Quasiinvariants of S_3

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Abstract

Let s_{ij} represent a transposition in S_n. A polynomial P in $\mathbb{Q}[X_n]$ is said to be m-quasiinvariant with respect to S_n if $(x_i - x_j)^{2m+1}$ divides $(1 - s_{ij})P$ for all $1 \leq i, j \leq n$. We call the ring of m-quasiinvariants, $QI_m[X_n]$. We describe a method for constructing a basis for the quotient $QI_m[X_3]/(e_1, e_2, e_3)$. This leads to the evaluation of certain binomial determinants that are interesting in their own right.

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The symmetric group S_n acts on the ring of polynomials $\mathbb{Q}[X_n]$ by permuting indices. That is for any permutation $\sigma \in S_n$

$$\sigma P(x_1, \ldots, x_n) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

A polynomial P is said to be S_n-invariant or symmetric if and only if $\sigma(P) = P$ for all $\sigma \in S_n$. The fundamental theorem of symmetric functions [9, p. 292] states that any invariant of S_n can be written as a polynomial in $\{e_1, e_2, \ldots, e_n\}$ where

$$e_k = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

For S_3 we have

$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 \\ e_3 &= x_1x_2x_3. \end{aligned}$$

A generalization of invariance known as "*quasiinvariance*" has been studied in the recent literature [1, 2, 3]. In the rest of this paper we will use the notation s_{ij} to denote the transposition (i, j) and will let QI_m denote $QI_m[X_n]$ for convenience.

Definition 1. *A polynomial P is m-quasiinvariant if and only if $(1 - s_{ij})P$ is divisible by $(x_i - x_j)^{2m+1}$ for all pairs $1 \le i < j \le n$.*

This definition is not vacuous because $(1-s_{ij})P$ is antisymmetric with respect to the transposition s_{ij} thus setting $x_i = x_j$ will yield zero. Hence $(x_i - x_j)$ divides $(1 - s_{ij})P$ and the antisymmetry forces an odd power of $(x_i - x_j)$ to divide it. We should note that an analogous condition defines m-quasiinvariance for any Coxeter group. In the general definition, the linear forms giving the equations of the reflecting hyperplanes play the role of the differences $x_i - x_j$.

It is easily seen that the divided difference operator $\Delta_{ij} = \frac{1-s_{ij}}{x_i - x_j}$ is a twisted derivation [6, pp. 192-194] which means that

$$\Delta_{ij}(Q_1Q_2) = \Delta_{ij}(Q_1)Q_2 + s_{ij}(Q_1)\Delta_{ij}(Q_2).$$

Thus if $\Delta_{ij}(Q_1)$ and $\Delta_{ij}(Q_2)$ are both divisible by $(x_i - x_j)^{2m}$ then so is $\Delta_{ij}(Q_1Q_2)$. The operator $(1 - s_{ij})$ is also linear which means that each QI_m is a ring. Furthermore $(1 - s_{ij})P$ will be divisible by $(x_i - x_j)^{2m+1}$ for arbitrarily large m if and only if $(1 - s_{ij})P = 0$ which means that all QI_m contain Λ_n (the ring of symmetric polynomials). We thus have the inclusions

$$\mathbb{Q}[x_1, \ldots, x_n] = QI_0 \supset QI_1 \supset QI_2 \supset \cdots \supset QI_\infty = \Lambda_n$$

A classic result states that $\mathbb{Q}[x_1, \ldots, x_n]$ is a free module of rank $n!$ over the ideal $(e_1, \ldots, e_n)$. Furthermore, the action on the quotient precisely gives the regular representation of S_n [6, p. 247].

This means that there exists a basis of $n!$ polynomials $\{\eta_1, \ldots, \eta_{n!}\}$ such that any n-variable polynomial can be written as a unique linear combination

$$\sum_{i=1}^{n!} A_i \eta_i$$

where the A_i's are symmetric polynomials. For example, any polynomial in $\mathbb{Q}[x_1, x_2, x_3]$ can be written uniquely as

$$A_1 + A_2 x_2 + A_3 x_3 + A_4 x_2 x_3 + A_5 x_3^2 + A_6 x_2 x_3^2$$

where $A_1, \ldots, A_6$ are symmetric polynomials. The polynomial ring can be thought of as the ring of 0-quasiinvariants and recently [3], an analogous result has been proven for the rings of m-quasiinvariants for $m > 0$. Namely, any element of QI_m can be written uniquely as a sum

$$\sum_{i=1}^{n!} A_i(e_1, \ldots, e_n) \cdot \eta_i$$

where the A_i's are polynomials and the η_i's are elements of QI_m.

These η_i's are therefore a basis for $QI_m / \langle (e_1, e_2, \ldots e_n) \rangle$, a space which has been shown [2] to have the following Hilbert series:

$$\sum_{i=1}^{n!} q^{degree(\eta_i)} = \sum_{T \in ST(n)} f_{\lambda(T)} \; q^{m\left(\binom{n}{2} - content(\lambda(T))\right) + cocharge(T)}$$

In the case that $n = 3$, this gives that the Hilbert series of $QI_m / \langle (e_1, e_2, e_3) \rangle$ (QI_m will always signify $QI_m[X_3]$ from here on out) is

$$q^0 + 2q^{3m+1} + 2q^{3m+2} + q^{6m+3}. \tag{1}$$

Note also by the respective degrees of e_1, e_2, and e_3 that the Hilbert series of QI_m is

$$\frac{q^0 + 2q^{3m+1} + 2q^{3m+2} + q^{6m+3}}{(1-q)(1-q^2)(1-q^3)}. \tag{2}$$

It is easily shown that the Vandermonde determinant $\Delta(x) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ raised to the power $2m+1$ accounts for the term q^{6m+3} and clearly the constants account for q^0. So the interesting problem arises to construct the four m-quasiinvariants that account for the terms $2q^{3m+1}+2q^{3m+2}$. The explicit construction of these four m-quasiinvariants is the goal and motivating force which led to the results of this paper. It developed that this construction required the evaluation of two binomial determinants which are interesting in their own right and deserve a special mention here. The two resulting identities may be stated as follows.

