Cluster Algebras, Somos Sequences and Exchange Graphs

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Abstract

In this thesis, we will investigate the theory of cluster algebras, a recently created combinatorial theory that is still developing. Cluster algebras are not only intrinsically interesting, but have useful applications to the theory of Somos sequences and Laurent polynomials, generalized associahedra and many other fields. We will concentrate on an axiomatic development of cluster algebras, motivating them by their aforementioned applications. We will end with several open problems and conjectures. This exposition will utilize semisimple Lie algebras and root systems; however, the necessary results from these mathematical areas will be presented here and developed as needed. This should be accessible to anyone familiar with graph theory and recurrence relations.

Contents

1 Introduction

This thesis surveys the work of Sergey Fomin and Andrei Zelevinksy in the development of cluster algebras. Let us spend a moment explaining the significance of this theory. Their theory of cluster algebras is a unifying framework which has produced more and more applications the more it is developed. Prior to the theory of cluster algebras, the sequences Somos-4 and Somos-5 had been proven to be integer sequences by several people including Janice Malouf and George Bergman [10]. The integrality of Somos-6 and Somos-7 had been proven by Raphael Robinson [10]. However, the method of cluster algebras provides a unified proof for the integrality of Somos-4 through Somos-7 as well as the integrality of a number of other sequences as described in [8]. More importantly, cluster algebras hint at a deep connection between this solution in the area of Laurent polynomial theory and their solution to a problem concerning the explicit factorization of totally positive matrices into elementary Jacobi matrices [6, 25]. Connections between cluster algebras and algebraic topological objects such as the associahedron have also been discovered more recently [3]. Though the theory has surprising applications, Zelevinsky (personal communication) has stated that he is most excited by the intrinsic beauty and elegance of the theory; they are an interesting object of study in their own right.

Fomin and Zelevinsky were motivated to create cluster algebras based on empirical properties of the dual canoncial bases found in total positivity theory. We will discuss this connection more in the appendix, however, our main focus will be the applications to Somos sequences and the properties of exchange graphs.

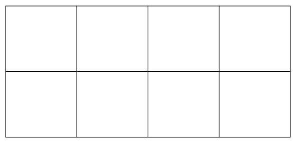
Acknowledgements. I am deeply indebted to my thesis advisor, Richard Stanley, for all of his guidance. For all of his suggestions and editorial suggestions, I have great gratitude. Also I very much appreciate Sergey Fomin's and Andrei Zelevinsky's willingness to discuss their theory more in depth with me. Sergey Fomin helped me better understand the beautiful geometry behind the theory of cluster algebras, and Zelevinsky showed me the intrinsic elegance of this developing theory. This year, I have been involved with a research group, REACH (Research Experiences in Algebraic Combinatorics at Harvard). My experience during this project has been invaluable for the completion of this exposition. I wish to thank all of the members of REACH, especially Jim Propp, who's encouragement as the group leader and expertise assisted me greatly. His insights concerning Somos sequences were particularly helpful. In addition, I am very thankful for the aid of David Speyer, another member of REACH, and his insights about Laurent polynomials. Lastly, I would like to thank my friends Eiichi Miyasaka and Harvey Wun for their editorial and technical support.

2 Laurentness and Somos Sequences

Consider the sequence $f_n = \frac{f_{n-1}^2+1}{f_{n-2}}$ $(f_n f_{n-2} = f_{n-1}^2 + 1)$. At first glance, this sequence appears to be a sequence of non-integral rational numbers, even if one lets $f_0 = f_1 = 1$. However, after computing several terms of the sequence, one finds that $f_2 = 2, f_3 = 5, f_4 = 13, f_5 = 34, \ldots$. Not only are these all integers, but they are every other Fibonacci number. One might believe this pattern continues despite the denominator in the recursion.

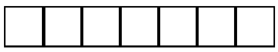
In fact this pattern will continue. There is a trivial proof by induction, but for our purposes, a proof by combinatorial interpretation is more edifying.

We define $G_{m,n}$ to be the $m \times n$ grid graph where there are m vertices in each column and each row has n vertices.



The grid graph $G_{3,5}$.

Let f_n be the number of perfect matchings in $G_{2,2(n-1)}$ such as

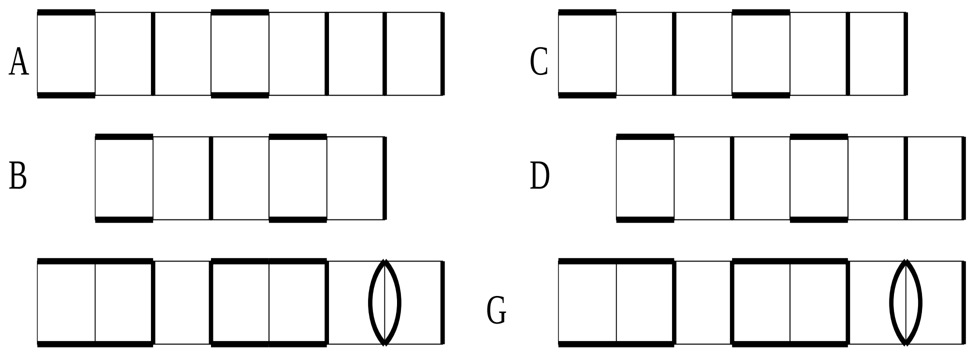


for $n = 4$. By convention we will set $f_0 = f_1 = 1$ and one can readily check that $f_2 = 2$. One can show using Eric Kuo's technique of *graphical condensation* that f_n satisfies the recurrence $f_n f_{n-2} = f_{n-1}^2 + 1$ [17].

The following proof is from [20] based on [17]. Consider the set of ordered pairs $(A, B) \in T_n \times T_{n+2}$ where T_n is the set of perfect matchings of $G_{2,2(n-1)}$. Since $|T_n| = f_n$, the number of such pairs is exactly $f_n f_{n+2}$. Similarly the set of pairs (C, D) from $T_{n+1} \times T_{n+1}$ will have cardinality f_{n+1}^2.

