Cluster Algebras and Brane Tilings

Gregg Musiker (University of Minnesota)

University of Connecticut Colloquium

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http://math.umn.edu/∼musiker/UCONN16.pdf

Outline.

1. Introduction to Cluster Algebras
2. What is a Brane Tiling
3. The Del Pezzo 3 Quiver and Lattice
4. Gale-Robinson Sequences (work of Jeong-M-Zhang)
5. Aztec Castles and Beyond (work of Leoni-Neel-Turner and Lai-M)

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Part of this work done during 2011-2013 REU in Combinatorics at University of Minnesota, Twin Cities.

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Cluster algebras are a certain class of commutative rings which have a distinguished set of generators that are grouped into overlapping subsets, called clusters, each having the same cardinality.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

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The set of all such generators are known as Cluster Variables, and the initial pattern of exchange relations (described as a *valued* quiver, i.e. a directed graph) determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Example: Rank 2 Cluster Algebras

Let $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$, $b, c \in \mathbb{Z}_{>0}$. $(\{x_1, x_2\}, B)$ is a seed for a cluster algebra $\mathcal{A}(b, c)$ of rank 2.

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$$\mu_1(B) = \mu_2(B) = -B \text{ and } x_1 x_1' = x_2^c + 1, \quad x_2 x_2' = 1 + x_1^b.$$

Thus the cluster variables in this case are

$$\{x_n : n \in \mathbb{Z}\} \text{ satisfying } x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 \text{ if } n \text{ is odd} \\ x_{n-1}^c + 1 \text{ if } n \text{ is even} \end{cases}.$$

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If we let $x_1 = x_2 = 1$, we obtain $\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}$.

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What is a Brane Tiling (in Physics & Algebraic Geometry)

In physics, **Brane Tilings** are combinatorial models that are used to

Decribe the world volume of both D_3 and M_2 branes, and describe certain $(3+1)$-dimensional superconformal field theories arising in string theory (Type II B).

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In Algebraic Geometry, they are used to

Probe certain toric Calabi-Yau singularities, and relate to non-commutative crepant resolutions and the 3-dimensional McKay correspondence.

Certain examples of path algebras with relations (Jacobian Algebras) can be constructed by a quiver and potential coming from a brane tiling.

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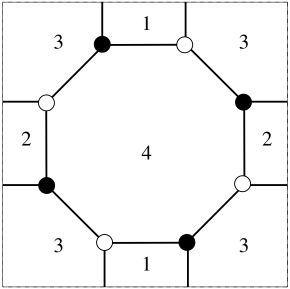
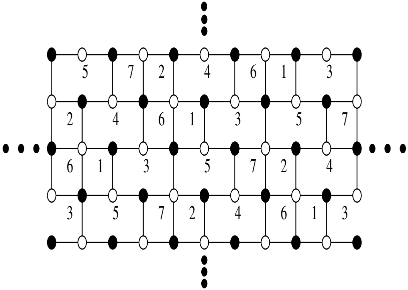
We view such a tiling as a doubly-periodic tiling of its universal cover, the Euclidean plane.

Examples: 

, 

Brane Tilings from a Quiver Q with Potential W

A **Brane Tiling** can be associated to a pair (Q, W), where Q is a quiver and W is a potential (called a superpotential in the physics literature).

A quiver Q is a directed graph where each edge is referred to as an arrow, and multiple edges are allowed.

A potential W is a linear combination of cyclic paths in Q (possibly an infinite linear combination).

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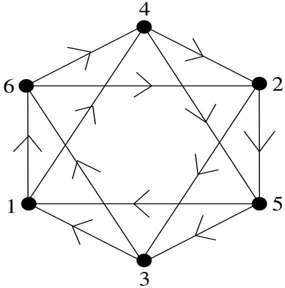
For combinatorial purposes, we assume other conditions on (Q, W), such as

• Each arrow of Q appears in one term of W with a positive sign, and one term with a negative sign.

• The number of terms of W with a positive sign equals the number with a negative sign. All coefficients in W are ± 1.

Brane Tilings from a Quiver Q with Potential W

Example (The dP_3 Quiver): $Q_{dP_3} = Q =$  ,

$$W = A_{16}A_{64}A_{42}A_{25}A_{53}A_{31} + A_{14}A_{45}A_{51} + A_{23}A_{36}A_{62}$$
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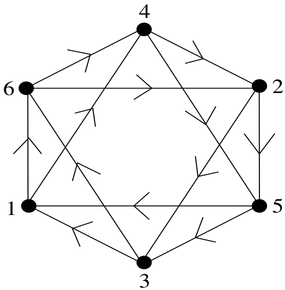
Example (The dP_3 Quiver): $Q_{dP_3} = Q =$ 

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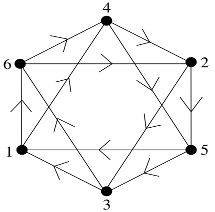
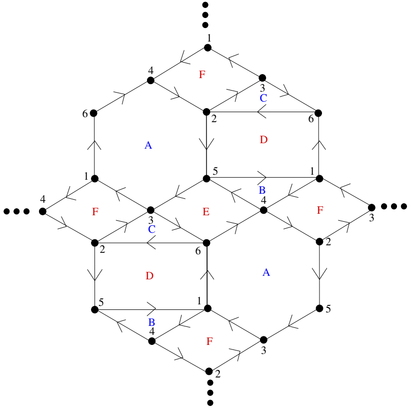
We now unfold Q onto the plane, letting the three positive (resp. negative) terms in W depict clockwise (resp. counter-clockwise) cycles on $\widetilde{Q}$.

Example (continued):

$Q =$



unfolds to $\widetilde{Q} =$

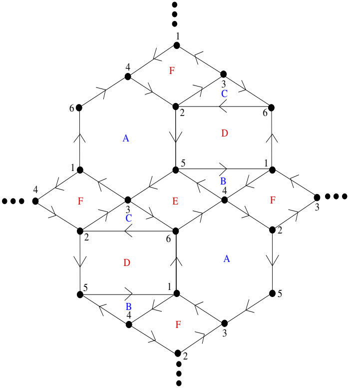


$$W \;=\; A_{16}A_{64}A_{42}A_{25}A_{53}A_{31}(A) + A_{14}A_{45}A_{51}\ (B) \ + \ A_{23}A_{36}A_{62}\ (C)$$
$$-\ A_{16}A_{62}A_{25}A_{51}\ (D) \ - \ A_{36}A_{64}A_{45}A_{53}\ (E) \ - \ A_{14}A_{42}A_{23}A_{31}(F).$$

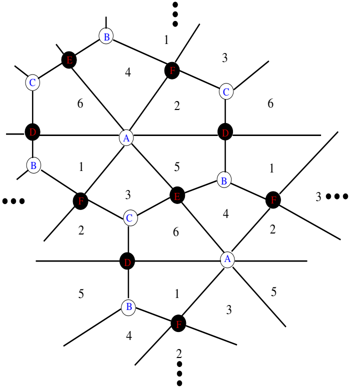
Locally, the configurations around vertices of Q and $\widetilde{Q}$ are identical.

Brane Tilings from a Quiver Q with Potential W

Taking the planar dual yields a bipartite graph on a torus (**Brane Tiling**):



$$\widetilde{Q} \longrightarrow \mathcal{T}_Q =$$



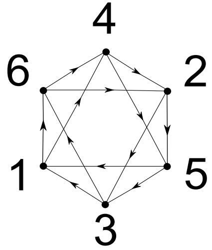
Negative Term in $W \longleftrightarrow$ Counter-Clockwise cycle in $\widetilde{Q} \longleftrightarrow \bullet$ in $\mathcal{T}_Q$

Positive Term in $W \longleftrightarrow$ Clockwise cycle in $\widetilde{Q}$ $\longleftrightarrow \circ$ in $\mathcal{T}_Q$

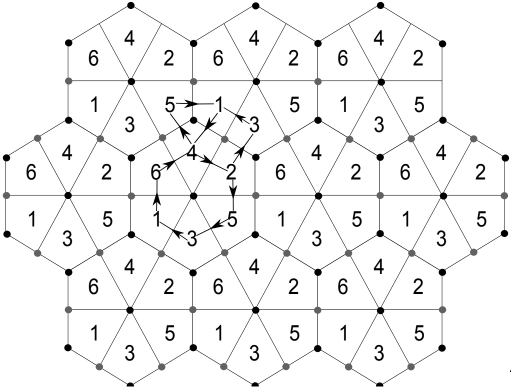
(To obtain $\widetilde{Q}$ from $\mathcal{T}_Q$, we dualize edges so that white is on the right.)