Theorem 1.

$$\det\left|\binom{C+\alpha i}{E+\beta j}-\binom{D-\alpha i}{E+\beta j}\right|_{i,j=1}^{k}=\frac{\binom{C+D}{E+\beta}\binom{C+D}{E+2\beta}\cdots\binom{C+D}{E+n\beta}}{\binom{C+D}{C+\alpha}\binom{C+D}{C+2\alpha}\cdots\binom{C+D}{C+n\alpha}}\cdot|\mathcal{F}| \tag{3}$$

where $\mathcal{F}$ denotes the collection of k-tuples of non-intersecting lattice paths respectively joining the points

$$\{(D-k\alpha,D-k\alpha),(D-(k-1)\alpha,D-(k-1)\alpha),\ldots,(D-\alpha,D-\alpha)\}$$

to

$$\{(0,C+D-E-k\beta),(0,C+D-E-(k-1)\beta),\ldots,(0,C+D-E-\beta)\}$$

and throughout remaining strictly below the line $y=-x+C+D$.

It is also worthy of notice the fact that the entries of the determinant in (3) are differences of binomial coefficients where the tops are different and the bottoms are the same. A literature search found no determinant results covering this particular case. Nevertheless, a manipulation suggested by an argument of Gessel and Viennot in [5] enabled us to derive Theorem 1 from the following general result:

Theorem 2. *For any integers a,b,c,d,e, the determinant*

$$\det\left|\binom{a+bi}{c+dj}-\binom{a+bi}{e-dj}\right|_{i,j=1}^{n}$$

is the number of families of non-intersecting lattice paths with NORTH and WEST steps, respectively joining the points

$$\{(c+d,c+d),(c+2d,c+2d),\ldots(c+nd,c+nd)\}$$

to

$$\{(0,a+b),(0,a+2b),\ldots,(0,a+nb)\}$$

and throughout avoiding the line $y=-x+(c+e)$.

Our main result is that a basis for the quotient of the m-quasiinvariants of S_3 can be found by computing the 1-dimensional null space of particular matrices.

The non-vanishing of the determinant (3) provides the crucial step in proving the null space in question is indeed 1-dimensional.

Our presentation is divided into four parts. In the first part we show (non-constructively) that quasiinvariants of a certain nice form exist. In the second part, we find a system of equations that the coefficients of these quasiinvariants must satisfy. In the third part, we show that we can solve this system by computing a 1-dimensional null space. In the final part we complete the construction, and prove the elements we've constructed complete a basis for the quotient.

We should mention that Feigin and Veselov in [1] have given explicit module bases for the m-quasiinvariants of all Dihedral groups D_n. But so far there are no other Coxeter groups for which explicit constructions have been given. The Feigin-Veselov construction is based on complex number techniques that are very suitable in the dihedral case. Although D_3 m-quasiinvariants can be easily converted into S_3 m-quasiinvariants, our work efforts have been guided by the need of developing methods that can be extended to the general case. Our results may be taken as an instance of such methods. Extensions of the present construction to S_n will be the topic of a forthcoming publication.

1 Quasiinvariants with a nice form

We begin by defining the following elements of the group algebra of S_3:

$$[S_3] = \frac{1}{6}\sum_{\sigma\in S_3}\sigma, \qquad [S_3]' = \frac{1}{6}\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)\sigma$$

$$\pi_1 = \frac{1}{3}(1+s_{23})(1-s_{12}), \qquad \pi_2 = \frac{1}{3}(1+s_{12})(1-s_{23})$$

These defined, the following identities are easily verified:

$$(\pi_1)^2 = \pi_1, (\pi_2)^2 = \pi_2 \tag{4}$$

$$[S_3]'\pi_1 = \pi_1\pi_2 = \pi_2\pi_1 = 0 \tag{5}$$

$$[S_3] + \pi_1 + \pi_2 + [S_3]' = 1 \tag{6}$$

$$s_{23}\pi_1 = \pi_1 \tag{7}$$

$$\pi_2 s_{12}\pi_1 = -s_{13}\pi_1 \tag{8}$$

We now show that there exist quasiinvariants satisfying certain symmetry and independence conditions.

Lemma 1. *For all $m \geq 0$, there exist non-symmetric m-quasiinvariants A_1, A_2 of degrees $3m+1, 3m+2$, respectively, such that $s_{23}(A_i) = A_i$ and in the quotient $QI_m/\big\langle(e_1, e_2, e_3)\big\rangle$, the image of A_i and $s_{12}(A_i)$ are linearly independent. Further all four of these will be independent of $\Delta^{2m+1}(x)$.*

Proof. It is easy to see that the image of $[S_3]$ in the quotient is the constant terms. We also note that any polynomial in the image of $[S_3]'$ is alternating and any alternating m-quasiinvariant must be divisible by $\Delta^{2m}(x)$, which has degree $6m$. Thus, from the Hilbert series (1), there must exist quasiinvariants B_i of degree $3m+i$, ($i \in \{1,2\}$) such that if we apply equation (6) to B_i we have

$$\pi_1(B_i) + \pi_2(B_i) \neq 0. \tag{9}$$

Assume without loss that $\pi_1(B_i) \neq 0$, and set

$$A_i = \pi_1(B_i). \tag{10}$$

Equation (7) immediately gives that $s_{23}(A_i) = A_i$. Now suppose we had symmetric functions S, T such that

$$SA_i + T(s_{12}A_i) = 0 \tag{11}$$

Applying π_2 to this gives (by (5) and (8)):

$$T\pi_2 s_{12}\pi_1 B_i = 0 \tag{12}$$
$$-Ts_{13}\pi_1 B_i = 0 \tag{13}$$
$$-Ts_{13}A_i = 0 \tag{14}$$

