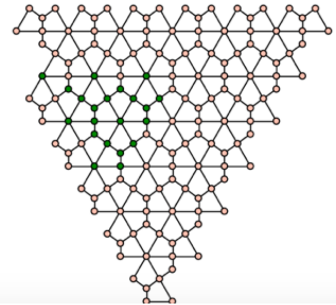
Based on conversations with David Speyer, it was conjectured that cluster variables corresponding to the lattice points corresponding to self-intersecting contours (which form a light cone or hour-glass as k varies) would correspond to **mixed dimers**.





Sketch of Proof for Self-intersecting Contours

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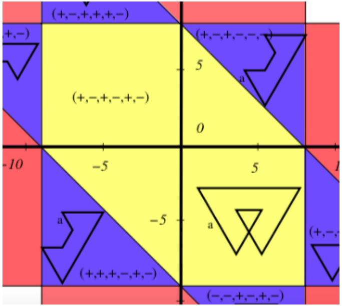
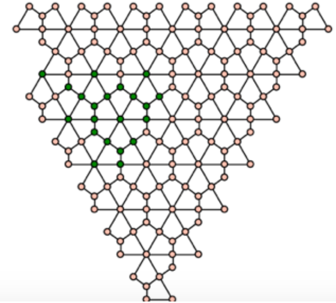
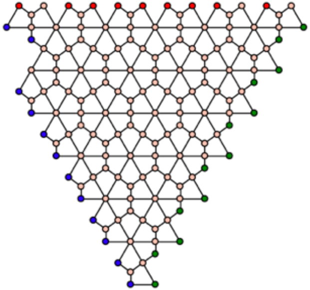


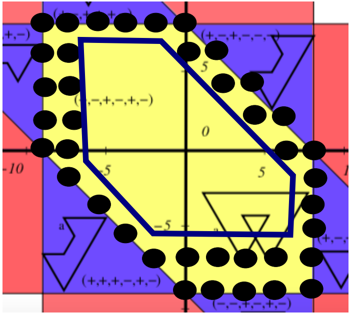


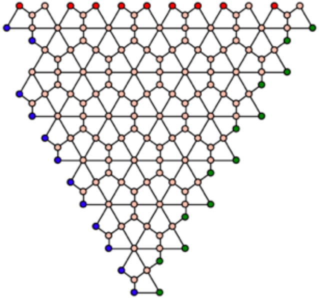


However, our work instead proves a combinatorial interpretation as **double dimers with nodes on the boundary**. (Example of $(-2,-1,6) \rightarrow (5,-3,4,-6,8,-7)$ shown.)

Sketch of Proof for Self-intersecting Contours

Lemma [Jenne-Lai-M 2020+]: For fixed k, the toric cluster variables $z_{i,j,k}$ for (i,j,k) on the rim of the hexagonal region have weighted enumeration formulas **simultaneously** in terms of dimers and double dimers with nodes on the boundary. (We show an explicit bijection.)





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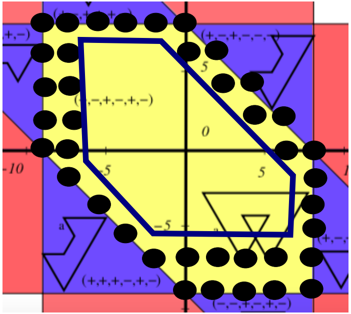
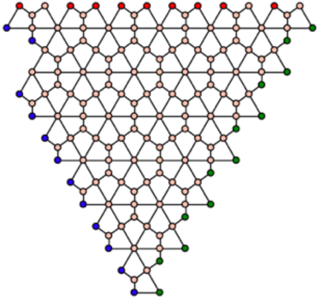




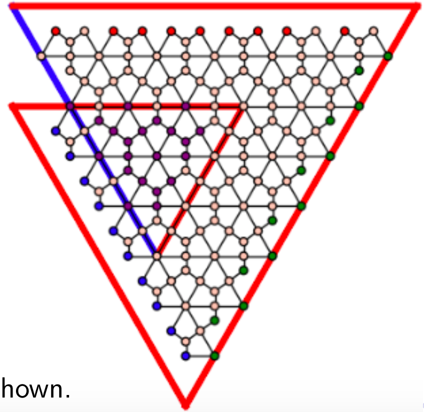
To obtain the double dimer interpretations (with nodes) for the **remaining self-intersecting contours** in the center of this hexagonal region, we use the dimer interpretations of [Lai-M 2017] as a base case, and then proceed by induction via Jenne condensation.