Brane Tilings from a Quiver Q with Potential W

Summarizing the dP_3 Example:



Q



$\mathcal{T}_Q$

Negative Term in $W \longleftrightarrow$ Counter-Clockwise cycle in $\widetilde{Q} \longleftrightarrow \bullet$ in $\mathcal{T}_Q$

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(To obtain $\widetilde{Q}$ from $\mathcal{T}_Q$, we dualize edges so that white is on the right.)

Brane Tilings in Physics

Face $\longleftrightarrow$ $U(N)$ Gauge Group

Edge $\longleftrightarrow$ Bifundamental Chiral Fields (Representations)

Vertex $\longleftrightarrow$ Gauge-invariant operator (Term in the Superpotential)

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Together, this data yields a **quiver gauge theory**. One can apply Seiberg duality to get a different quiver gauge theory.

Combinatorial connection:
Seiberg duality corresponds to mutation in cluster algebra theory.

Description of Seiberg Duality (from physics)

From **"Brane Dimers and Quiver Gauges Theories (2005)** by Franco, Hanany, Kennaway, Wegh, and Wecht:

After picking a node to dualize at: "Reverse the direction of all arrows entering or exiting the dualized node. This is because Seiberg duality requires that the dual quarks transform in the conjugate flavor representations to the originals. ...

Next, draw in ... bifundamentals which correspond to composite (mesonic) operators. ... the Seiberg mesons are promoted to the fields in the bifundamental representation of the gauge group. ...

It is possible that this will make some fields massive, in which case the appropriate fields should then be integrated out."

Description of Seiberg Duality (rephrased combinatorially)

Pick a vertex j of the quiver Q (equiv. face of the brane tiling $\mathcal{T}_Q$) at which to mutate. Then, reverse the direction of all arrows incident to j, i.e. $A_{ij} \to A^*_{ji}$. Next, for every two-path $i \to j \to k$, *"meson"*, in Q draw in a new arrow $i \to k$, *"the Seiberg mesons are promoted to the fields"*. Let Q' denote this new quiver.

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We similarly alter the superpotential W to get W'. For every 2-path $i \to j \to k$ in Q, we replace any appearance of the product $A_{ij}A_{jk}$ in W with the singleton A_{ik}, and add or subtract a new degree 3-term, $A_{ik}A^*_{kj}A^*_{ji}$.

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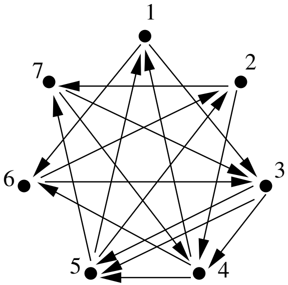
This is in fact Mutation of Quivers with potential from cluster algebras (as defined by Derksen-Weyman-Zelevinsky)!

In the special case, that we are mutating at a vertex with two arrows in and out, a **toric vertex**, this corresponds to a Urban Renewal of a square face in the brane tiling.

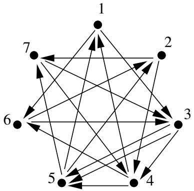


Example ($Q_7^{(2,3)}$): with potential

$$W = A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} + A_{35}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} - A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}A_{51} - A_{27}A_{73}A_{35}A_{52} - A_{45}A_{57}A_{74}.$$

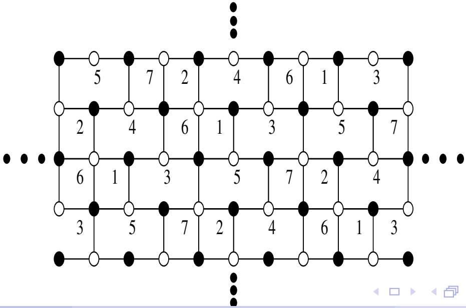
Consider the corresponding Brane Tiling $\mathcal{T}_7^{(2,3)}$ and mutation of (Q, W) at the toric vertex labeled 1. **(Associated to Gale-Robinson Sequence)**

Description of Seiberg Duality (on the Brane Tiling)

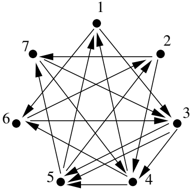
Example ($Q_7^{(2,3)}$): 

with potential

$$\begin{aligned} W &= A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}^{(V)}A_{51} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}^{(H)}A_{51} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}. \end{aligned}$$

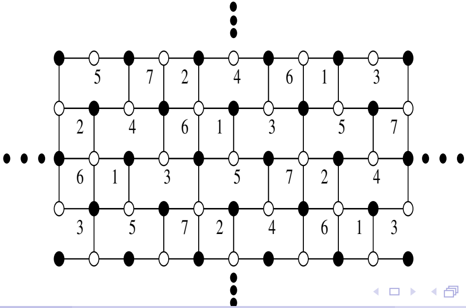


Description of Seiberg Duality (on the Brane Tiling)

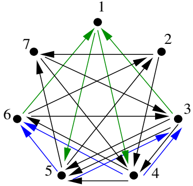


Example ($Q_7^{(2,3)}$): Rotate potential terms containing 1

$$W = A_{41}A_{13}A_{34} + A_{51}A_{16}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62}$$
$$- A_{41}A_{16}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{51}A_{13}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}.$$

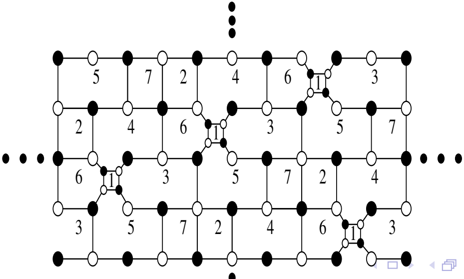


Example ($Q_7^{(2,3)}$):

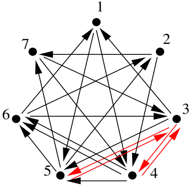


Mutating at 1 yields

$$
\begin{aligned}
W' \;=\;& A_{43}A_{34} + A_{56}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\
-\;& A_{46}^{(D)}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{53}^{(H)}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74} \\
+\;& A_{14}^{*}A_{46}^{(D)}A_{61}^{*} + A_{15}^{*}A_{53}^{(H)}A_{31}^{*} - A_{14}^{*}A_{43}A_{31}^{*} - A_{15}^{*}A_{56}A_{61}^{*}.
\end{aligned}
$$

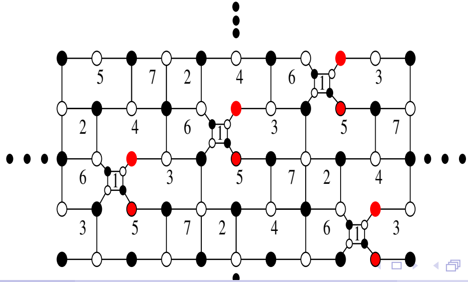


Example ($Q_7^{(2,3)}$):

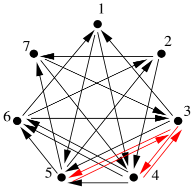


Highlighting Massive terms

$$\begin{aligned} W' &= A_{43}A_{34} + A_{56}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A_{46}^{(D)}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{53}^{(H)}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74} \\ &+ A_{14}^{*}A_{46}^{(D)}A_{61}^{*} + A_{15}^{*}A_{53}^{(H)}A_{31}^{*} - A_{14}^{*}A_{43}A_{31}^{*} - A_{15}^{*}A_{56}A_{61}^{*}. \end{aligned}$$

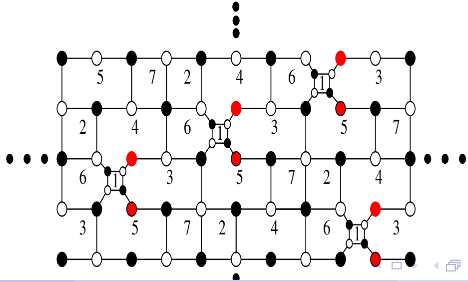


Description of Seiberg Duality (on the Brane Tiling)



Example ($Q_7^{(2,3)}$): Highlighting complementary terms

$$\begin{aligned} W' &= A_{43}A_{34} + A_{56}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A_{46}^{(D)}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{53}^{(H)}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74} \\ &+ A_{14}^{*}A_{46}^{(D)}A_{61}^{*} + A_{53}^{(H)}A_{31}^{*}A_{15}^{*} - A_{43}A_{31}^{*}A_{14}^{*} - A_{15}^{*}A_{56}A_{61}^{*}. \end{aligned}$$