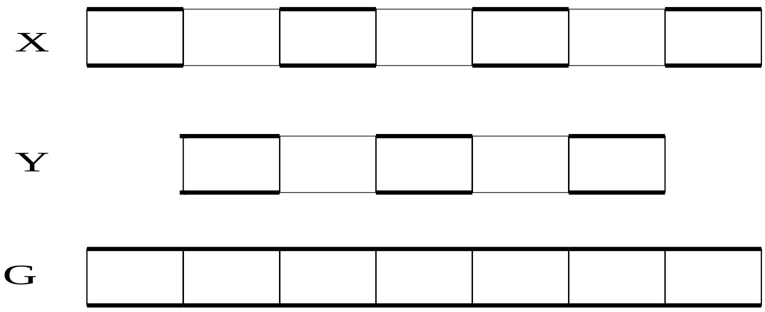
Lemma 1 *There is a bijection from $T_n \times T_{n+2} - (X, Y)$ to $T_{n+1} \times T_{n+1}$ where X and Y are specific instances of perfect matchings as pictured below.*

Sketch of Proof. One can superimpose a matching A and a matching B onto a $2 \times 2(n + 1)$ grid graph G with distinguished edge set M_{AB} (allowing double edges) so that the matching B is centered on G. Each vertex of G (except those on the outer boundary) will have two distinguished edges emanating from it. Similarly one can superimpose matchings C and D onto the same $2 \times 2(n + 1)$ grid graph G with distinguished edge set M_{CD} where C is left-justified and D is right-justified with respect to G.



An example of such a superposition.

Notice that the double matching on graph G can be decomposed into the matchings (A, B) or the matchings (C, D). This decomposition is not unique. However the number of decomposition into (A, B) is the same as the number of decompositions into a pair (C, D).

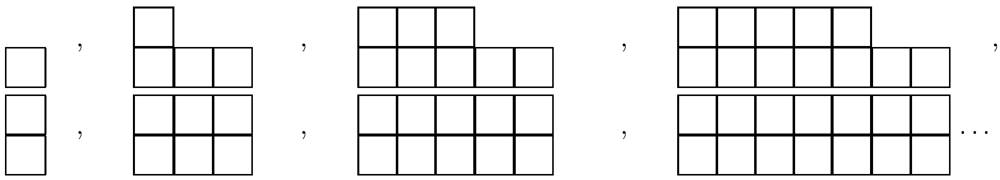


An undecomposable pair.

The correspondence between decompositions will be valid for all pairs of matchings but one, (X, Y). For the pair (X, Y), it is not possible to superimpose (X, Y) together on G and then decompose it into two matchings of left- and right- justified graphs. For all other pairs, there is a bijection (counting multiplicities) between the two decompositions, hence there is a bijection between $T_n \times T_{n+2} - (X, Y)$ and $T_{n+1} \times T_{n+1}$.

Since f_n is a function that counts an actual object, it is clear that f_n must be a nonnegative integer for all $n \geq 1$.

Similar techniques work for sequences such as $g_n g_{n-3} = g_{n-1} g_{n-2} + 1$ where $g_0 = g_1 = g_2 = 1$, $\{g_n : n \geq 3\} = 2, 3, 7, 11, 26, 41, 97, 153, \ldots$. In fact, this counts the number of perfect matchings of the family of graphs:



Ira Gessel [11] noticed that the sequence $\{g_{2n}\} = (1, 3, 11, 41, \ldots)$ appeared on Neil Sloane's website, the Encyclopedia of Integer Sequences [22]. On this site, the sequence was noted to have the combinatorial interpretation of counting domino tilings of a $3 \times 2(n-1)$ rectangle, which implies it counts the number of perfect matchings of $G_{3,2(n-1)}$. Eric Kuo noted that the terms $\{g_{2n+1}\} = (1, 2, 7, 26, 97, \ldots)$ counting the number of "mutilated" $3 \times 2(n-1)$ grid graphs [16]. By mutilated $3 \times 2(n-1)$ grid graphs, we mean graphs resembling the ones in the top row of the previous figure, i.e. they are $3 \times 2(n-1)$ grid graphs where the rightmost two vertices in the top row, along with their incident edges, have been removed.

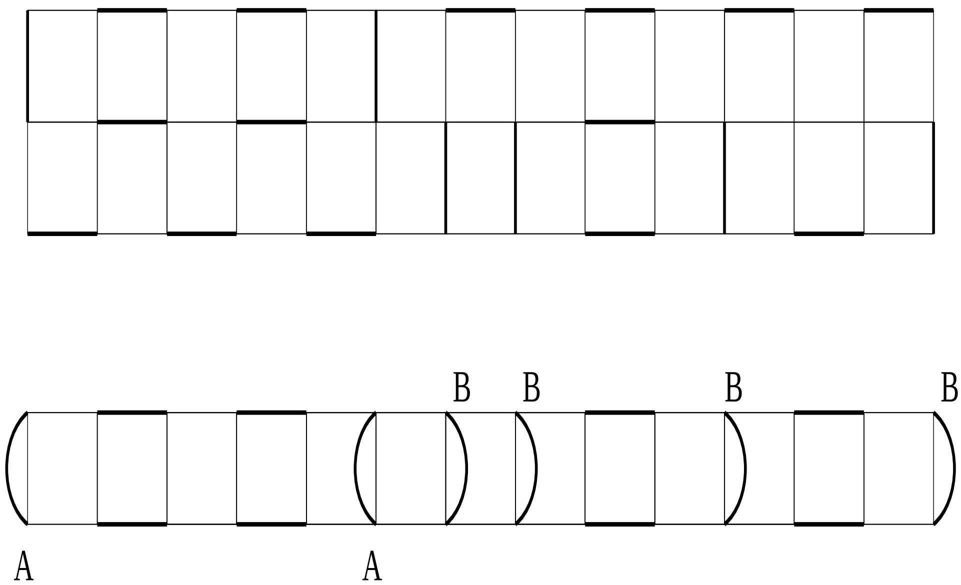
The following is an original proof of a direct bijection between the perfect matchings of $G_{3,2(n-1)}$ and the perfect matchings of $\tilde{G}_{2,2(n-1)}$, a $2 \times 2(n-1)$ grid multi-graph where each vertical edge has been replaced with two vertical edges (labeled A and B) and the vertical edges are paired off so that each pair of consecutive vertical edges in the matching use the same label.

The bijection is as follows: whenever a vertical edge appears in a matching M of $G_{3,2(n-1)}$, it will either be an edge from the 2nd row to the 3rd row, or the 1st row to the 2nd. If it is from the 2nd to the 3rd, then the corresponding matching of the $2 \times 2(n-1)$ grid multi-graph $\tilde{G}_{2,2(n-1)}$ has the vertical edge labeled A in the corresponding column. If it is from the 1st to the 2nd, use the edge labeled B.

Claim 1 *Once these vertical edges have been specified there is a unique choice of horizontal edges that will complete M to a perfect matching.*

Claim 2 *Each consecutive pair of vertical edges will be in the same row.*

These claims are easily verified by studying the possible perfect matchings of $G_{3,2(n-1)}$.



A pair of corresponding matchings of $G_{3,14}$ and $\tilde{G}_{2,14}$.