Double Dimer Interpretations with Nodes

Theorem In Progress [Jenne-Lai-M 2020+]: For the case of the dP3 Quiver (of Model I), we complete the assignment of **combinatorial** interpretations to **all** toric cluster variables.

Double Dimer Interpretations with Nodes

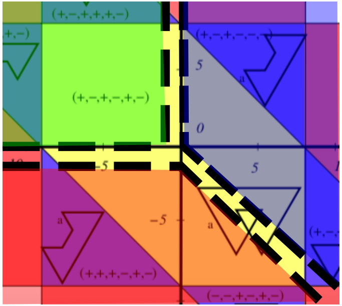
Theorem In Progress [Jenne-Lai-M 2020+]: For the case of the dP3 Quiver (of Model I), we complete the assignment of **combinatorial** interpretations to **all** toric cluster variables. In particular, for those parametrized by lattice points (i, j, k) associated to self-intersecting contours, we express Laurent expansions of such cluster variables as weighted enumeration of **double dimers with nodes on the boundary**.

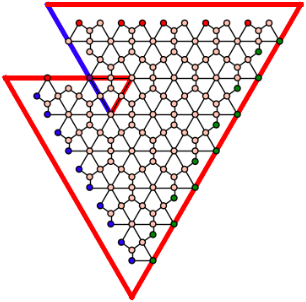


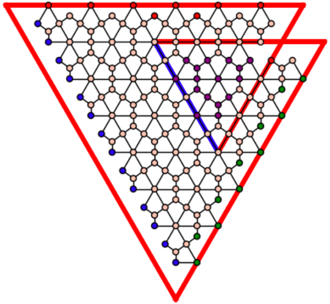
Example of $(-2, -1, 6) \rightarrow (5, -3, 4, -6, 8, -7)$ shown.

Double Dimer Configurations More Precisely

Theorem In Progress [Jenne-Lai-M 2020+]: For a fixed value of $k \geq 1$, we split up the hexagon of lattice points corresponding to self-intersecting contours into three rhombi; cut-out by the lines ($y = -1$ and $y = -x - 1$), ($y = -x$ and $x = 0$), as well as ($x = -1$ and $y = 0$).







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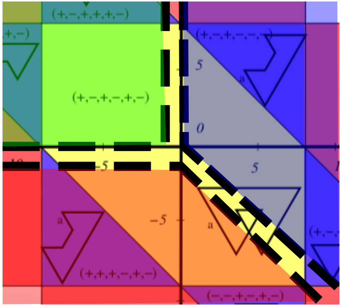

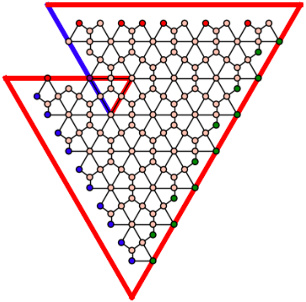

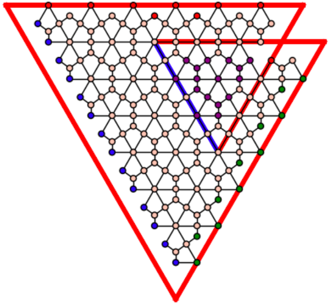

For the SW rhombic region, the blue and green nodes satisfy a regular pattern of being all boundary vertices of degree 2 along edges d and e, respectively.