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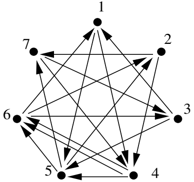
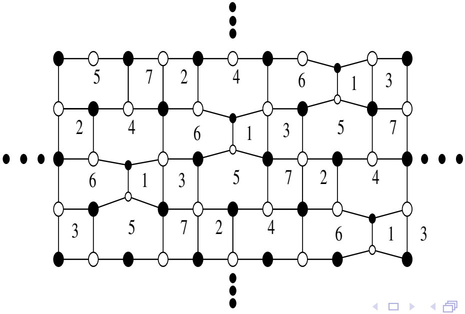
Example ($Q_7^{(2,3)}$):



Reduces the potential to

$$W'' = A_{56}A_{63}A_{35}^{(V)} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} - A_{46}^{(D)}A_{62}A_{24} - A_{27}A_{73}A_{35}^{(V)}A_{52}$$
$$- A_{45}A_{57}A_{74} + A_{14}^{*}A_{46}^{(D)}A_{61}^{*} - A_{15}^{*}A_{56}A_{61}^{*} - A_{46}A_{63}A_{31}^{*}A_{14}^{*} + A_{31}^{*}A_{15}^{*}A_{57}A_{73}.$$



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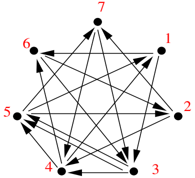
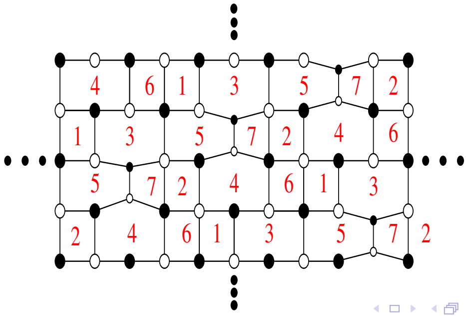
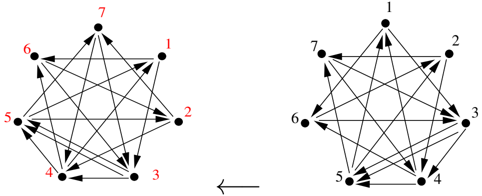


Example ($Q_7^{(2,3)}$): If we cyclically permute vertices

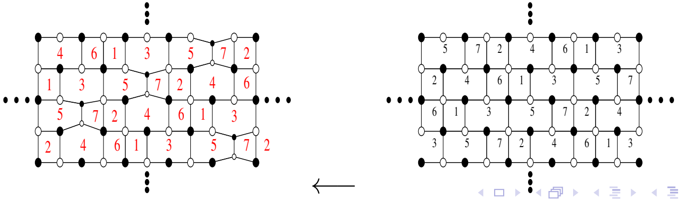
$$
\begin{aligned}
W'' \quad = \quad & A_{45}A_{52}A_{24}^{(V)} + A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} - A_{35}^{(D)}A_{51}A_{13} - A_{16}A_{62}A_{24}^{(V)}A_{41} \\
- \quad & A_{34}A_{46}A_{63} + A^*_{73}A_{35}^{(D)}A^*_{57} - A^*_{74}A_{45}A^*_{57} - A_{35}A_{52}A^*_{27}A^*_{73} + A^*_{27}A^*_{74}A_{46}A_{62}.
\end{aligned}
$$





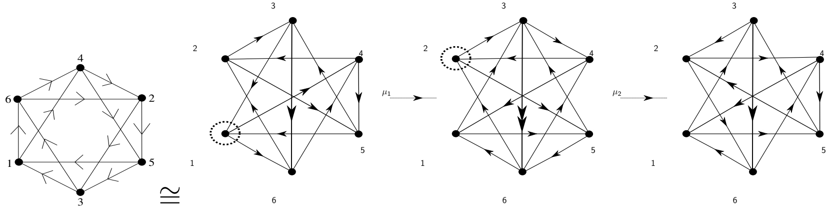
Example ($Q_7^{(2,3)}$):
The cyclic permutation yields the **original** Brane Tiling and (Q, W)!

$$\begin{aligned}
W'' &= A_{45}A_{52}A_{24}^{(V)} + A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} - A_{35}^{(D)}A_{51}A_{13} - A_{16}A_{62}A_{24}^{(V)}A_{41} \\
&- A_{34}A_{46}A_{63} + A_{73}^{*}A_{35}^{(D)}A_{57}^{*} - A_{74}^{*}A_{45}A_{57}^{*} - A_{35}A_{52}A_{27}^{*}A_{73}^{*} + A_{27}^{*}A_{74}^{*}A_{46}A_{62} \\
W &= A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}^{(V)}A_{51} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\
&- A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}^{(H)}A_{51} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}.
\end{aligned}$$



Enter Combinatorics

The quiver Q_{dP_3} has a similar **periodicity** property.



If we mutate Q_{dP_3} by $1, 2, 3, 4, 5, 6, 1, 2, \ldots$, after the first two mutations, we obtain same quiver back up to cyclically permuting the vertex labels.

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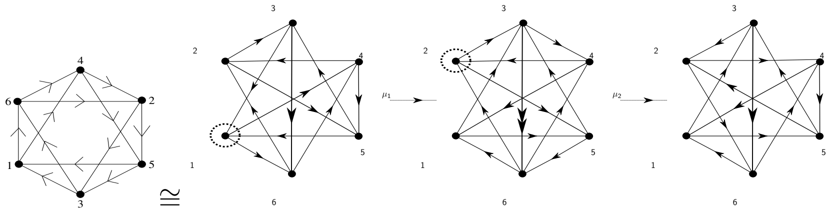

If we mutate Q_{dP_3} by $1, 2, 3, 4, 5, 6, 1, 2, \ldots,$ after the first two mutations, we obtain same quiver back up to cyclically permuting the vertex labels.

Point: Mutating once in the $Q_N^{(r,s)}$ case, or twice in the Q_{dP_3} case, yields a quiver with potential that is equivalent up to cyclic rotation.

Such quivers are called periodic in the Fordy-Marsh sense.

Cluster Variable Mutation

In addition to the mutation of quivers, there is also a complementary **cluster mutation** that can be defined.

Cluster mutation yields a sequence of Laurent polynomials in $\mathbb{Q}(x_1, x_2, \ldots, x_n)$ known as **cluster variables**.

Given a quiver Q (the potential is irrelevant here) and an initial cluster $\{x_1, \ldots, x_N\}$, then mutating at vertex 1 yields a new cluster variable x_{N+1}

$$\text{defined by} \qquad x_{N+1} = \left(\prod_{1 \to i \in Q} x_i + \prod_{i \to 1 \in Q} x_i \right) \Big/ x_1.$$

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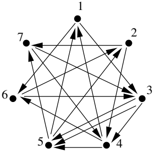
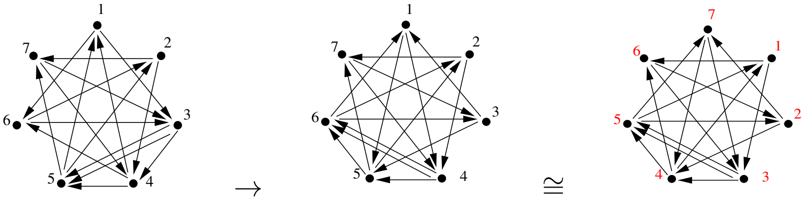
Example ($Q_N^{(r,s)}$): In Q, $1 \to r+1,\ N-r+1$ and $1 \leftarrow s+1,\ N-s+1$.



For $r = 2, s = 3, N = 7$, we get $x_8 = (x_3 x_6 + x_4 x_5) / x_1$.

Example ($Q_N^{(r,s)}$): (e.g. $r = 2,\ s = 3,\ N = 7$)



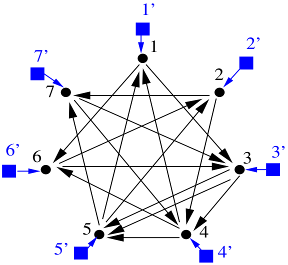
Mutating at $1, 2, 3, \ldots, N, 1, 2, \ldots$ yields the same quiver, up to cyclic permutation, at each step, hence we obtain the infinite sequence of $x_{N+1}, x_{N+2}, \ldots$ satsifying

$$x_n = \left(x_{n-r}x_{n-N+r} + x_{n-s}x_{n-N+s}\right)/x_{n-N} \quad \text{for} \quad n > N.$$

Known as the **Gale-Robinson Sequence** of Laurent polynomials.