Furthermore, the number of perfect matchings in $\tilde{G}_{2,2(n-1)}$ is the same as the weighted number of perfect matchings in $G_{2,2(n-1)}$ where we give a matching that uses m vertical edges weight 2^m. Let $\tilde{f}_n$ be the number of perfect matchings in $\tilde{G}_{2,2(n-1)}$.

Using graphical condensation, one can show that just as f_n satisfies the recurrence $f_n f_{n-2} = f_{n-1}^2 + 1$, $\tilde{f}_n$ satisfies the recurrence $\tilde{f}_n \tilde{f}_{n-2} = \tilde{f}_{n-1}^2 + 2$.

In fact if $f_{n,w}$ is the weighted number of perfect matchings in $G_{2,2(n-1)}$ where we give a matching that uses m vertical edges weight w^m, then $f_{n,w}f_{n-2,w} = f^2_{n-1,w}+w$. Consequently, every other term of the sequence g_n ($g_{2n} = 1, 3, 11, 41, \dots$) satisfies the recurrence $g_{2n}g_{2n-4} = g^2_{2n-2} + 2$.

2.1 Somos Sequences

So far we've seen two rational recurrences give rise to integer sequences. What about the sequence

$$s_n s_{n-4} = s_{n-1}s_{n-3} + s^2_{n-2}$$

where $s_1 = s_2 = s_3 = s_4 = 1$? This sequence is called Somos-4 where a general Somos-k sequence is a sequence of the form $S_n S_{n-k} = S_{n-1}S_{n-k+1} + S_{n-2}S_{n-k+2} + \dots$. Such sequences were discovered by Michael Somos while he was studying recurrences resembling relations found among elliptic functions. More can be found about Somos sequences in David Gale's article [10] or Jim Propp's website [19]. Somos-4 is in fact a sequence of positive integers, however assigning a combinatorial interpretation to s_n (like in the case of f_n or g_n) was an open problem until recently.[1]

As mentioned in the introduction, one way to prove the integrality for the sequence Somos-4 involves using cluster algebras. Fomin and Zelevinsky in fact can prove a much more general result using their technique [8]. Before describing the use of cluster algebras to prove Laurentness, we will consider a simpler problem based on the work of David Speyer, an example which is also a special case of the Laurent phenomenon discussed in [8]. The following is David Speyer's proof from an email to REACH [23].

Consider a sequence x_n that satisfies the recurrence $x_n x_{n-2} = p(x_{n-1})$ for $n \geq 3$ where $p(t)$ is a univariate polynomial.

Definition 1 A *Laurent polynomial* over the variables $x_1, \dots, x_n$ is a finite sum of terms where the variables $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ appear rather than just $x_1, \dots, x_n$ as in the case of a polynomial.

Another way to think of a Laurent polynomial is as a rational function in $x_1, \dots, x_n$ where the denominator consists of a single monomial. Let $R = \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}]$ be the ring of Laurent polynomials in the variables x_1 and x_2 with coefficients in $\mathbb{Q}$.

$$\text{When is } x_n \in R \ \ \forall\, n \geq 1?$$

Proposition 1 *As long as $p(0) \neq 0$, all of the $x_n \in R$ if and only if $p(t) = c \cdot t^{\deg p} \cdot p\left(\frac{p(0)}{t}\right)$ for some $c \in \mathbb{Q}$.*

[1] As of March, 2002, this was solved by members of REACH in work to be written up. Like f_n and g_n, the combinatorial interpretation of s_n involves perfect matchings of a family of graphs. Bousquet-Mélou and West also just recently found a combinatorial interpretation using an earlier suggestion from Jim Propp.

Proof. Assume $x_n \in R \ \forall\, n \geq 1$ but that $p(t) \neq c \cdot t^{\deg p} \cdot p\left(\frac{p(0)}{t}\right) \ \forall \ c \in \mathbb{Q}$. Then,

$$x_5 = \frac{p\left(\frac{p\left(\frac{p(x_2)}{x_1}\right)}{x_2}\right)}{\frac{p(x_2)}{x_1}}$$

is Laurent only if $x_1 \cdot p\left(\frac{p\left(\frac{p(x_2)}{x_1}\right)}{x_2}\right) \equiv 0 \mod p(x_2)$ in R. Using $p(x_2) \equiv 0$, this requirement reduces to $x_1 \cdot p\left(\frac{p(0)}{x_2}\right) \equiv 0 \mod p(x_2)$. Since x_1 and x_2 are units in R, we obtain $p(x_2) \Big| \left(x_1^{k_1} x_2^{k_2} \cdot p\left(\frac{p(0)}{x_2}\right)\right)$ for some choices of k_1, k_2. The variable x_1 does not appear in $p(x_2)$ so k_1 must equal 0. Furthermore, the degrees (in terms of x_2) of $p(x_2)$ and $x_2^{k_2} \cdot p\left(\frac{p(0)}{x_2}\right)$ only match if $k_2 = \deg p$. However, in this case, $p(x_2)|x_2^{\deg p} p\left(\frac{p(0)}{x_2}\right)$ implies there exists a $c \in \mathbb{Q}$ such that $p(t) = c \cdot t^{\deg p} \cdot p\left(\frac{p(0)}{t}\right)$, a contradiction.

Now assume $p(t) = c \cdot t^{\deg p} \cdot p\left(\frac{p(0)}{t}\right)$ for some $c \in \mathbb{Q}$. $x_3 = \frac{p(x_2)}{x_1}$ and $x_4 = \frac{p\left(\frac{p(x_2)}{x_1}\right)}{x_2}$ are in R along with x_1 and x_2. From this base case of four elements we will inductively show that all $x_n \in R$.