Double Dimer Configurations More Precisely

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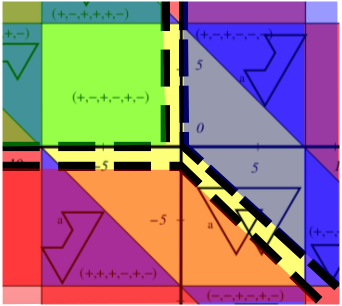
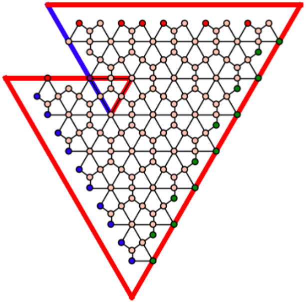
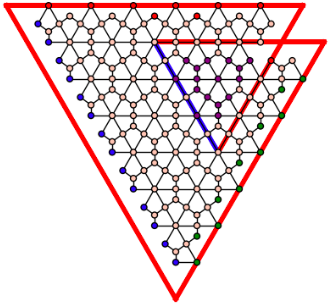


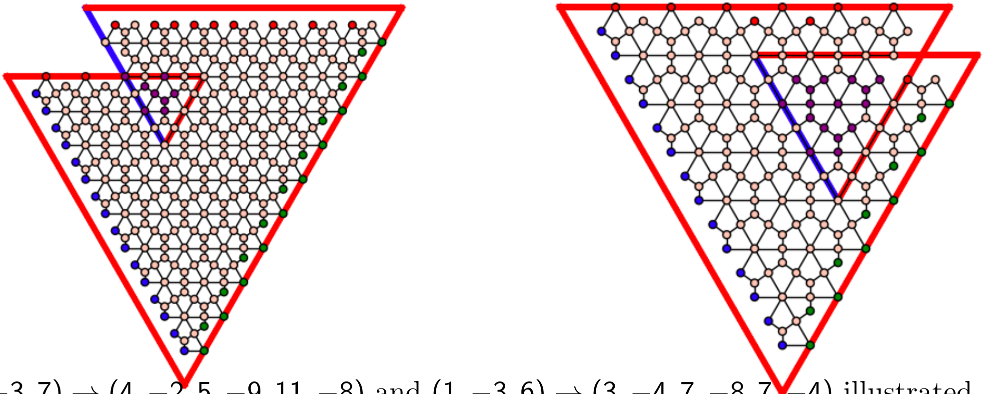




For the SW rhombic region, the blue and green nodes satisfy a regular pattern of being all boundary vertices of degree 2 along edges d and e, respectively. The red nodes are placed by a more complicated (semi)-regular pattern along edges c and f.

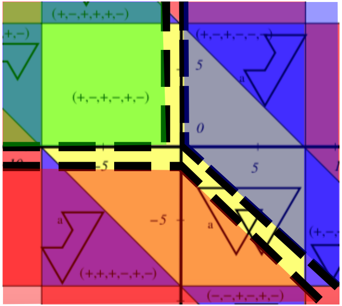
Double Dimer Configurations More Precisely

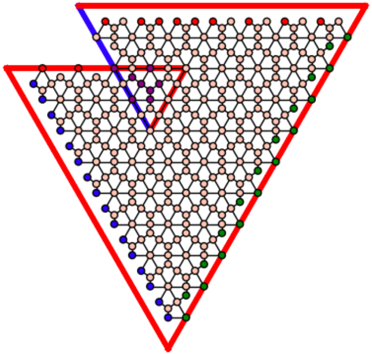
For a fixed value of $k \geq 1$, and (i, j, k) in the SW rhombic region, i.e. $j \leq -1$ and $i + j \leq -1$, we place red nodes in the folllowing (semi)-regular pattern along edges c and f: For $i < 0$, the leftmost $-i$ red nodes of **side f** are as usual, followed by $(k + i + j)$ extra red nodes. For $i \geq 0$, the rightmost $(i + 1)$ red nodes of **side c** are as usual, and then $(j + k - 1)$ extra red nodes.