The Gale-Robinson Sequence (with coefficients)

Example ($Q_N^{(r,s)}$): (e.g. $r = 2,\ s = 3,\ N = 7$)



We add N frozen vertices to $Q_N^{(r,s)}$ with incoming arrows. Let y_i denote the **cluster variable** corresponding to vertex i'.

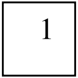
Mutating again at $1, 2, 3, \ldots, N, 1, 2, \ldots$ (never at frozen vertices) yields a infinite sequence of cluster variables with a more complicated recurrence:

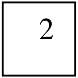
$$x_n x_{n-N} = x_{n-r} x_{n-N+r} + \prod_{i=1}^{n} y_i^{d(N-n-i,s,n-s)}\ x_{n-s} x_{n-N+s} \quad \text{for} \quad n > N.$$

where $d(M, s, s') = \#$ ways to write M as $A \cdot s + B \cdot s'$ with $A, B \in \mathbb{Z}_{\geq 0}$

Gale-Robinson Sequence Example

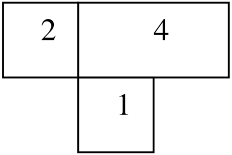
For $Q_7^{(2,3)}$, $x_8 = \frac{x_4x_5y_1 + x_3x_6}{x_1}$, $x_9 = \frac{x_5x_6y_2 + x_4x_7}{x_2}$, $x_{10} = \frac{x_1x_6x_7y_1y_3 + x_4x_5^2y_1 + x_3x_5x_6}{x_1x_3}$,
$x_{11} = \frac{x_2x_4x_5x_7y_1y_2y_4 + x_2x_3x_6x_7y_2y_4 + x_1x_5x_6^2y_2 + x_1x_4x_6x_7}{x_1x_2x_4}$, ...

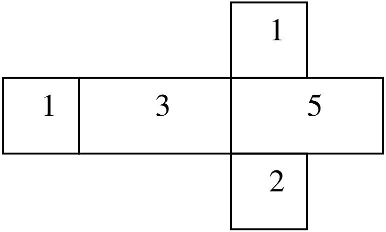
$x_8 \leftrightarrow$ 

, $x_9 \leftrightarrow$ 

, $x_{10} \leftrightarrow$ 

,

$x_{11} \leftrightarrow$ 

, $x_{12} \leftrightarrow$ 

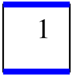
, ...

Gale-Robinson Sequence Example (continued)

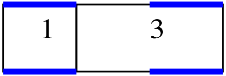
With **Minimal Matchings** Highlighted:

For $Q_7^{(2,3)}$, $x_8 = \frac{x_4x_5y_1 + x_3x_6}{x_1}$, $x_9 = \frac{x_5x_6y_2 + x_4x_7}{x_2}$, $x_{10} = \frac{x_1x_6x_7y_1y_3 + x_4x_5^2y_1 + x_3x_5x_6}{x_1x_3}$,

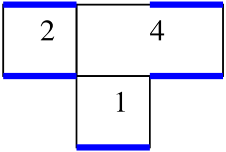
$x_{11} = \frac{x_2x_4x_5x_7y_1y_2y_4 + x_2x_3x_6x_7y_2y_4 + x_1x_5x_6^2y_2 + x_1x_4x_6x_7}{x_1x_2x_4}$, ...

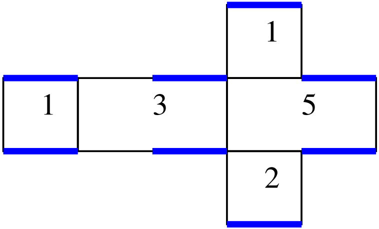
$x_8 \leftrightarrow$ 

, $x_9 \leftrightarrow$ 

, $x_{10} \leftrightarrow$ 

,

$x_{11} \leftrightarrow$ 

, $x_{12} \leftrightarrow$ 

, ...

Theorem (Jeong-M-Zhang) (FPSAC Proceedings 2013)

For certain periodic quivers Q, which include the **Gale-Robison** quiver family, the dP_3 quiver, and some other 2-periodic quivers, we can use the Brane Tiling $\mathcal{T}_Q$ to obtain combinatorial formulas for an infinite sequence of cluster variables in $\mathcal{A}_Q$.

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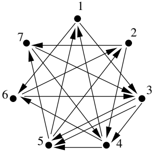
For certain periodic quivers Q, which include the **Gale-Robison** quiver family, the dP_3 quiver, and some other 2-periodic quivers, we can use the Brane Tiling $\mathcal{T}_Q$ to obtain combinatorial formulas for an infinite sequence of cluster variables in $\mathcal{A}_Q$.

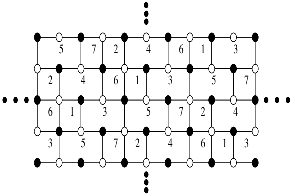
$$\text{For } n > N, \quad x_n = cm(G_n) \sum_{M = \text{ perfect matching of } G_n} x(M)y(M), \text{ where}$$

$\{G_n : n > N\}$'s are a collection of subgraphs of $\mathcal{T}_Q$, $x(M) = \prod_{\text{edge } e \in M} \frac{1}{x_i x_j}$ (for edge e straddling faces i and j), $y(M) =$ height of M (recording what faces need to be twisted to obtain matching M starting from the minimal matching, and $cm(G_n) =$ the covering monomial of the graph G_n (which records what face labels are contained in G_n and along its boundary).

Remark: This weighting scheme is a reformulation of schemes appearing in works of Speyer ("Octahedron Recurrence") and Goncharov-Kenyon.

Gale-Robinson Example ($Q_7^{(2,3)}$, Mutating $1, 2, \ldots, 7, \ldots$)

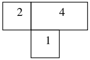
$Q =$ 

$\mathcal{T}_Q =$ 

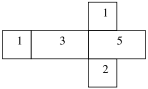
$x_8 \leftrightarrow$ 

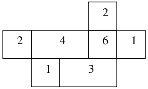
, $x_9 \leftrightarrow$ 

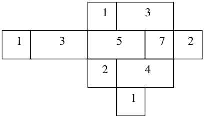
, $x_{10} \leftrightarrow$ 

, $x_{11} \leftrightarrow$ 

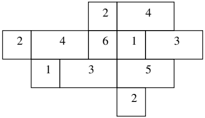
,

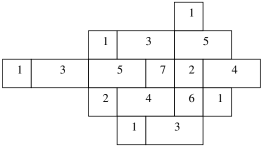
$x_{12} \leftrightarrow$ 

, $x_{13} \leftrightarrow$ 

, $x_{14} \leftrightarrow$ 

,

$x_{15} \leftrightarrow$ 

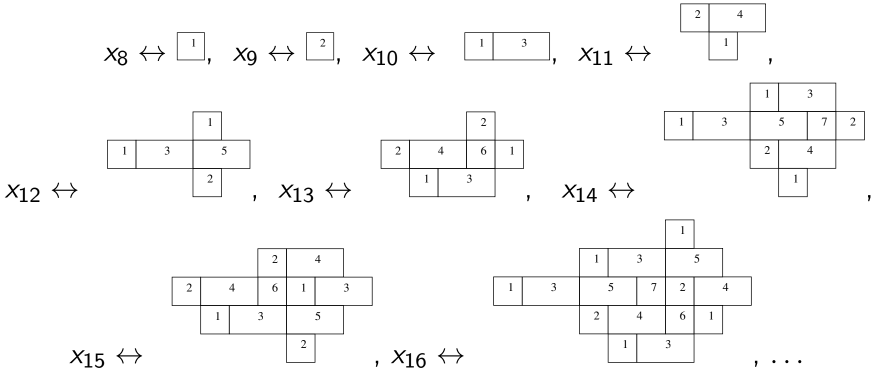
, $x_{16} \leftrightarrow$ 

, ...

Gale-Robinson Example ($Q_7^{(2,3)}$, Mutating $1, 2, \ldots, 7, \ldots$)

Obtain **pinecone graphs** from Bousquet-Mélou, Propp, and West in terms of Brane Tilings Terminology.