Claim 3 *Suppose* $x_{n+1}, x_{n+2}, x_{n+3}$, *and* $x_{n+4} \in R$. *Then* $x_{n+5} \in R$.
By the defining recurrence of the sequence, $x_{n+2}x_{n+4} = p(x_{n+3})$ which is equivalent to $p(0) \mod x_{n+3}$. The term $p(0)$ is nonzero and rational therefore $p(0)$ is a unit which implies that x_{n+2} and x_{n+4} are also units. Now we can divide freely by x_{n+2} and x_{n+4}. Thus

$$p(x_{n+4}) \equiv p\left(\frac{p(0)}{x_{n+2}}\right) \equiv \frac{1}{c x_{n+2}^{\deg p}} p(x_{n+2}) \equiv$$

$$\frac{1}{c x_{n+2}^{\deg p}} x_{n+1} x_{n+3} \equiv 0 \mod x_{n+3}.$$

Consequently $x_{n+5} = \frac{p(x_{n+4})}{x_{n+3}} \in R$. Given this claim, $x_n \in R \ \forall\, n \geq 1$. □

We will later show this result holds if we let $R = A[x_1^{\pm 1}, x_2^{\pm 1}]$ where A is any unique factorization domain. In particular, we could allow A to be $\mathbb{Z}$ and the first two terms x_1, x_2 to be 1. In this case we recover integrality. Thus Laurentness is a more general condition than integrality. Fomin and Zelevinsky's result concerns the question of whether or not all terms of a sequence are Laurent polynomials in terms of the k initial terms. Thus they are able to prove that a sequence satisfying the Somos-4 recurrence $x_0 x_4 = x_1 x_3 + x_2^2$ is a sequence of Laurent polynomials in the initial four terms. In the case that $x_1 = x_2 = x_3 = x_4 = 1$, we get that Somos-4 is a sequence of integers. To understand their proof, we will now introduce the theory of cluster algebras.

2.2 Fomin and Zelevinsky's Definitions

Unless otherwise noted, the material from this section is directly from or based on [7]. Fomin and Zelevinsky define a *cluster algebra* $\mathcal{A}$ as "a commutative ring with unit and no zero divisors, equipped with a distinguished family of generators called *cluster variables*" [7, pg. 1]. The cluster algebra is a (non-disjoint) union of a distinguished collection of subsets called *clusters*. Each of the subsets in this collection have equal size, and this size is known as the rank of $\mathcal{A}$. For every cluster $X = \{x_1, \ldots, x_n\} \subset \mathcal{A}$ in a cluster algebra $\mathcal{A}$ of rank n there exist n clusters $Y_1, Y_2, \ldots, Y_n \subset \mathcal{A}$ adjacent to X. These clusters are adjacent to X because X and each $Y_i = \{x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n\}$ are related by a *binomial exchange relation*

$$x_i y_i = M_i(X) + M_i(Y_i), \tag{1}$$

where $M_i(X)$ and $M_i(Y_i)$ are two relatively prime monomials in the $n-1$ variables $X - \{x_i\}$. For example, the monomial[2] $M_i(X)$ is given by

$$M_i(X) = c_i(X) \prod_{1 \le j \le n, j \ne i} x_j(X)^{b_{ij}(X)}$$

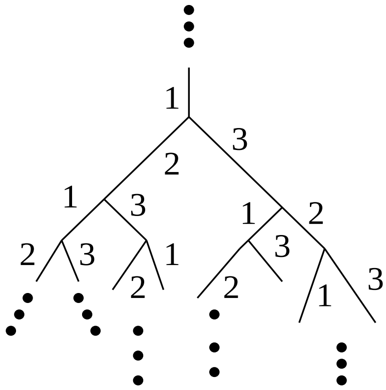
and for a general cluster C, the associated monomial is

$$M_i(C) = c_i(C) \prod_{z \in C} z^{b_{i,z}(C)}.$$

Furthermore, one can switch between any two clusters of $\mathcal{A}$ by a series of such exchanges. Besides the condition that $M_i(C)$ cannot depend on the i^{th} variable of the cluster C, the choice of a family of monomials is restricted by specific axioms which we will explain after some initial definitions.

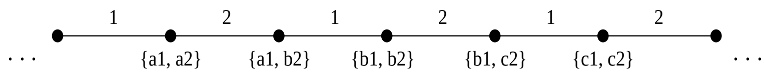
For any cluster algebra of rank n we define an n-regular graph called an *exchange graph* whose vertices are the different clusters, and whose edges correspond to the exchanges between two clusters. If $\mathcal{A}$ is a rank 1 cluster algebra, the only possible exchange graph is a 1-regular graph consisting of two vertices. When $n \ge 2$ and there are no relations between the variables of the various clusters, this graph will be an *exchange tree*, an infinite n-degree graph such that each of the n edges coming out of a given vertex have a unique label out of $\{1, \ldots, n\}$.

[2]Fomin and Zelevinsky allow the coefficients $c_i(X)$ to be chosen from a torsion-free multiplicative abelian group $\mathbb{P}$. However, for the remainder of this exposition, we will assume all of the $c_i(X)$'s are 1, i.e. that $\mathbb{P} = \{1\}$.



An exchange tree for a rank 3 cluster algebra.

Also, an exchange tree $\mathbb{T}$ for a rank 2 cluster algebra is a line.



Notice that whenever $\{x_1, y_2\}$ connects to $\{w_1, z_2\}$ via an edge labeled 1, then $y_2 = z_2$ and if an edge labeled 2 connects them, $x_1 = w_1$. Furthermore, we can define an *exchange pattern* β, a family of exchange binomials $\{B_t\}$, so that $x_1 w_1 = B(y_2)$ for some $B \in \beta$ when edge 1 connects them and $y_2 z_2 = B'(x_1)$ for $B' \in \beta$ when edge 2 connects them. Here I emphasize that the dependent variable of the binomial is determined by the edge label.

One can more formally define the possible exchange patterns that can be associated to a cluster algebra. Here we will assume that the exchange graph is an undirected tree of degree n. We will let $\mathcal{T}$ be the set of vertices in the exchange tree, and will use the notation $E_i(t, t')$ to signify that vertices t and t' are connected by an edge labeled i.

Then if $E_i(t, t')$ we will let the exchange binomial associated with this edge be $M_i(t) + M_i(t')$ where we will let the vertices t, t' stand for the associated clusters. For $\mathcal{A}$ to be a cluster algebra, the exchange pattern $\{M_i(t) : i \in \{1, \ldots, n\},\ t \in \mathcal{T}\}$ must satisfy the following axioms:

$$\text{If } E_j(t_1, t_2), \text{ then } x_i(t_1) = x_i(t_2) \text{ when } i \neq j, \quad (2)$$

$$\text{and } x_j(t_1) x_j(t_2) = M_j(t_1) + M_j(t_2). \quad (3)$$

$$\text{For } t_1 \in \mathcal{T},\ x_j \nmid M_j(t_1). \quad (4)$$

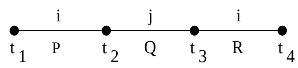
$$\text{If } E_i(t_1, t_2) \text{ and } x_i | M_j(t_1) \text{ then } x_i \nmid M_j(t_2). \quad (5)$$

$$\text{If } E_i(t_1, t_2) \text{ and } E_j(t_2, t_3) \text{ then } x_j | M_i(t_1) \text{ if and only if } x_i | M_j(t_2). \quad (6)$$