Since A_i was assumed to be non-zero, this gives $T = 0$ and (11) gives $S = 0$. Now assume there was a nontrivial relationship between these and $\Delta^{2m+1}(x)$

$$c_1A_1 + c_2(s_{12}A_1) + c_3A_2 + c_4(s_{12}A_2) + c_5\Delta^{2m+1}(x) = 0 \tag{15}$$

Applying $[S_3]'$ gives (by (5))

$$c_2[S_3]'(s_{12}A_1) + c_4[S_3]'(s_{12}A_2) + c_5\Delta^{2m+1}(x) = 0 \tag{16}$$

But $[S_3]'s_{12} = s_{12}[S_3]'$ and $[S_3]'A_i = 0$ so (16) gives $c_5 = 0$. $\square$

Since $s_{23}(A_i) = A_i$, A_i is symmetric with respect to x_2 and x_3. This means that we can write the m-quasiinvariants A_1 and A_2 as

$$A_1 = \sum_{0\leq i\leq j\leq i+j\leq d} C_{[i,j]}x_1^{d-i-j}m_{[i,j]}(x_2,x_3). \tag{17}$$

and

$$A_2 = \sum_{0\leq i\leq j\leq i+j\leq d} \tilde{C}_{[i,j]}x_1^{d-i-j}m_{[i,j]}(x_2,x_3). \tag{18}$$

for $d = 3m+1$ or $3m+2$, respectively. In fact we can make the following stronger statement about the form of the A_i:

Lemma 2. *There exist m-quasiinvariants A_1 and A_2, satisfying the conditions of Lemma 1, of the form*

$$A_1 = \sum_{0\le i\le j\le m} C_{[i,j]} x_1^{3m+1-i-j} m_{[i,j]}(x_2, x_3)$$

and

$$A_2 = \sum_{0\le i\le j\le m+1} \tilde{C}_{[i,j]} x_1^{3m+2-i-j} m_{[i,j]}(x_2, x_3).$$

Proof. We first prove this result for A_1. By grouping together monomials with similar exponent sequences, we can rewrite the above sum (17) as

$$\sum_{\substack{0\le i<j<k\\ i+j+k\ =\ 3m+1}} \Big(C_{[i,j]}(x_1^k x_2^i x_3^j + x_1^k x_2^j x_3^i) \ + \ C_{[j,k]}(x_1^i x_2^j x_3^k + x_1^i x_2^k x_3^j)$$
$$+ \ C_{[i,k]}(x_1^j x_2^k x_3^i + x_1^j x_2^i x_3^k) \Big) + \sum_{\substack{0\le i,j\\ 2i+j\ =\ 3m+1}} C_{[i,j]}(x_1^i x_2^i x_3^j \ + \ x_1^i x_2^j x_3^i)$$
$$+ \ C_{[i,i]} x_1^j x_2^i x_3^i.$$

Using this decomposition, we find that $(1 - s_{13})A_1$ is the sum

$$\sum_{\substack{0\le i<j<k\\ i+j+k\ =\ 3m+1}} \Big((C_{[j,k]} \ - \ C_{[i,j]})(x_1^i x_2^j x_3^k \ - \ x_1^k x_2^j x_3^i) \ +$$
$$(C_{[i,k]} \ - \ C_{[j,k]})(x_1^j x_2^k x_3^i \ - \ x_1^i x_2^k x_3^j) \ +$$
$$(C_{[i,j]} \ - \ C_{[i,k]})(x_1^k x_2^i x_3^j \ - \ x_1^j x_2^i x_3^k) \Big) \ +$$
$$\sum_{\substack{0\le i,j\\ 2i+j\ =\ 3m+1}} \Big((C_{[i,j]} \ - \ C_{[i,i]})(x_1^i x_2^i x_3^j \ - \ x_1^j x_2^i x_3^i) \Big).$$

We can now discover properties of the coefficients by focusing on one summand at a time. For instance, given a specific composition $[i,j,k]$ of $3m+1$ such that $0 \le i < j < k$, the fact that $i + j + k > 3m$ means that the largest exponent, namely k, will be greater than m. However, A_1 m-quasiinvariant means that $(x_1 - x_3)^{2m+1} \Big| (1 - s_{13})A_1$ and thus the highest power of x_2 that can appear in $(1 - s_{13})A_1$ will be $(3m + 1) - (2m + 1) = m$. Thus x_2^k cannot appear in a term of $(1 - s_{13})A_1$ with a nonzero coefficient, and thus we obtain $C_{[i,k]} = C_{[j,k]}$. If both the exponents j and k happen to be greater than m, then by similar logic we conclude that $C_{[i,j]} = C_{[i,k]} = C_{[j,k]}$. Finally, if we are given the composition $[i,i,j]$ with $i > m$, we see that $C_{[i,j]} = C_{[i,i]}$. We summarize these conditions here:

$$
\begin{aligned}
&C_{[i,k]} = C_{[j,k]} && \text{when } \; i < j < k && && (19)\\
&C_{[i,j]} = C_{[i,k]} = C_{[j,k]} && \text{when } \; i < j < k, && j > m && (20)\\
&C_{[i,j]} = C_{[i,i]} && \text{when } \; m < i < j. && && (21)
\end{aligned}
$$

The idea now will be to subtract certain symmetric functions from A_1 in order to get rid of exponents of x_2 and x_3 greater than m, without changing the equivalence class of A_1 in the quotient. For every triplet $\{i, j, k\}$ of exponents with $i < j < k, j \leq m$, we see that $A_1 - C_{[i,k]}m_{i,j,k}$ has

$$(C_{[i,j]} - C_{[i,k]})(x_1^k x_2^i x_3^j + x_1^k x_2^j x_3^i) \tag{22}$$

as the only monomials with exponent sequence a permutation of (i, j, k), by (19). (Here $m_{i,j,k}$ is the monomial symmetric function with exponents i, j, k). For every triplet $\{i, j, k\}$ of exponents with $i < j < k, j > m$, we have that $A_1 - C_{[i,k]}m_{i,j,k}$ has no monomials with exponent sequence a permutation of (i, j, k), by (20). For every remaining triplet $\{i, i, j\}$ of exponents we see that $A_1 - C_{[i,j]}m_{i,i,j}$ has

$$(C_{[i,i]} - C_{[i,j]})x_1^j x_2^i x_3^i \tag{23}$$

as the only monomial with exponent sequence a permutation of (i, i, j), which by (21) is only nonzero when $i \leq m$. Thus, after subtracting appropriate symmetric functions we are left with a sum containing only monomials such that the exponents of x_2 and x_3 are less than or equal to m. This gives the stated result for A_1.