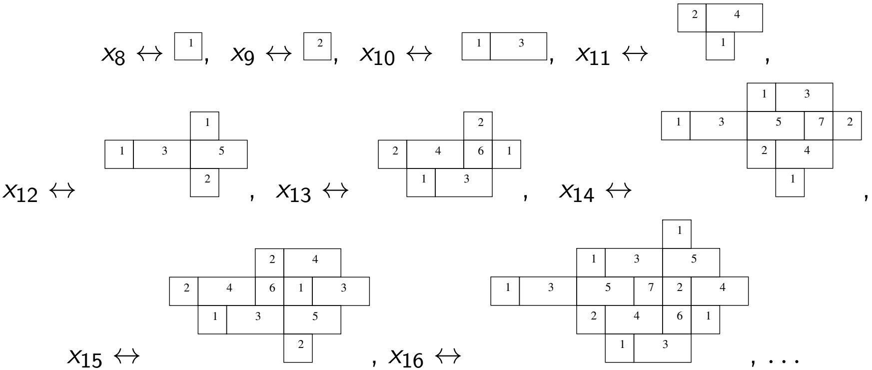
Furthermore, to get cluster variable formulas with coefficients, need only use weights (Goncharov-Kenyon, Speyer) and heights (Kenyon-Propp-...)



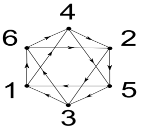
Gale-Robinson Example ($Q_7^{(2,3)}$, Mutating $1, 2, \ldots, 7, \ldots$)

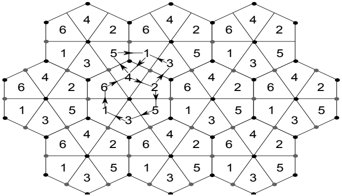
Similar connections (without principal coefficients) also observed in "Brane tilings and non-commutative geometry" by Richard Eager.

Eager uses physics terminology where he looks at $Y^{p,q}$ and $L^{a,b,c}$ quiver gauge theories, and their periodic Seiberg duality (i.e. quiver mutations).



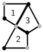
dP_3 Example (Mutating $1, 2, 3, 4, 5, 6, 1, 2, \ldots$)



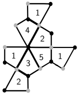


$Q \longrightarrow \mathcal{T}_Q$:

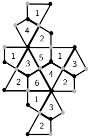
$D_{\frac{1}{2}}$



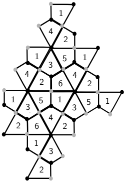
D_2



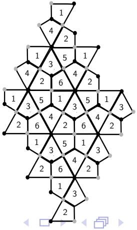
$D_{\frac{3}{2}}$



D_2



$D_{\frac{5}{2}}$

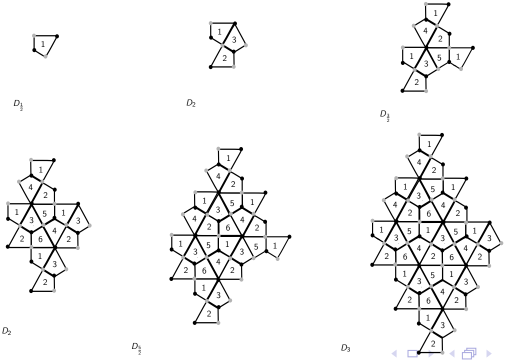


D_3

dP_3 Example (Mutating $1, 2, 3, 4, 5, 6, 1, 2, \ldots$)

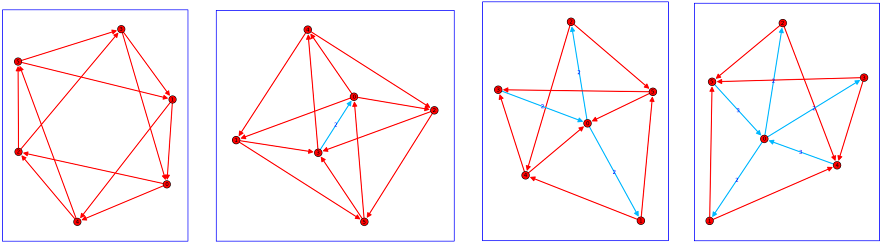
These subgraphs appear in work by Cottrell-Young and a subsequence of them appear in M. Ciucu's work "Perfect matchings and perfect powers", where they are called **Aztec Dragons**.

S. Zhang proved weighted enumerations of perfect matchings in Aztec Dragons yield the Laurent expansions of cluster variables. **(REU 2012)**



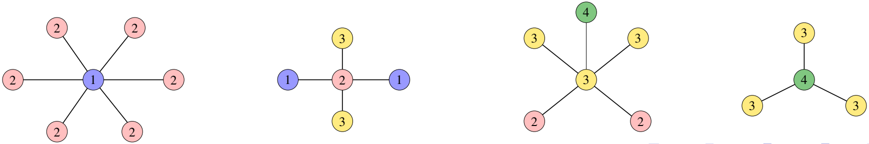
Non-periodic mutation sequences in the dP_3 Lattice

Toric mutations take place at vertices with in-degree and out-degree 2.



Starting with any of these four models of the dP_3 quiver, any sequence of toric mutations yields a quiver that is graph isomorphic to one of these.

Figure 20 of Eager-Franco (Incidences betweeen these Models):

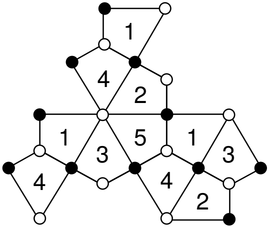


Goal: Combinatorial Formula for Toric Cluster Variables

Example from M. Leoni, S. Neel, and P. Turner (2013 REU):
Mutations at antipodal vertices of dP_3 quiver yield τ-mutation sequences. Resulting Laurent polynomials correspond to Aztec Castles under appropriate weighted enumeration of perfect matchings.

e.g. $1, 2, 3, 4, 1, 2, 5, 6$ yields cluster variable

$$\begin{aligned} & (x_1x_2^2x_3^3x_5^4 + x_2^3x_3^2x_4x_5^4 + 2x_1^2x_2x_3^3x_5^3x_6 + 4x_1x_2^2x_3^2x_4x_5^3x_6 + 2x_2^3x_3x_4^2x_5^3x_6 + x_1^3x_3^3x_5^2x_6^2 \\ + \quad & 5x_1^2x_2x_3^2x_4x_5^2x_6^2 + 5x_1x_2^2x_3x_4^2x_5^2x_6^2 + x_2^3x_4^3x_5^2x_6^2 + 2x_1^3x_3^2x_4x_5x_6^3 + 4x_1^2x_2x_3x_4^2x_5x_6^3 \\ + \quad & 2x_1x_2^2x_4^3x_5x_6^3 + x_1^3x_3x_4^2x_6^4 + x_1^2x_2x_4^3x_6^4)/x_1^2x_2^2x_3^2x_4^2x_6 = \frac{(x_1x_3 + x_2x_4)(x_4x_6 + x_3x_5)^2(x_1x_6 + x_2x_5)^2}{x_1^2x_2^2x_3^2x_4^2x_6} \end{aligned}$$



Segway: $\mathbb{Z}^3$ Parameterization for Toric Cluster Variables

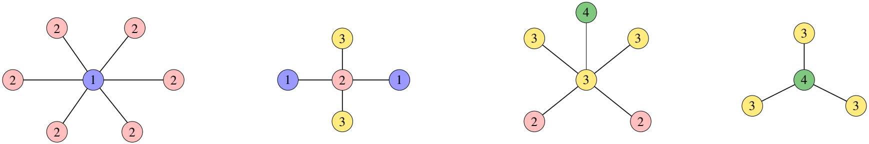
Theorem 1 [Lai-M 2015] Starting from the initial cluster $\{x_1, x_2, \ldots, x_6\}$, the set of cluster variables reachable via toric mutations can be parameterized by $\mathbb{Z}^3$.

Under this correspondence, the initial cluster bijects to

$$[(0,-1,1),(0,-1,0),(-1,0,0),(-1,0,0),(-1,0,1),(0,0,1),(0,0,0)]$$

and toric mutations transform the six-tuple in $\mathbb{Z}^3$ as we will illustrate.

Up to symmetry, enough to consider $\mu_1\mu_2$, $\mu_1\mu_4\mu_1\mu_5\mu_1$, and $\mu_1\mu_4\mu_3$.