$$\text{Suppose } E_i(t_1, t_2),\ E_j(t_2, t_3) \text{ and } E_i(t_3, t_4). \text{ Then } \frac{M_i(t_3)}{M_i(t_4)} = \left(\frac{M_i(t_2)}{M_i(t_1)} \right) \Bigg|_{x_j \leftarrow M_0 / x_j} \quad (7)$$

$$\text{where } M_0 = (M_j(t_2) + M_j(t_3))|_{x_i = 0}.$$

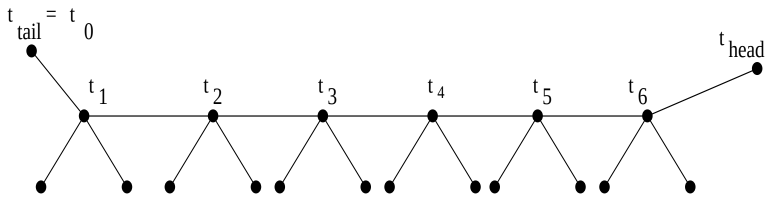
Axiom (7) is the most significant axiom. Axiom (7) will uniquely determine how to propagate the binomial exchanges. Letting $P = M_i(t_1) + M_i(t_2)$, $Q = M_j(t_2) + M_j(t_3)$ and $R = M_i(t_3) + M_i(t_4)$, axiom (7) implies that whenever



appears in the exchange graph or tree, then the exchange binomials P, Q and R satisfy the condition that there exists a Laurent monomial L and nonnegative integer b such that $L \cdot Q_0^b \cdot P = R|_{x_j \leftarrow \frac{Q_0}{x_j}}$ where $Q_0 = Q|_{x_i \leftarrow 0}$ [8, pgs. 8-9].

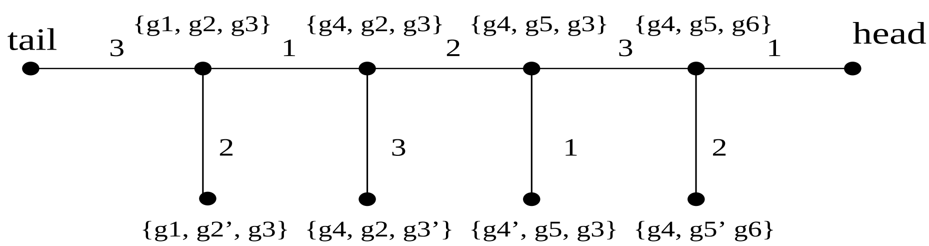
A Laurent monomial is a fraction consisting of a monomial over another monomial and the notation $P|_{x_i \leftarrow a}$ signifies the evaluation of polynomial P where a has been substituted for the variable x_i.

Exchange trees have another graph structure embedded in them: graphs $\mathbb{T}_{m,n}$ which Fomin and Zelevinsky call caterpillars. A caterpillar $\mathbb{T}_{m,n}$ for $m \geq 2$ is defined as a tree with a spine of m vertices of degree n and $m(n-2)+2$ vertices of degree 1. Of the degree-1 vertices, $m(n-2)$ of them will be referred to as feet and the remaining two (which must emanate from the extremities of the spine) will be called the head and the tail.



A caterpillar $\mathbb{T}_{6,4}$.

The clusters on the spine could represent a recursive sequence that we would like to propagate. For example, consider the sequence $g_n = \frac{g_{n-1}g_{n-2}+1}{g_{n-3}}$ Then the associated caterpillar $\mathbb{T}_{4,3}$ would look like



where $g_4 g_1 = g_2 g_3 + 1$, $g_5 g_2 = g_3 g_4 + 1$ and $g_6 g_3 = g_4 g_5 + 1$ are the exchange relations corresponding to the edges of the spine. The legs have different relations and our goal is to show that there are ways to define binomial exchanges corresponding to the leg edges that keep the caterpillar consistent with axioms (2-7) thereby making it part of the exchange graph for a cluster algebra.

To uphold these axioms in the above caterpillar, the polynomial relations associated with the legs will be $g_2' g_2 = g_1 + g_3, g_3' g_3 = g_2 + g_4,\ \ g_4' g_4 = g_3 + g_5$, and $g_5' g_5 = g_4 + g_6$.

Notice now that if we start with the cluster $\{g_1, g_2, g_3\}$ and then travel along edge 1, then edge 2 and edge 1, we get to

$\{\frac{g_2 g_3+1}{g_1}, g_2, g_3\} \to \{\frac{g_2 g_3+1}{g_1}, \frac{(g_2 g_3+1)g_3+g_1}{g_1 g_2}, g_3\}$

$\to \{\frac{g_1}{g_2 g_3+1} \frac{(g_2 g_3+1)g_3+g_1+g_1 g_2 g_3}{g_1 g_2}, \frac{(g_2 g_3+1)g_3+g_1}{g_1 g_2}, g_3\} = \{\frac{g_1+g_3}{g_2}, \frac{(g_2 g_3+1)g_3+g_1}{g_1 g_2}, g_3\}$

$= \{g_4', g_5, g_3\}$

We will study this example in more depth later. As of now, it is noteable that a priori one might expect g_4' to be more complicated then $g_4 = \frac{g_2 g_3+1}{g_1}$ just as g_5 is more complicated than g_2 but it is in fact still a Laurent polynomial. It is just as simple if not simpler than g_4. Furthermore, even though g_5 is more complicated, it also is a Laurent polynomial in the variables x_1, x_2 and x_3. One could construct similar caterpillars with larger spines, $\mathbb{T}_{m,3}$ for arbitrarily large m, and allow the exchange binomial $xy + 1$ to be associated to all of the edges of the spine. We thereby would extend the sequence of g_n and g_n'. It is natural to ask: *Will all of the g_n and g_n' turn out to be Laurent polynomials in terms of the initial variables?* One can answer this in the affirmative and we will find this is the corollary of a more general result.

2.3 The Caterpillar Lemma

The following results and proofs come from [7] and [8]. The motivation for the Caterpillar Lemma is the following observation by Fomin and Zelevinsky.

> One of the main structural features of cluster algebras established in the present paper is the following *Laurent phenomenon*: any cluster variable x viewed as a rational function in the variables of any given cluster is in fact a Laurent polynomial. This property is quite surprising: in most cases, the numerators of these Laurent polynomials contain a huge number of monomials, and the numerators for x moves into the denominator when we compute the cluster variable x' obtained from x by an exchange (1). The magic of the Laurent phenomenon is that, at every stage of the recursive process, a cancellation will inevitably occur, leaving a single monomial in the denominator [7, pg. 3].