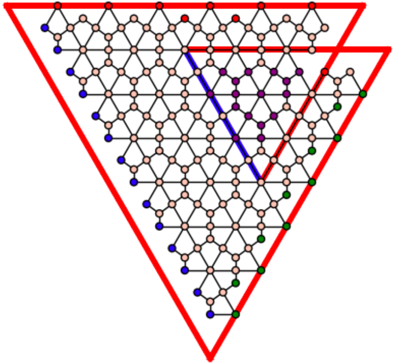


$(-2, -3, 7) \rightarrow (4, -2, 5, -9, 11, -8)$ and $(1, -3, 6) \rightarrow (3, -4, 7, -8, 7, -4)$ illustrated, resp.

Double Dimer Configurations More Precisely



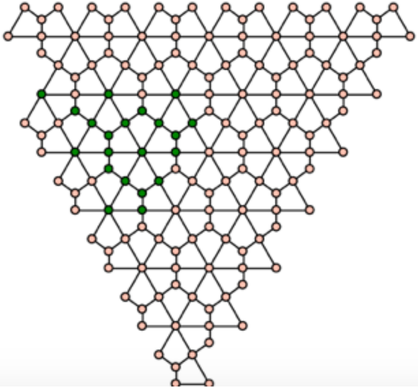


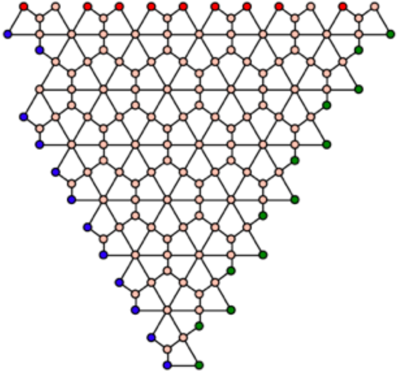


For the NE and NW rhombic regions, we rotate the graphs 120° or 240° degrees and rotate our node coloring rules accordingly. The cases of $k \leq 0$ are similarly reflections of the above.

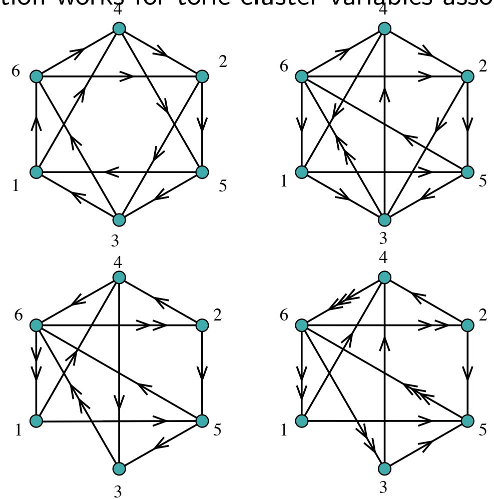
Further Work in Progess

Conjecture: There exist (weighted) bijections that map our double dimer configurations, which have nodes only on the boundary, to mixed dimer configurations where there is a internal region where every vertex has valence two, and the remaining region has vertices of valence one.

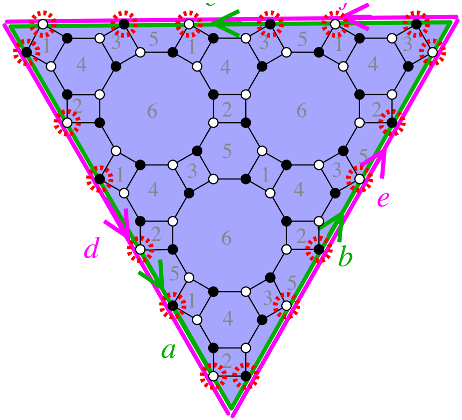


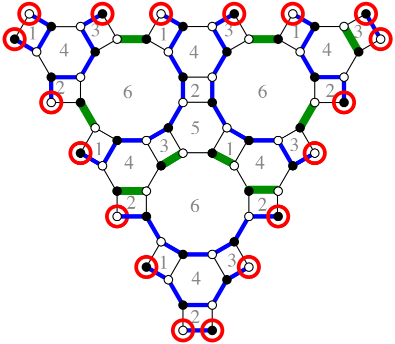


Conjecture: For the cases of the dP3 Quivers (of Model II, III, and IV), a similar double dimer with node interpretation works for toric cluster variables associated to self-intersecting contours.