Algebraic Formula for Toric Cluster Variables for dP_3

$$\text{Let} \quad A = \frac{x_3x_5 + x_4x_6}{x_1x_2}, \quad B = \frac{x_1x_6 + x_2x_5}{x_3x_4}, \quad C = \frac{x_1x_3 + x_2x_4}{x_5x_6},$$

$$D = \frac{x_1x_3x_6 + x_2x_3x_5 + x_2x_4x_6}{x_1x_4x_5}, \text{ and } E = \frac{x_2x_4x_5 + x_1x_3x_5 + x_1x_4x_6}{x_2x_3x_6}.$$

Let $z_i^{j,k}$ be the cluster variable corresponding to $(i,j,k) \in \mathbb{Z}^3$

Theorem 2 [Lai-M 2015] (Extension of [LMNT 2013] and [Lai 2014]):

$$z_i^{j,k} = x_r \;\; A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} \; B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} \; C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} \; D^{\lfloor \frac{(k-1)^2}{4} \rfloor} \; E^{\lfloor \frac{k^2}{4} \rfloor}$$

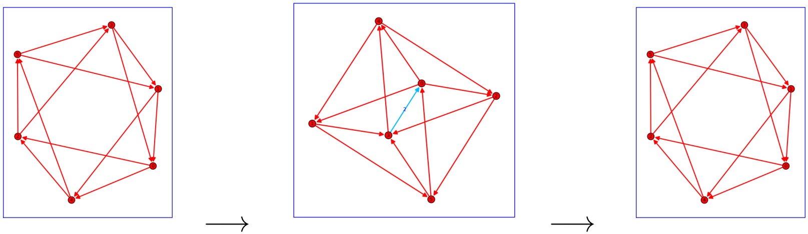
where, working **modulo** 6, we have (cyclically around the dP_3 Quiver)

$$r = 6 \text{ if } 2(i-j)+3k \equiv 0, \quad r = 4 \text{ if } 2(i-j)+3k \equiv 1,$$
$$r = 2 \text{ if } 2(i-j)+3k \equiv 2, \quad r = 5 \text{ if } 2(i-j)+3k \equiv 3,$$
$$r = 3 \text{ if } 2(i-j)+3k \equiv 4, \quad r = 1 \text{ if } 2(i-j)+3k \equiv 5.$$

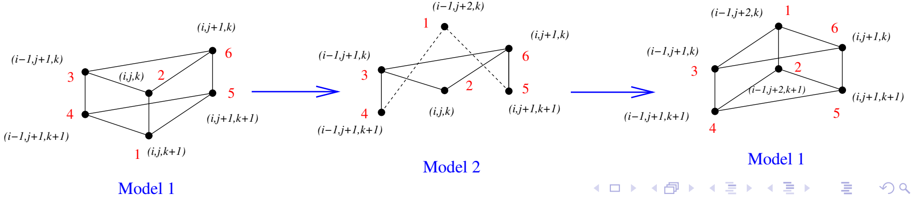
i.e. we **determine** x_r by looking at $(i-j)$ modulo 3 and k modulo 2.

Mutating Model I to Model II and back to Model I

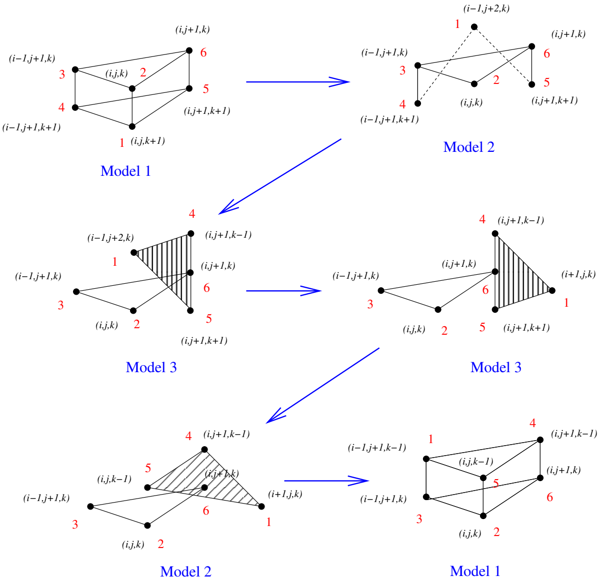
By applying $\mu_1 \circ \mu_2$, $\mu_3 \circ \mu_4$, or $\mu_5 \circ \mu_6$, we mutate the quiver (up to graph isomorphism):



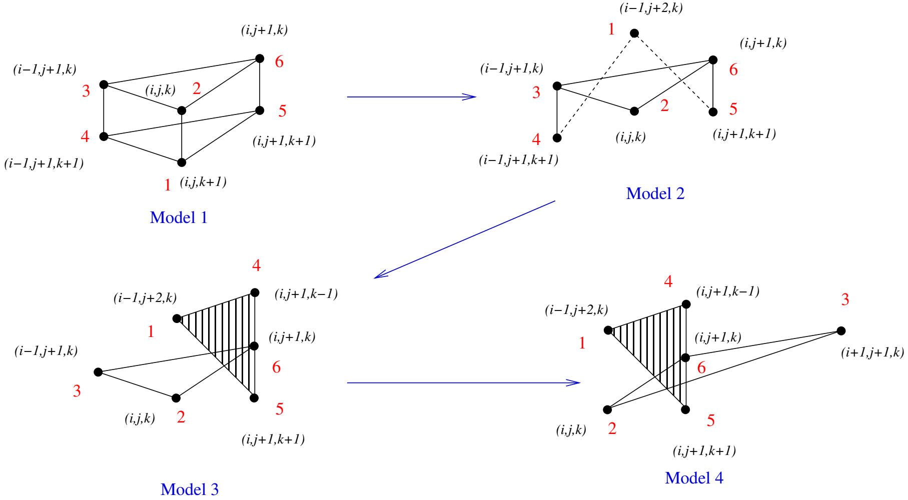
Corresponding action in $\mathbb{Z}^3$ (on triangular prisms):



Illustrating the mutation sequence $\mu_1\mu_4\mu_1\mu_5\mu_1$



Illustrating the mutation sequence $\mu_1\mu_4\mu_3$



Theorem 3 [Lai-M 2015]

Theorem (Reformulation of [Leoni-M-Neel-Turner 2014]): Let $Z^S = [z_1, z_2, \ldots, z_6]$ be the cluster obtained after applying a toric mutation sequence S to the initial cluster $\{x_1, x_2, \ldots, x_6\}$.

Let $w(G) = cm(G) \sum_{M \text{ a perfect matching of } G} x(M)$.

Let $\mathcal{G}(\mathcal{C}_i)$ be the subgraph cut out by the contour $\mathcal{C}_i$.

Then $\mathbf{Z^S} = [\mathbf{w}(\mathcal{G}(\mathcal{C}^{\mathbf{S}}_{\mathbf{1}}), \mathbf{w}(\mathcal{G}(\mathcal{C}^{\mathbf{S}}_{\mathbf{2}}), \ldots, \mathbf{w}(\mathcal{G}(\mathcal{C}^{\mathbf{S}}_{\mathbf{6}})]$ where $\mathcal{C}^{S_1}, \mathcal{C}^{S_2}, \ldots, \mathcal{C}^{S_6}$ are defined as follows:

1) Start with the six-tuple
$[(0,-1,1),(0,-1,0),(-1,0,0),(-1,0,0),(-1,0,1),(0,0,1),(0,0,0)]$ in $\mathbb{Z}^3$.
2) Toric Mutations transform this six-tuple as illustrated earlier.
3) Map from $\mathbb{Z}^3$ to $\mathbb{Z}^6$:

$$(i,j,k) \to (a,b,c,d,e,f) = (j+k, -i-j-k, i+k, j-k+1, -i-j+k-1, i-k+1)$$

and use these six six-tuples to define the contours $\mathcal{C}^{S_1}, \mathcal{C}^{S_2}, \ldots, \mathcal{C}^{S_6}$.