Theorem 1 *In a cluster algebra, any cluster variable is expressed in terms of any given cluster as a Laurent polynomial with coefficients in $\mathbb{Z}$.* [3]

[3]The statement of this theorem differs from Fomin and Zelevinsky's formulation in the fact that Fomin and Zelevinsky allow the cluster variables to be written in terms of coefficients from the group ring $\mathbb{ZP}$ but since we previously set $\mathbb{P} = \{1\}$ we only allow for integer coefficients.

Remark 1 Fomin and Zelevinsky conjecture that all of the cluster variables can be expressed using *nonnegative* integer coefficients.

To prove this theorem, we will prove a generalization from [8]. First, we will need to generalize our definition of exchange pattern.

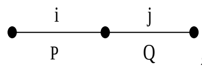
Definition 2 Let $\mathbb{A}$ be a unique factorization domain, and assume that a nonzero polynomial $P \in \mathbb{A}[x_1, \ldots, x_n]$ that does not depend on x_k is associated with every edge such that $E_k(t, t')$ in the exchange tree $\mathbb{T}$. This will be called a *generalized exchange pattern.*

These generalized exchange patterns are analogous to the exchange patterns relying on binomials and $x_k(t)x_k(t') = P(x(t))$.

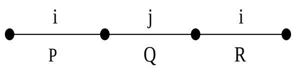
We will label the vertices on the spine of a caterpillar, $\mathbb{T}_{m,n}$, t_1 through t_m and label the tail t_{tail} as t_0.

Lemma 2 *(Caterpillar Lemma) Assume that that a generalized exchange pattern on $\mathbb{T}_{m,n}$ satisfies the following conditions:*

• *For any edge labeled k, the associated exchange polynomial P does not depend on x_k, and is not divisible by any $x_i \in \{x_1, \ldots, x_n\}$.*

• *If two consecutive edges have P and Q associated to them,* 

, then the polynomials P and $Q_0 = Q|_{x_i \leftarrow 0}$ are coprime elements of $\mathbb{A}[x_1, \ldots, x_n]$.

• *If three consecutive edges have P, Q and R associated to them,* 

, then there exists a nonnegative integer b and Laurent monomial L coprime with P with coefficients in $\mathbb{A}$ such that $L \cdot Q_0^b \cdot P = R|_{x_j \leftarrow \frac{Q_0}{x_j}}$ where $Q_0 = Q|_{x_i \leftarrow 0}$.

If those conditions are satisfied, then for every $i \in \{1, \ldots, n\}$, $t \in \mathbb{T}_{m,n}$, $x_i(t)$ is a Laurent polynomial in $X(t_0) = \{x_1(t_0), \ldots, x_n(t_0)\}$ with coefficients in $\mathbb{A}$.

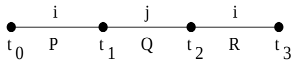
Remark 2 As mentioned previously, this third axiom resembles axiom (7) except now P, Q and R are allowed to be polynomials rather than just binomials.

Proof. For every $t \in \mathbb{T}_{m,n}$, let

$$\mathcal{L}(t) = \mathbb{A}[x_1(t)^{\pm 1}, \ldots, x_n(t)^{\pm 1}]$$

be the Laurent polynomial ring of the cluster $X(t)$ with coefficients in $\mathbb{A}$. We will treat $\mathcal{L}(t)$ as a subring of the field of rational functions of $\mathbb{A}(X(t_0))$.

It suffices to prove that every cluster $X(t) \in \mathcal{L}(t_0) = \mathcal{L}_0$. Since $\mathcal{L}_0$ is a unique factorization domain, elements have a gcd defined up to units of $\mathbb{A}$. We will prove all $X(t) \in \mathcal{L}_0$ by induction on the size of the spine, m. The case $m = 1$ is trivial so we can assume there exists an M such that for all $m \leq M$, the caterpillar lemma is true. Now assume $m \geq 2$. We will prove that $X(t_{head}) \in \mathcal{L}_0$ and be done since t_{head} will be the vertex of the caterpillar furthest from t_0. We will assume $E_i(t_0, t_1)$ and $E_j(t_1, t_2)$. Letting $t_3 \in \mathbb{T}_{m,n}$ be the vertex so that $E_i(t_2, t_3)$, we have



$X(t_1) \cup X(t_2) \cup X(t_3) = X(t_0) \cup \{x_i(t_1), x_j(t_2), x_i(t_3)\}$ and similar to Speyer's proof,

$$x_i(t_1) = \frac{P(x_j(t_0))}{x_i(t_0)}$$

and

$$x_j(t_2) = \frac{Q\big(\frac{P(x_j(t_0))}{x_i(t_0)}\big)}{x_j(t_0)}$$

are clearly in $\mathcal{L}_0$. We now must show that all of the clusters $X(t_1), X(t_2)$ and $X(t_3)$ are contained in $\mathcal{L}_0$ as subsets. This it suffices to prove:

$$\begin{aligned} x_i(t_3) &\in \mathcal{L}_0, & (8)\\ \gcd\big(x_i(t_1), x_j(t_2)\big) &= 1, & (9)\\ \gcd\big(x_i(t_1), x_i(t_3)\big) &= 1. & (10) \end{aligned}$$

By the third axiom stated in the lemma (previously axiom 7), $R\big(\frac{Q(0)}{x_j(t_0)}\big) = L\big(x_j(t_0)\big)Q(0)^b P\big(x_j(t_0)\big)$ where $L\big(x_j(t_0)\big) = L|_{x_j \leftarrow x_j(t_0)}$.

$$x_i(t_3) = \frac{R\big(\frac{Q(x_i(t_1))}{x_j(t_0)}\big)}{x_i(t_1)} = \frac{R\big(\frac{Q(x_i(t_1))}{x_j(t_0)}\big) - R\big(\frac{Q(0)}{x_j(t_0)}\big)}{x_i(t_1)} + \frac{R\big(\frac{Q(0)}{x_j(t_0)}\big)}{x_i(t_1)}.$$

The polynomial $Q(x_i(t_1))$ minus its constant term $Q(0)$ is divisible by $x_i(t_1)$ and extending this property we obtain

$$\frac{R\big(\frac{Q(x_i(t_1))}{x_j(t_0)}\big) - R\big(\frac{Q(0)}{x_j(t_0)}\big)}{x_i(t_1)} \in \mathcal{L}_0 \quad \text{and}$$

$$\frac{R\big(\frac{Q(0)}{x_j(t_0)}\big)}{x_i(t_1)} = \frac{L\big(x_j(t_0)\big)Q(0)^b P\big(x_j(t_0)\big)}{x_i(t_1)} = L\big(x_j(t_0)\big)Q(0)^b x_i(t_0) \in \mathcal{L}_0,$$

thus (8) is true. $x_j(t_2) = \frac{Q\left(x_i(t_1)\right)}{x_j(t_0)} \equiv \frac{Q(0)}{x_j(t_0)} \mod x_j(t_0)$ and $x_i(t_0), x_j(t_0)$ are invertible in $\mathcal{L}_0$ so $\gcd\big(x_i(t_1), x_j(t_2)\big) = \gcd\big(P(x_j(t_0)), Q(0)\big) = 1$ by the the second axiom of generalized exchange patterns. Thus (9) is proved.