Since A_2 has degree $3m+2$, the highest power of x_2 that can appear in $(1-s_{13})A_2$ is $m + 1$. Thus any composition $[i, j, k]$ such that $0 \leq i < j < k$ and $i + j + k = 3m + 2$ will have to satisfy $k > m + 1$, which will allow us to equate certain coefficients as above. Any composition where $0 \leq i, j$ and $2i + j = 3m + 2$ will only yield three terms, two of which have the same coefficient. Either way, we will analogously be able to use appropriate symmetric functions to subtract from A_2 so that monomials with powers of x_2 or x_3 exceeding $m+1$ will disappear. □

Later on, we will demonstrate that, for A_2, we can strengthen the result of Lemma 2. Namely we will prove that there exists a quasiinvariant A_2 of degree $3m + 2$ that satisfies the properties of Lemma 1 and is of the form

$$A_2 = \sum_{0 \leq i \leq j \leq m} \tilde{C}_{[i,j]} x_1^{3m+2-i-j} m_{[i,j]}(x_2, x_3). \tag{24}$$

Note that the indices of the sum are now less than $m + 1$. The proof of this will require the explicit construction of A_1, and will be necessary to explicitly construct A_2.

2 Relations satisfied by the coefficients $C_{[i,j]}$

In this section we show the $C_{[i,j]}$ satisfy certain relations. We begin by setting $d = 3m + 1$, and

$$A_{i,j,k,l} = \begin{cases} \binom{i}{k}\binom{d-i-k}{l} - \binom{i}{k}\binom{2i-k}{l} & \text{if } i = j, \\ \binom{i}{k}\binom{d-j-k}{l} + \binom{j}{k}\binom{d-i-k}{l} - \left(\binom{i}{k} + \binom{j}{k}\right)\binom{i+j-k}{l} & \text{otherwise.} \end{cases}$$

We can now state the main result of this section.

Lemma 3. *The coefficients $C_{[i,j]}$ satisfy the linear equations*

$$\sum_{0 \le j \le i \le m} A_{i,j,k,l} C_{[i,j]} = 0 \tag{25}$$

for $k \in \{0, \ldots, m\}$ and $l \in \{1, 3, 5, \ldots, 2m-1\}$.

Proof. By definition, if $i > j$, then $C_{[i,j]}$ is the coefficient of

$$x_1^{d-i-j} m_{[i,j]}(x_2, x_3) = x_1^{d-i-j}\left(x_2^i x_3^j + x_2^j x_3^i\right).$$

If instead $i = j$, then $C_{[i,j]}$ is the coefficient of $x_1^{d-2i} x_2^i x_3^i$. Consequently, inside of $(1 - s_{13})A_1$, $C_{[i,j]}$ is the coefficient of the polynomial

$$x_1^{d-i-j} x_2^i x_3^j + x_1^{d-i-j} x_2^j x_3^i - x_1^i x_2^j x_3^{d-i-j} - x_1^j x_2^i x_3^{d-i-j}$$

if $i > j$ and

$$x_1^{d-2i} x_2^i x_3^i - x_1^i x_2^i x_3^{d-2i}$$

if $i = j$. Using the substitutions $y_1 = x_2 - x_1$ and $y_2 = x_1 - x_3$, we rewrite these polynomials. For the case $i = j$ we have

$$\begin{aligned}
(1 - s_{13})A_1\Big|_{C_{[i,i]}} &= x_1^{d-2i}(y_1 + x_1)^i x_3^i - x_1^i (y_1 + x_1)^i x_3^{d-2i} \\
&= \sum_{k=0}^{i} \binom{i}{k} x_1^{d-i-k} y_1^k x_3^i - \binom{i}{k} x_1^{2i-k} y_1^k x_3^{d-2i} \\
&= \sum_{k=0}^{i} \binom{i}{k} (y_2 + x_3)^{d-i-k} y_1^k x_3^i - \binom{i}{k} (y_2 + x_3)^{2i-k} y_1^k x_3^{d-2i} \\
&= \sum_{k=0}^{i} \binom{i}{k} \left(\sum_{l=0}^{d-i-k} \binom{d-i-k}{l} y_1^k y_2^l x_3^{d-k-l} - \sum_{l=0}^{2i-k} \binom{2i-k}{l} y_1^k y_2^l x_3^{d-k-l} \right) \\
&= \sum_{k=0}^{i} \sum_{l=0}^{\max(d-i-k,\ 2i-k)} \binom{i}{k} \left(\binom{d-i-k}{l} - \binom{2i-k}{l} \right) y_1^k y_2^l x_3^{d-k-l} \\
&= \sum_{k=0}^{i} \sum_{l=0}^{\max(d-i-k,\ 2i-k)} A_{i,i,k,l} y_1^k y_2^l x_3^{d-k-l}
\end{aligned}$$