Example 1: mutation sequence $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6$

We start at the initial prism $C_1, C_2, \ldots C_6$. Applying the mutation sequence $\mu_1, \mu_2, \mu_3\mu_4\mu_5\mu_6$ corresponds to the transformations

$$C_1 = (0,0,1,-1,1,0),\ C_2 = (-1,1,0,0,0,1),\ C_3 = (0,1,-1,1,0,0),$$
$$C_4 = (1,0,0,0,1,-1),\ C_5 = (1,-1,1,0,0,0),\ C_6 = (0,0,0,1,-1,1).$$

Example 1: mutation sequence $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6$

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$$C_4 = (1,0,0,0,1,-1),\ C_5 = (1,-1,1,0,0,0),\ C_6 = (0,0,0,1,-1,1).$$

$\longrightarrow$

$$C_1' = (2,-1,0,1,0,-1), C_2' = (1,0,-1,2,-1,0), C_3 = (0,1,-1,1,0,0),$$

$$C_4 = (1,0,0,0,1,-1), C_5 = (1,-1,1,0,0,0), C_6 = (0,0,0,1,-1,1).$$

Example 1: mutation sequence $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6$

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$$C_1 = (0,0,1,-1,1,0),\ C_2 = (-1,1,0,0,0,1),\ C_3 = (0,1,-1,1,0,0),$$
$$C_4 = (1,0,0,0,1,-1),\ C_5 = (1,-1,1,0,0,0),\ C_6 = (0,0,0,1,-1,1).$$

$\longrightarrow$

$$C_1' = (2,-1,0,1,0,-1), C_2' = (1,0,-1,2,-1,0), C_3 = (0,1,-1,1,0,0),$$

$$C_4 = (1,0,0,0,1,-1), C_5 = (1,-1,1,0,0,0), C_6 = (0,0,0,1,-1,1).$$

$\longrightarrow$

$$C_1' = (2,-1,0,1,0,-1), C_2' = (1,0,-1,2,-1,0), C_3' = (1,-1,0,2,-2,1),$$
$$C_4' = (2,-2,1,1,-1,0), C_5 = (1,-1,1,0,0,0), C_6 = (0,0,0,1,-1,1).$$

Example 1: mutation sequence $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6$

We start at the initial prism $C_1, C_2, \ldots C_6$. Applying the mutation sequence $\mu_1, \mu_2, \mu_3\mu_4\mu_5\mu_6$ corresponds to the transformations

$$C_1 = (0,0,1,-1,1,0),\ C_2 = (-1,1,0,0,0,1),\ C_3 = (0,1,-1,1,0,0),$$
$$C_4 = (1,0,0,0,1,-1),\ C_5 = (1,-1,1,0,0,0),\ C_6 = (0,0,0,1,-1,1).$$

$\longrightarrow$

$$C_1' = (2,-1,0,1,0,-1), C_2' = (1,0,-1,2,-1,0), C_3 = (0,1,-1,1,0,0),$$
$$C_4 = (1,0,0,0,1,-1), C_5 = (1,-1,1,0,0,0), C_6 = (0,0,0,1,-1,1).$$

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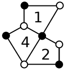
$$C_4' = (2,-2,1,1,-1,0), C_5' = (3,-2,0,2,-1,-1), C_6' = (2,-1,-1,3,-2,0).$$

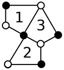
Example 1: mutation sequence $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6$

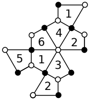
$C_1' = (2,-1,0,1,0,-1)$, $C_2' = (1,0,-1,2,-1,0)$, $C_3' = (1,-1,0,2,-2,1)$,
$C_4' = (2,-2,1,1,-1,0)$, $C_5' = (3,-2,0,2,-1,-1)$, $C_6' = (2,-1,-1,3,-2,0)$.

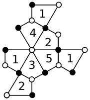

$\dfrac{x_4x_6 + x_3x_5}{x_2}$

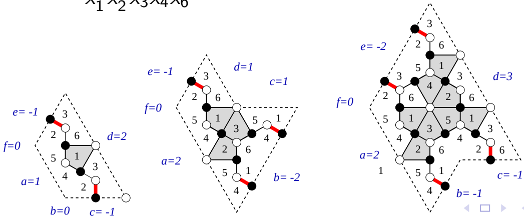

$\dfrac{x_3x_5 + x_4x_6}{x_1}$


$\dfrac{x_2x_3x_5^2 + x_1x_3x_5x_6 + x_2x_4x_5x_6 + x_1x_4x_6^2}{x_1x_2x_4}$


$\dfrac{x_2x_3x_5^2 + x_1x_3x_5x_6 + x_2x_4x_5x_6 + x_1x_4x_6^2}{x_1x_2x_3}$


$\dfrac{(x_2x_5 + x_1x_6)(x_1x_3 + x_2x_4)(x_3x_5 + x_4x_6)^2}{x_1^2x_2^2x_3x_4x_6}$


$\dfrac{(x_2x_5 + x_1x_6)(x_1x_3 + x_2x_4)(x_3x_5 + x_4x_6)^2}{x_1^2x_2^2x_3x_4x_5}$

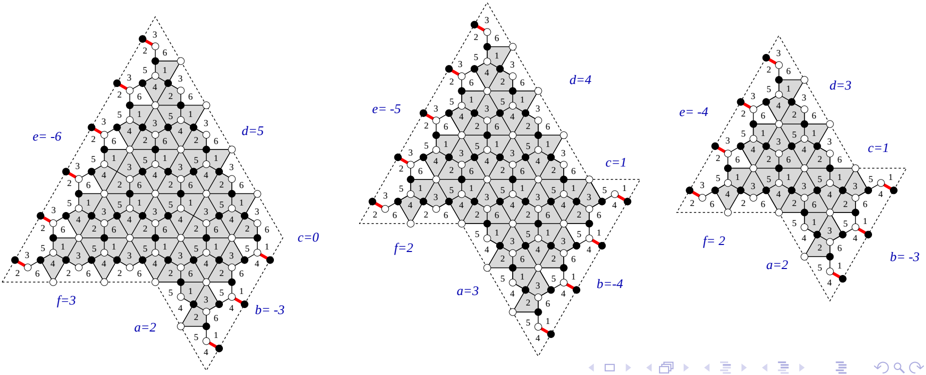


Example 2: $S = \tau_1\tau_2\tau_3\tau_1\tau_2\tau_3\tau_2\tau_1\tau_4$

We reach $\{(1,3),(1,2),(0,3)\}$ from applying $\tau_1\tau_2\tau_3\tau_1\tau_2\tau_3\tau_2\tau_1$ ($\tau_1 = \mu_1\mu_2$, $\tau_2 = \mu_3\mu_4$, and $\tau_3 = \mu_5\mu_6$) and then $\tau_4 = \mu_1\mu_4\mu_1\mu_5\mu_1$ yields $\mathcal{C}^S = [\sigma^{-1}\mathcal{C}_1^3, \mathcal{C}_1^3, \mathcal{C}_1^2, \sigma^{-1}\mathcal{C}_1^2, \sigma^{-1}\mathcal{C}_0^3, \mathcal{C}_0^3] =$

$$[(2,-3,0,5,-6,3),(3,-4,1,4,-5,2),(2,-3,1,3,-4,2),$$

$$(1,-2,0,4,-5,3),(2,-2,-1,5,-5,2),(3,-3,0,4,-4,1)].$$



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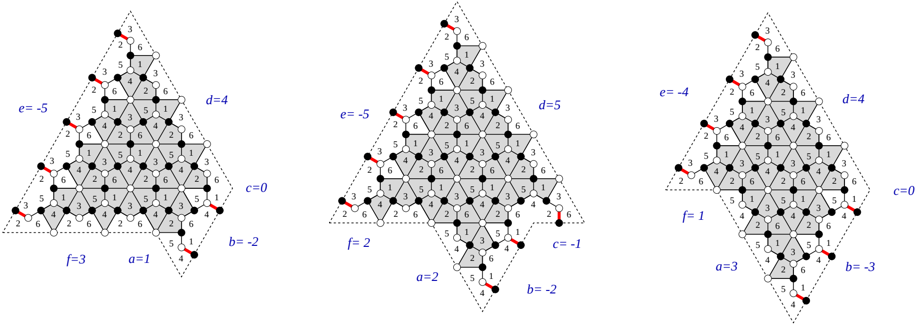
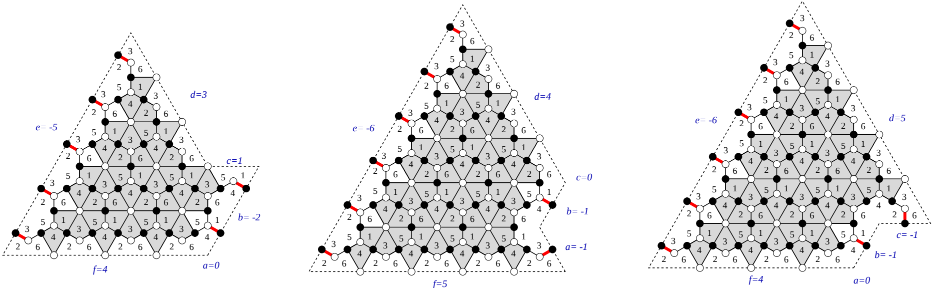