To prove (10), we use the fact that

$$x_i(t_3) = \frac{R\big(\frac{Q(x_i(t_1))}{x_j(t_0)}\big) - R\big(\frac{Q(0)}{x_j(t_0)}\big)}{x_i(t_1)} + L\big(x_j(t_0)\big)Q(0)^b x_i(t_0)$$

and taking the limit $x_i(t_1) \to 0$, and applying calculus, we arrive at the equality

$$x_i(t_3) \equiv R'\left(\frac{Q(0)}{x_j(t_0)}\right) \cdot \frac{Q'(0)}{x_j(t_0)} + L(x_j(t_0)Q(0)^b x_i(t_0) \mod x_i(t_1).$$

Since $\gcd\big(L(x_j(t_0)Q(0)^b, P(x_j(t_0))\big) = 1$ thus $\gcd\big(x_i(t_1), x_i(t_3)\big) = 1$.

By the inductive step, the subset $X(t_{head})$ is contained in both $\mathcal{L}(t_1)$ and $\mathcal{L}(t_3)$ since the length of the spine between t_{head} and t_1 or t_3 is less than the distance to t_0. Thus for any $x \in X(t_{head})$, $x = \frac{f_1}{x_i(t_1)^a} = \frac{f_3}{x_j(t_2)^b x_i(t_3)^c}$ for some $f_1, f_3 \in \mathcal{L}_0$ and nonnegative integers a, b, c. By (10), the denominators are relatively prime, hence $x \in \mathcal{L}_0$. $\square$

This Lemma is a generalization of Theorem 1 for the following reasons, as explained in [7].

• $\mathbb{T}_{m,n}$ can be embedded in $\mathbb{T}_n$.

• We are allowing polynomials with appropriate restrictions instead of binomials.

• We are allowing any unique factorization domain $\mathbb{A}$ instead of just $\mathbb{Z}$.

In the special case of $k = 2$ and $\mathbb{A} = \mathbb{Q}$, we get the condition $L_1 \cdot P\left(\frac{P(0)}{t_2}\right) \equiv P(t_2) \mod t_3$, $L_1 \in \mathbb{Q}[t_2^{\pm 1}]$, which exactly matches Speyer's result. However, unlike David Speyer's proof, the converse of the caterpillar lemma does not hold.

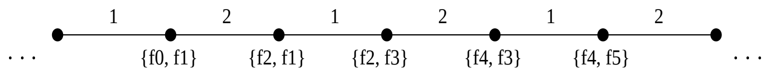
2.4 Sample Proofs for Laurentness of Sequences

The following proofs are based on proofs given in [8]. More details have been included below. Consider the sequence $f_n f_{n-2} = f_{n-1}^2 + 1$ from section 2. We can show that for all $n \geq 1$, f_n is a Laurent polynomial where only a monomial of the form $f_0^a f_1^b$ appears in the denominator.

Proof. We can create the following sequence of clusters

$$\{f_0, f_1\}, \{f_2, f_1\}, \{f_2, f_3\}, \{f_4, f_3\}, \ldots$$

and make an exchange tree of rank 2 using the clusters as vertices and label edges with an alternating pattern of 1 and 2. We will use the exchange binomial $P(t) = t^2 + 1$ for all edges. Thus $f_2 f_0 = P(f_1)$, $f_3 f_1 = P(f_2)$, etc.



By the caterpillar lemma, it suffices to show that such a choice of an exchange pattern is consistent with the axioms of a cluster algebra. We have an alternating pattern •—1— •—2— •—1—• thus axiom (7) requires there exists a Laurent monomial $L = c \cdot t^d$ s.t.

$$P(t) = L \cdot P\left(\frac{P(0)}{t}\right)$$

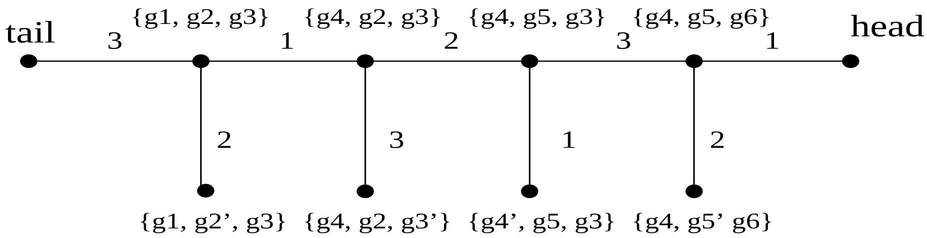
In fact, letting $L = t^2$,

$$L \cdot \left(\frac{0^2 + 1}{t}\right)^2 + 1 = 1 + t^2 = P(t).$$

We can similarly prove that $\{g_n\}$ from section 2 is a Laurent sequence. We build the clusters

$$\{g_0, g_1, g_2\}, \{g_3, g_1, g_2\}, \{g_3, g_4, g_2\}, \{g_3, g_4, g_5\}, \ldots$$

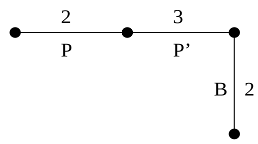
and the caterpillar



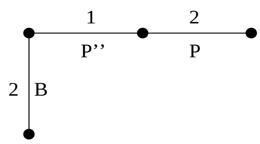
where each edge on the spine will have the exchange binomial $P(x, y) = xy + 1$. However, now it is important to specify the domain variables: letting n be an arbitrary integer with the property $n \equiv 0 \mod 3$ an edge labeled 1 will use g_{n+2} and g_n as domain variables, an edge labeled 2 will use g_{n+1} and g_n, and an edge labeled 3 will use g_{n+1} and g_{n+2}.

From each vertex on the spine, there will be three edges stemming from it, but only two of those edges will be contained in the spine. To show $\{g_n\}$ is Laurent, it suffices to show that the remaining edges, the *legs* can be chosen so that the axioms of a cluster algebra are satisfied.