For $i > j$ we have

$$\begin{aligned}
(1-s_{13})A_1\Big|_{C_{[i,j]}} &= x_1^{d-i-j}(y_1+x_1)^i x_3^j + x_1^{d-i-j}(y_1+x_1)^j x_3^i \\
&\quad - x_1^i(y_1+x_1)^j x_3^{d-i-j} - x_1^j(y_1+x_1)^i x_3^{d-i-j} \\
&= \sum_{k=0}^{i} \binom{i}{k} x_1^{d-j-k} y_1^k x_3^j + \binom{j}{k} x_1^{d-i-k} y_1^k x_3^i \\
&\quad - \binom{i}{k} x_1^{i+j-k} y_1^k x_3^{d-i-j} - \binom{j}{k} x_1^{i+j-k} y_1^k x_3^{d-i-j} \\
&= \sum_{k=0}^{i} \binom{i}{k} (y_2+x_3)^{d-j-k} y_1^k x_3^j + \binom{j}{k} (y_2+x_3)^{d-i-k} y_1^k x_3^i \\
&\quad - \binom{i}{k} (y_2+x_3)^{i+j-k} y_1^k x_3^{d-i-j} - \binom{j}{k} (y_2+x_3)^{i+j-k} y_1^k x_3^{d-i-j} \\
&= \sum_{k=0}^{i} \sum_{l=0}^{\max\{d-j-k,\ i+j-k\}} \left(\binom{i}{k}\binom{d-j-k}{l} + \binom{j}{k}\binom{d-i-k}{l} \right. \\
&\quad \left. - \binom{i}{k}\binom{i+j-k}{l} - \binom{j}{k}\binom{i+j-k}{l} \right) y_1^k y_2^l x_3^{d-k-l} \\
&= \sum_{k=0}^{i} \sum_{l=0}^{\max\{d-j-k,\ i+j-k\}} A_{i,j,k,l} y_1^k y_2^l x_3^{d-k-l}
\end{aligned}$$

By definition, A_1 is m-quasiinvariant if and only if $(1-s_{13})A_1$ is divisible by y_2^{2m+1}. Solving the equations implies that $(1-s_{13})A_1\Big|_{C_{[i,j]}}$ has even order or order greater than $2m-1$ with respect to y_2. Since $(1-s_{13})A_1$ is divisible by an odd power of (x_1-x_3), we make the following statement: for fixed $k \in \{0,\ldots,m\}$ and fixed odd $l < 2m+1$, we must have

$$\sum_{0\le j\le i\le m} A_{i,j,k,l} C_{[i,j]} y_1^k y_2^l x_3^{d-k-l} = 0.$$

The lemma is an immediate consequence. $\square$

3 The coefficients have a one-dimensional solution space

Once we verify that the relations in (25) have a one-dimensional solution space, it is a straightforward (although time-intensive) process to find a representative solution. This will allow us to explicitly construct A_1, for which we currently have only an existence proof. We begin by computing the determinants of certain matrices, beginning with Theorem 2, stated in the introduction.

Proof of Theorem 2. We first show that the number of lattice paths from $(c + jd, c + jd)$ to $(0, a + ib)$ which avoid the line $y = -x + (c + e)$ is $\binom{a+bi}{c+dj} - \binom{a+bi}{e-dj}$. Consider the following two diagrams:

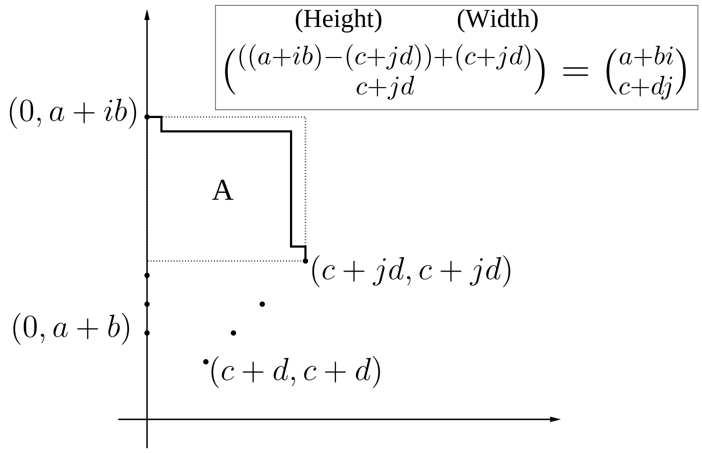

Figure 1: Counting paths from $(c + jd, c + jd)$ to $(0, a + ib)$.

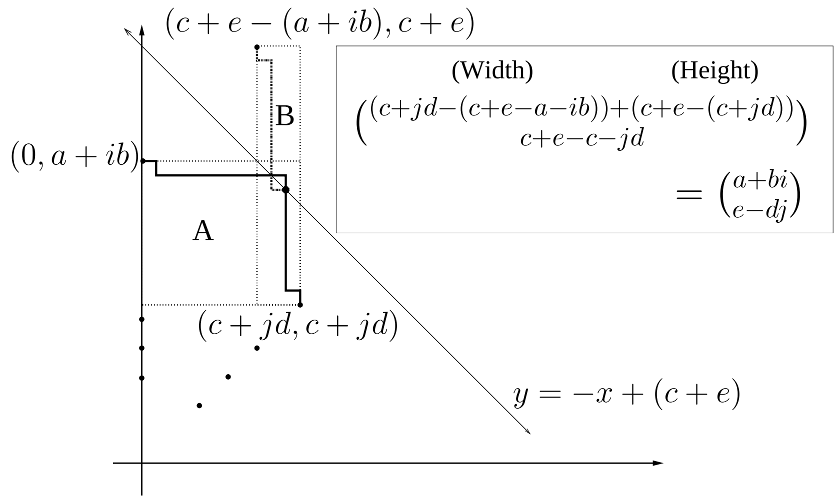

Figure 2: 'Bad' paths from $(c + jd, c + jd)$ to $(0, a + ib)$.

The number of bad paths in rectangle A, namely the ones that go through the forbidden line, is in bijection with the number of total paths in rectangle B; we replace WEST steps with NORTH steps and NORTH steps with WEST steps following the first touch of the forbidden line. This is known as André's

Reflection Principle [4]. Thus the number of good paths in rectangle A is exactly the correct difference of binomials.

This shown, a classical involution of Lindström [8] and Gessel-Viennot [5] shows that when the entries of a matrix count paths, the determinant counts families of non-intersecting paths. This completes the proof. $\square$

We now are in a position to prove Theorem 1, as stated in the introduction.