Example 3: $S = \tau_1\tau_2\tau_3\tau_1\tau_3\tau_2\tau_1\tau_4\tau_5$

$$[(0,-2,1,3,-5,4),(-1,-1,0,4,-6,5),(0,-1,-1,5,-6,4),$$

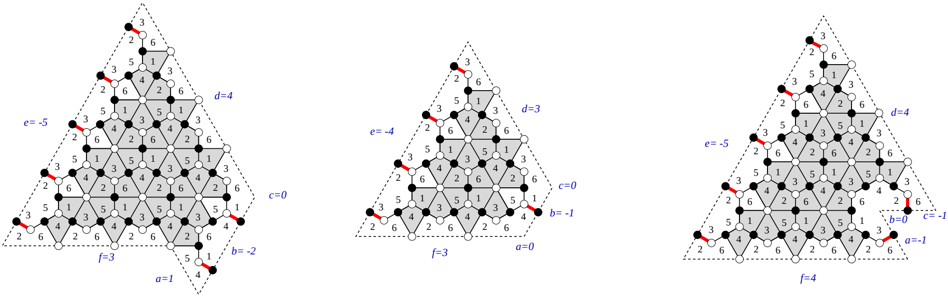
$$(1,-2,0,4,-5,3),(0,-1,0,3,-4,3),(-1,0,-1,4,-5,4)].$$



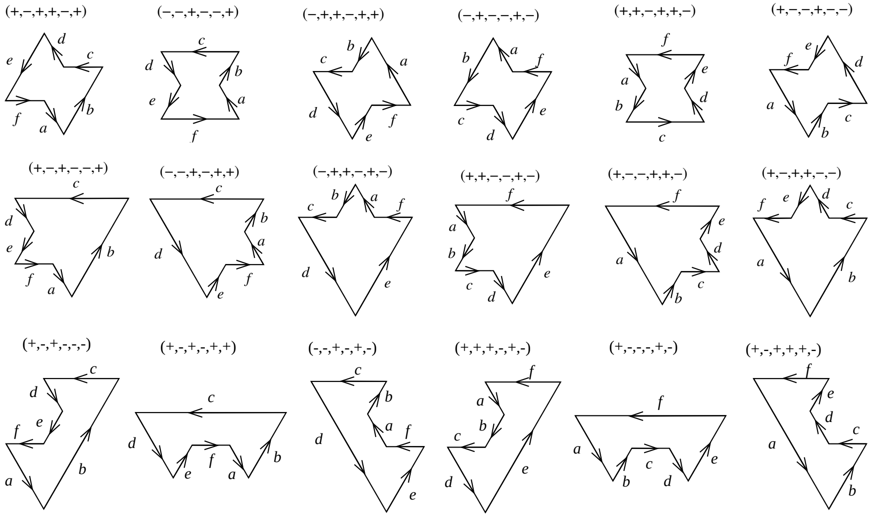
Example 3: $S = \tau_1\tau_2\tau_3\tau_1\tau_3\tau_2\tau_1\tau_4\tau_5$

$$[(0,-2,1,3,-5,4),(-1,-1,0,4,-6,5),(0,-1,-1,5,-6,4),$$

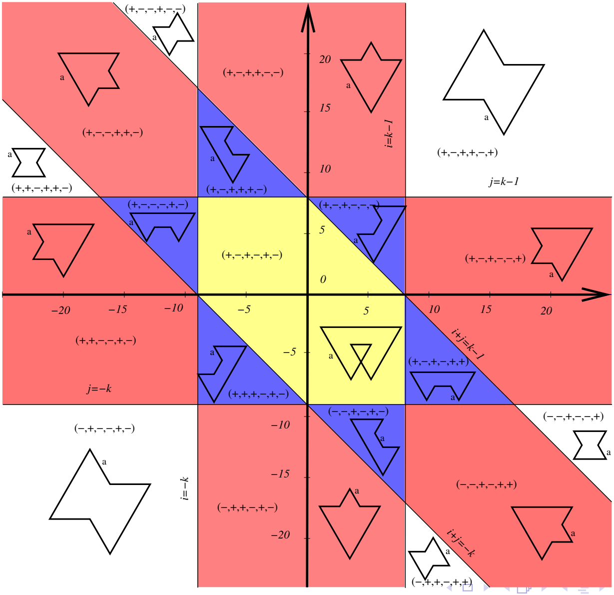
$$(1,-2,0,4,-5,3),(0,-1,0,3,-4,3),(-1,0,-1,4,-5,4)].$$



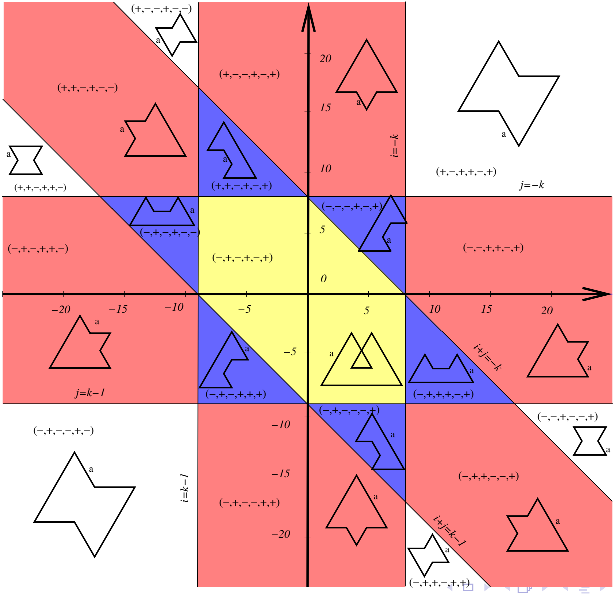
Possible Shapes of Aztec Castles



Cross-section when k positive



Cross-section when k negative

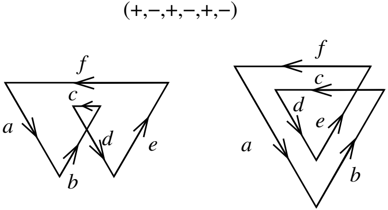


Future Work: Self-intersecting Contours

Algebraic formula

$$z_i^{j,k} = x_r \quad A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} \; B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} \; C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} \; D^{\lfloor \frac{(k-1)^2}{4} \rfloor} \; E^{\lfloor \frac{k^2}{4} \rfloor}$$

still works for (a, b, c, d, e, f) when alternating in signs but combinatorial formula for such cases open.



Work in progress (with David Speyer): Conjectural Double-Dimer combinatorial interpretation for self-intersecting contours.

Additional Open Questions

Question: Work of Di Francesco and Soto-Garrido studied arctic curves from T-systems. Can we adapt these methods to obtain Limit Shapes for the graphs arising from toric mutations sequences for the dP_3 quiver?

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Question: Finally, we focused on cluster expansions assuming the initial cluster was Model I. What if we start from a different model. It appears that it the initial cluster is of Model IV that one gets Hexagonal dungeons. T. Lai and I plan to do further work on **Dungeons and Dragons**.

Thanks for Coming (Slides at http://math.umn.edu/∼musiker/UCONN16.pdf)

- Richard Eager and Sebastian Franco, *Colored BPS Pyramid Partition Functions, Quivers and Cluster Transformations*, arXiv:1112.1132.
- Eric Kuo, *Applications of Graphical Condensation for Enumerating Matchings and Tilings*, *Theoretical Computer Science*, 319:29–57.
- Sicong Zhang, *Cluster Variables and Perfect Matchings of Subgraphs of the dP_3 Lattice*, 2012 REU Report, arXiv:1511.06055.
- Tri Lai, *A Generalization of Aztec Dragons*, arXiv:1504.00303, to appear in *Graphs and Combinatorics*.

- *Gale-Robinson Sequences and Brane Tilings* (with In-Jee Jeong and and Sicong Zhang), *Discrete Mathematics and Theoretical Computer Science Proc.* **AS** (2013), 737-748.
- *Aztec Castles and the dP3 Quiver* (with Megan Leoni, Seth Neel, and Paxton Turner), *Journal of Physics A*: Math. Theor. 47 474011, arXiv:1308.3926.
- *Beyond Aztec Castles: Toric Cascades in the dP_3 Quiver* (with Tri Lai), arXiv:1512.00507.