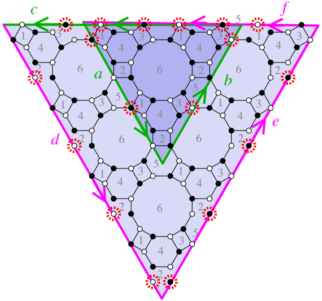
Conjecture: For the cases of the dP3 Quivers (of Model II, III, and IV), a similar double dimer with node interpretation works for toric cluster variables associated to self-intersecting contours. (Model IV Example from [Lai-M 2020]; A_5 of [Kenyon-Pemantle 2012])

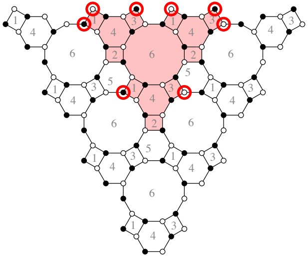




Further Work in Progess

In particular, we have successfully recast some examples of mixed dimer interpretations described in Section 8 of [Lai-M 2020] as double dimer (with boundary nodes) interpretations instead. (Model IV Example from [Lai-M 2020]; B_4 of [Kenyon-Pemantle 2012])





Further Work in Progess

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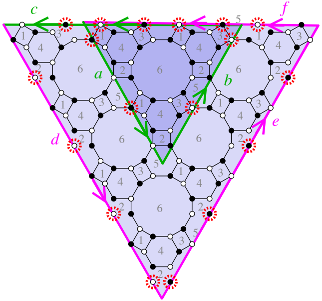

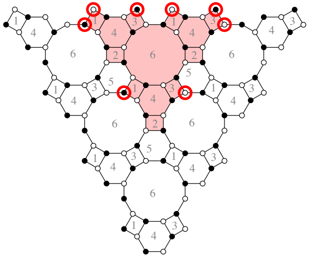

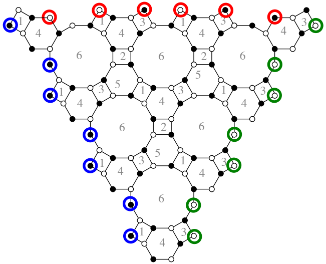

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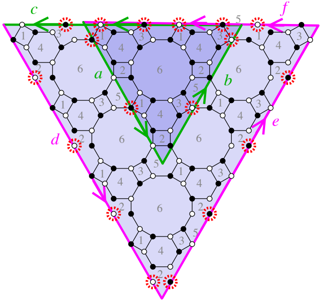

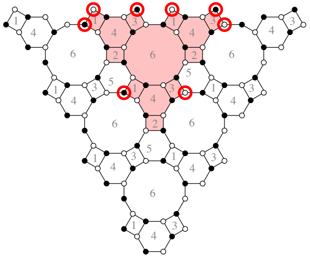

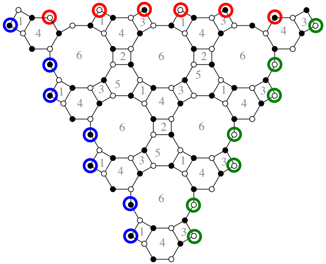

Conjecture: Other cluster algebras arising from Newton polygons with six sides also have toric cluster variables with combinatorial interpretations in terms of double dimers with boundary nodes.

Thanks for Listening http://www-users.math.umn.edu/∼musiker/OCAS20.pdf

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