Proof of Theorem 1. We begin by considering a more general form of this matrix and factoring it. This factorization was suggested by an argument of Gessel and Viennot [5]:

$$\det\left|\binom{a_i}{b_j}-\binom{c-a_i}{b_j}\right|_{i,j=1}^{k}=$$

$$\det\left|\frac{\binom{c}{b_{k-i+1}}}{\binom{c}{a_{k-j+1}}}\cdot\left(\binom{c-b_{k-i+1}}{c-a_{k-j+1}}-\binom{c-b_{k-i+1}}{a_{k-j+1}}\right)\right|_{i,j=1}^{k}=$$

$$\frac{\binom{c}{b_1}\cdots\binom{c}{b_k}}{\binom{c}{a_1}\cdots\binom{c}{a_k}}\cdot\det\left|\binom{c-b_{k-i+1}}{c-a_{k-j+1}}-\binom{c-b_{k-i+1}}{a_{k-j+1}}\right|_{i,j=1}^{k}$$

Proposition 14 of [5] used an analogous factorization for the determinant of a matrix of single binomial coefficients. Our factorization also works by the symmetry $\binom{c}{a_i}=\binom{c}{c-a_i}$. This implies that the same quotient of binomials can be factored out of both terms that appear as a difference in our entries. Returning to the proof of Theorem 1, we let $a_i = C+\alpha i$, $b_j = E+\beta j$, and $c = C+D$ and find

$$\det\left|\binom{C+\alpha i}{E+\beta j}-\binom{D-\alpha i}{E+\beta j}\right|_{i,j=1}^{k}=\frac{\binom{C+D}{E+\beta}\binom{C+D}{E+2\beta}\cdots\binom{C+D}{E+n\beta}}{\binom{C+D}{C+\alpha}\binom{C+D}{C+2\alpha}\cdots\binom{C+D}{C+n\alpha}}\bullet$$

$$\det\left|\binom{C+D-E-(k-i+1)\beta}{D-(k-j+1)\alpha}-\binom{C+D-E-(k-i+1)\beta}{C+(k-j+1)\alpha}\right|_{i,j=1}^{k}.$$

Notice that now the tops of the binomial coefficients are the same and the bottoms are different. This allows us to apply Theorem 2 to obtain the result. $\square$

We now see how these results can help us with our system of equations. Notice that in (25) there are $\binom{m+2}{2}$ coefficients $C_{[i,j]}$ and $m(m+1)$ equations. We define B_m as the restriction of the matrix given by (25) to the $(\binom{m+2}{2}-1)\times(\binom{m+2}{2}-1)$ sub-matrix where $[i,j]\neq[m,m]$, $0\le k\le m-1$ and $l\in\{2m-2k-1,\ldots,2m-3,2m-1\}$ or $k=m$ and $l\in\{1,3,5,\ldots,2m-1\}$.

Lemma 4. *The matrix B_m is nonsingular*

Proof. By using an ordering for the pairs (k, l) where the k's increase and the l's decrease while lexicographically ordering the $[i, j]$s, the matrix B_m becomes block triangular. Furthermore, there is one block of size 1, one block of size 2, ..., one block of size $m-1$, and two blocks of size m. This block triangularity follows from the fact that for i, j such that $0 \le j \le i < k$, then $\binom{i}{k} = \binom{j}{k} = 0$ and thus the $A_{i,j,k,l}$'s of equation (25) are all zero.

Furthermore, the entries of B_m inside these blocks, where j runs over the interval $0 \le j \le i = k$, are much simpler than the general case. For such i, j's, the $A_{i,j,k,l}$'s of equation (25) simplify to

$$A_{k,j,k,l} = \begin{cases} \binom{k}{k}\binom{d-2k}{l} - \binom{k}{k}\binom{k}{l} & \text{if } j = k, \\ \binom{k}{k}\binom{d-j-k}{l} + \binom{j}{k}\binom{d-2k}{l} - \left(\binom{k}{k} + \binom{j}{k}\right)\binom{j}{l} & \text{otherwise.} \end{cases}$$

But since $\binom{k}{k} = 1$ and $\binom{j}{k} = 0$ if $j < k$ we obtain

$$A_{k,j,k,l} = \binom{d-j-k}{l} - \binom{k}{l}. \tag{26}$$

For $f \in \{1, 2, \dots, m\}$, we let $B^{f,m}$ denote the f^{th} block matrix on the diagonal of B_m, which forces $f = k+1$, and set B^m to be the final block matrix. Setting $d = 3m+1$ and utilizing (26) allows us to describe the entries of these blocks as follows:

For $j \in \{0, \dots, f-1\}$ and $l \in \{2m-2f+1, 2m-2f-1, \dots, 2m-1\}$,

$$B^{f,m}_{l,j} = \binom{3m+1-j-(f-1)}{l} - \binom{j}{l}$$

and for $j \in \{0, \dots, m-1\}$ and $l \in \{1, 3, \dots, 2m-1\}$,

$$B^{m}_{l,j} = \binom{2m+1-j}{l} - \binom{j}{l}.$$

At this point, we re-index the matrix $B^{f,m}$, replacing the current indices of j and l with the standard indices $i, j \in \{1, \dots, f\}$. This gives

$$B^{f,m} = \left| \binom{3m+1-(j-1)-(f-1)}{2m+1-2i} - \binom{j-1}{2m+1-2i} \right|_{i,j=1}^{f} \tag{27}$$

and

$$B^m = \left| \binom{2m+1-(j-1)}{2m+1-2i} - \binom{j-1}{2m+1-2i} \right|_{i,j=1}^{m}. \qquad (28)$$

Applying Theorem 1 to the transpose of this matrix, we find the determinant of (27) is

$$\frac{\binom{3m+2-f}{2m-1}\binom{3m+2-f}{2m-3}\cdots\binom{3m+2-f}{2m-2f+1}}{\binom{3m+2-f}{3m+2-f}\binom{3m+2-f}{3m+1-f}\cdots\binom{3m+2-f}{3m-2f+3}} \cdot |\mathcal{F}|$$

where $\mathcal{F}$ is the set of families of non-intersecting lattice paths from $\{(0,0),(1,1),\ldots,(f-1,f-1)\}$ to $\{(0,m-f+3),(0,m-f+5),\ldots,(0,m+f+1)\}$ which stay below the line $y=-x+3m+2-f$. Since this family of paths is non-empty, we conclude that the matrices $B^{f,m}$ are non-singular for $f\in\{1,2,\ldots,m\}$. Similarly we find that the determinant of (28) is positive and thus B^m is also non-singular. Since the diagonal blocks of B_m are non-singular, the matrix B_m must also be. $\square$

An example may help to clarify things at this point. When $m=3$ and $d=10$, we have the matrix

$$\begin{bmatrix}
252 & 378 & 126 & 308 & 182 & 56 & 273 & 147 & 75\\
0 & 126 & 56 & 252 & 133 & 42 & 378 & 174 & 75\\
0 & 84 & 56 & 168 & 147 & 68 & 252 & 184 & 125\\
0 & 0 & 0 & 56 & 21 & 6 & 168 & 63 & 19\\
0 & 0 & 0 & 56 & 35 & 20 & 168 & 105 & 66\\
0 & 0 & 0 & 8 & 6 & 4 & 21 & 15 & 11\\
0 & 0 & 0 & 0 & 0 & 0 & 21 & 6 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 35 & 20 & 10\\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 5 & 3
\end{bmatrix}$$

This matrix is the matrix of coefficients $A_{i,j,k,l}$ where the columns are indexed by the $[i,j]$'s and the rows are indexed by the pairs (k,l). In this example, the columns have the order

$$[i,j]=[0,0],\ [1,0],\ [1,1],\ [2,0],\ [2,1],\ [2,2],\ [3,0],\ [3,1],\ [3,2]$$

and the rows have the order:

$$(k,l)=(0,5),\ (1,5),\ (1,3),\ (2,5),\ (2,3),\ (2,1),\ (3,5),\ (3,3),\ (3,1).$$

We also have the following block sub-matrices:

$$B^{1,3}=\begin{bmatrix}252\end{bmatrix}$$

$$B^{2,3}=\begin{bmatrix}126 & 56\\ 84 & 56\end{bmatrix}$$

$$B^{3,3} = \begin{bmatrix} 56 & 21 & 6 \\ 56 & 35 & 20 \\ 8 & 6 & 4 \end{bmatrix}$$

$$B^{3} = \begin{bmatrix} 21 & 6 & 1 \\ 35 & 20 & 10 \\ 7 & 5 & 3 \end{bmatrix}$$

Lemma 5. *The equations given in (25) have a solution that is unique up to scalar multiples.*

Proof. Since the system in (25) has an $\left(\binom{m+2}{2} - 1\right) \times \left(\binom{m+2}{2} - 1\right)$ nonsingular sub-matrix, it must be true that the rank of the system in (25) is $\geq \binom{m+2}{2} - 1$. Thus the null space has dimension ≤ 1. However, since we know by Lemma 2 that A_1 is a solution, the dimension of the null space must be exactly one. □

4 Constructing A_2 and a basis for the quotient

In the first section, we showed the existence of nonzero (in the quotient) m-quasiinvariants A_1, A_2 of degrees $3m+1$ and $3m+2$, respectively, that are both symmetric with respect to s_{23}. In the previous two sections we illustrated an explicit construction of the element A_1. We now give an explicit construction of the element A_2, which will be linearly independent of A_1. We have deferred this construction until now since this argument is dependent on the explicit form of A_1. We begin by strengthening Lemma 2.

Lemma 6. *There exists an m-quasiinvariant of degree $3m+2$, satisfying the conditions of Lemma 1, which has the form given in equation (24).*

Proof. First, we observe that the Hilbert series (1) and Lemma 2 tell us there is a nonzero m-quasiinvariant of degree $3m+1$,

$$A_1 = \sum_{0 \leq i \leq j \leq m} C_{[i,j]} x_1^{3m+1-i-j} m_{[i,j]}(x_2, x_3),$$

as well as a nonzero m-quasiinvariant of degree $3m+2$,

$$A_2 = \sum_{0 \leq i \leq j \leq m+1} \tilde{C}_{[i,j]} x_1^{3m+2-i-j} m_{[i,j]}(x_2, x_3).$$

We proved in the last section that the set of possible coefficient vectors $\langle C_{[i,j]} \rangle$ comprises a 1-dimensional space. Eliminating the last column of the matrix of entries $A_{i,j,k,l}$'s is like setting the coefficient $C_{[m,m]} = 0$. Since the sub-matrix B_m also lacks that column and is nonsingular we conclude that the nonzero m-quasiinvariant A_1 satisfies $C_{[m,m]} \neq 0$. Consequently, $e_1 A_1$ has a nonzero multiple of $x_1^{m+1} x_2^{m+1} x_3^{m}$ as one of its terms while at the same time the term $x_1^{m} x_2^{m+1} x_3^{m+1}$ will not appear. With no cancellation therefore possible, the quantity $(1-s_{13})e_1A_1$ will contain the term $C(x_1-x_3)^{2m+1}x_2^{m+1}$ for some nonzero C.

Since $(1-s_{13})A_2=(x_1-x_3)^{2m+1}(C'x_2^{m+1}+$ terms of lower order$)$, we find that $(1-s_{13})(C'e_1A_1-CA_2)$ contains no term with x_2^{m+1}. We thus re-define A_2 as the quantity $C'e_1A_1-CA_2$ (which still meets the conditions of Lemma 1). Recall that in the proof of Lemma 2, the crucial step that proved the result for A_1 was the fact that we could eliminate every term in $(1-s_{13})A_1$ containing a power of x_2 exceeding m. Now we can utilize this fact for A_2 also. The rest of the proof goes through as before and we conclude that A_2 can be written as

$$\sum_{0\leq i\leq j\leq m}\tilde{C}_{[i,j]}x_1^{3m+2-i-j}m_{[i,j]}(x_2,x_3). \tag{29}$$

□

We now examine how the construction of A_1 can be applied to construct A_2. In section 2, we used the fact that A_1 had the form

$$\sum_{0\leq i\leq j\leq m}C_{[i,j]}x_1^{3m+1-i-j}m_{[i,j]}(x_2,x_3).$$

to obtain a linear system of relations that the $C_{[i,j]}$'s satisfy. Since we now know that A_2 has an analogous form, namely (29), we can apply the same proof (setting $d=3m+2$) to obtain an analogous system for the $\tilde{C}_{[i,j]}$'s.

These coefficients can be explicitly computed by finding the null space of the matrix given by the linear system

$$\sum_{0\leq j\leq i\leq m}A_{i,j,k,l}\tilde{C}_{[i,j]}=0 \tag{30}$$

for $k\in\{0,\ldots,m\}$ and $l\in\{1,3,5,\ldots,2m-1\}$. As in the A_1 case, this null space is 1-dimensional and we prove this by showing that the matrix $\tilde{B}_m$ is nonsingular, where $\tilde{B}_m$ is the restriction of the matrix given by (30) to the $\left(\binom{m+2}{2}-1\right)\times\left(\binom{m+2}{2}-1\right)$ sub-matrix where $[i,j]\neq[m,m]$, $0\leq k\leq m-1$ and $l\in\{2m-2k-1,\ldots,2m-3,2m-1\}$ or $k=m$ and $l\in\{1,3,5,\ldots,2m-1\}$.

The matrix $\tilde{B}_m$ is block triangular and thus we prove that it is nonsingular by proving that its blocks

$$\tilde{B}^{f,m}=\left|\binom{3m+2-(j-1)-(f-1)}{2m+1-2i}-\binom{j-1}{2m+1-2i}\right|_{i,j=1}^{f} \tag{31}$$

for $f \in \{1, 2, \ldots, m\}$ as well the additional block

$$\tilde{B}^m = \left| \binom{2m+2-(j-1)}{2m+1-2i} - \binom{j-1}{2m+1-2i} \right|_{i,j=1}^{m} \tag{32}$$

are nonsingular. We proceed identically to our computation of the determinant of (27). We find that the determinant of (31) is a positive scalar multiplied by the number of families of non-intersecting lattice paths from $\{(0,0), (1,1), \ldots, (f-1, f-1)\}$ to $\{(0, m-f+4), (0, m-f+6), \ldots, (0, m+f+2)\}$ which stay below the line $y = -x + 3m + 3 - f$. Since such paths exist, this determinant is positive. Similarly we find that the determinant of (32) is positive and thus our construction of A_2 is valid.

Theorem 3. *The set* $\{1, A_1, s_{12}(A_1), A_2, s_{12}(A_2), \Delta^{2m+1}(x)\}$ *is a basis for the quotient* $QI_m / \langle (e_1, e_2, e_3) \rangle$.

Proof. It remains only to prove the independence of $\{A_1, s_{12}(A_1), A_2, s_{12}(A_2)\}$ in the quotient. By examining the Hilbert series of QI_m (2), we find that the subspace of $\mathbb{Q}[x_1, x_2, x_3]$ consisting of $3m+2$ dimensional m-quasiinvariants which are not symmetric is 4 dimensional. Thus it is spanned by $e_1 A_1, e_1(s_{12})A_1$ and two other elements. Since we have shown that A_i and $s_{12}A_i$ are linearly independent for $i \in \{1, 2\}$, it remains to show that there is no nontrivial collection of constants c_1, c_2, c_3, c_4 such that

$$c_1 e_1 A_1 + c_2 e_1 s_{12} A_1 + c_3 A_2 + c_4 s_{12} A_2 = 0. \tag{33}$$

We first note that

$$A_2 \neq c e_1 A_1 \quad \text{for any} \quad c. \tag{34}$$

This is seen by examining the terms containing x_2^{m+1} in each, as was done in the proof of Lemma 6. Now assume that (33) held. Applying $s_{13}\pi_2$ gives

$$c_2 e_1 A_1 + c_4 A_2 = 0 \tag{35}$$

which is in immediate contradiction of (34), unless $c_2 = c_4 = 0$. Returning to (33) gives

$$c_1 e_1 A_1 + c_3 A_2 = 0. \tag{36}$$

Again (34) forces $c_1 = c_3 = 0$. This completes the proof. $\square$

We have thus reduced the problem of finding a basis for the quasiinvariants of S_3 to finding the 1-dimensional nullspace of particular matrices, a computation easily carried out by computer. We have used this technique to explicitly compute the basis for several small values of m. We conclude with the following examples:

For $m = 1$,

$$A_1 = x_1^4 - 2x_1^3(x_2 + x_3) + 6x_1^2(x_2x_3)$$
$$A_2 = x_1^5 - \frac{5}{3}x_1^4(x_2 + x_3) + \frac{10}{3}x_1^3(x_2x_3).$$

For $m = 2$,

$$\begin{aligned}A_1 &= x_1^7 - \frac{7}{2}x_1^6(x_2 + x_3) + 14x_1^5(x_2x_3) + 72x_1^5(x_2^2 + x_3^2)\\ &\quad - \frac{35}{2}x_1^4(x_2^2x_3 + x_2x_3^2) + 35x_1^3x_2^2x_3^2\\ A_2 &= x_1^8 - \frac{16}{5}x_1^7(x_2 + x_3) + \frac{56}{5}x_1^6(x_2x_3) + \frac{14}{5}x_1^6(x_2^2 + x_3^2)\\ &\quad - \frac{56}{5}x_1^5(x_2^2x_3 + x_2x_3^2) + 14x_1^4x_2^2x_3^2.\end{aligned}$$

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