Specifically, we must be able to choose a binomial B such that



with $P = P(g_{n+3}, g_{n+1})$, $P' = P(g_{n+1}, g_{n+2})$ satisfies axiom (7) for some Laurent monomial $L = c \cdot x^a y^b$ and



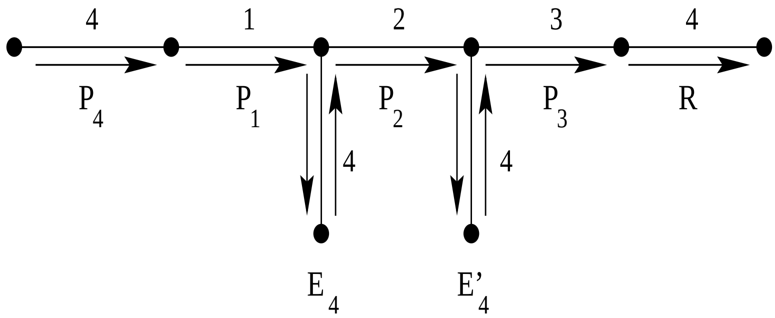
with $P'' = P(g_{n+2}, g_{n+3})$ satisfies axiom (7) for some Laurent monomial $L' = c' \cdot x^{a'} y^{b'}$.

The first relation uniquely determines B. In this particular example $B = x + y$ and plugging this B into the second relation we find that the second relation is indeed satisfied. Thus we conclude $\{g_n\}$ is a Laurent sequence.

Assigning $f_0 = f_1 = 1$ or $g_0 = g_1 = g_2 = 1$ we can conclude that the sequences f_n and g_n are integer sequences. For these two examples we already knew this since we have a combinatorial interpretation of f_n or g_n. However, the beauty of this Cluster Algebra method is that it can prove Laurentness even when there is no combinatorial interpretation of a sequence. The next proofs will demonstrate Laurentness for Somos-4, Somos-5, Somos-6, and Somos-7.

2.4.1 Proof of Laurentness for Several Somos Sequences

For the Somos-4 sequence s_n we create a spine of clusters of size 4 such that each cluster only contains a window $s_{n+1}, s_{n+2}, s_{n+3}, s_{n+4}$ for some n. We build the caterpillar starting arbitrarily at a vertex between edges labeled 4 and 1 on the spine, and create the associated leg with edge label 4. We then continue back onto the spine and find the exchange polynomial associated with the next edge labeled 4. This edge will also be a leg of the caterpillar. Finally, by using the spine edge labeled 3, we conclude that this series of exchanges is consistent with the axioms of the caterpillar lemma. Thus Somos-4 is a Laurent sequence, a sequence of Laurent polynomials in the first four terms.



A caterpillar for the Somos-4 sequence.

$$
\begin{aligned}
P_4 &= x_1x_3 + x_2^2 \quad & P_1 = x_2x_4 + x_3^2 \\
E_4 &= x_3^3 + x_2^2x_1 \quad & P_2 = x_3x_1 + x_4^2 \\
E_4' &= x_3x_2^2 + x_1^3 \quad & P_3 = x_4x_2 + x_1^2 \\
R &= x_1x_3 + x_2^2 \quad \square &
\end{aligned}
$$

The summary of the calculations for Somos-5, Somos-6, and Somos-7 are below. They are also calculated constructing associated caterpillars according to axiom (7). Since P_5, P_6 and P_7 respectively equal R for each calculation, these sequences are also Laurent sequences. The program **Maple** was used for the calculations.

$$
\begin{array}{rcll}
P_5 & = & x_1x_4 + x_2x_3 & P_1 = x_2x_5 + x_3x_4 \\
E_5 & = & x_4^2 + x_2x_1 & P_2 = x_3x_1 + x_4x_5 \\
E_5' & = & x_4^2x_2 + x_1^2x_3 & P_3 = x_4x_2 + x_5x_1 \\
E_5'' & = & x_3x_4 + x_1^2 & P_4 = x_5x_3 + x_1x_2 \\
R & = & x_1x_4 + x_2x_3 & \square
\end{array}
$$

$$
\begin{array}{rcl}
P_6 & = & x_1x_5 + x_2x_4 + x_3^2 \\
P_1 & = & x_2x_6 + x_3x_5 + x_4^2 \\
E_6 & = & x_5^2x_3 + x_5x_4^2 + x_4x_2x_1 + x_3^2x_1 \\
P_2 & = & x_3x_1 + x_4x_6 + x_5^2 \\
E_6' & = & x_5^2x_3x_2 + x_5x_4^2x_2 + x_1^2x_4x_3 + x_1x_4x_5^2 + x_3^2x_1x_2 \\
P_3 & = & x_4x_2 + x_5x_1 + x_6^2 \\
E_6'' & = & x_4x_1x_2^2 + x_4x_5x_3^2 + x_1^2x_4x_3 + x_2x_1^2x_5 + x_5^2x_3x_2 \\
P_4 & = & x_5x_3 + x_6x_2 + x_1^2 \\
E_6''' & = & x_5x_3^2 + x_1^2x_3 + x_4x_2x_5 + x_2^2x_1 \\
P_5 & = & x_6x_4 + x_1x_3 + x_2^2 \\
R & = & x_1x_5 + x_2x_4 + x_3^2 \quad \square
\end{array}
$$

$$
\begin{array}{rcl}
P_7 & = & x_1x_6 + x_2x_5 + x_3x_4 \\
P_1 & = & x_2x_7 + x_3x_6 + x_4x_5 \\
E_7 & = & x_3x_6^2 + x_6x_5x_4 + x_5x_2x_1 + x_3x_4x_1 \\
P_2 & = & x_3x_1 + x_4x_7 + x_5x_6 \\
E_7' & = & x_3x_6^2x_2 + x_6x_5x_4x_2 + x_5^2x_1x_6 + x_1^2x_3x_5 + x_4x_2x_1x_3 \\
P_3 & = & x_4x_2 + x_5x_1 + x_6x_7 \\
E_7'' & = & x_4x_2x_1 + x_6^2x_2 + x_6x_5x_3 + x_1^2x_5 \\
P_4 & = & x_5x_3 + x_6x_2 + x_7x_1 \\
E_7''' & = & x_2x_1x_5x_3 + x_2^2x_1x_6 + x_6^2x_2x_4 + x_6x_5x_3x_4 + x_1^2x_5x_4 \\
P_5 & = & x_6x_4 + x_7x_3 + x_1x_2 \\
E_7'''' & = & x_6x_3x_4 + x_1^2x_4 + x_6x_5x_2 + x_2x_1x_3 \\
P_6 & = & x_7x_5 + x_1x_4 + x_2x_3 \\
R & = & x_1x_6 + x_2x_5 + x_3x_4 \quad \square
\end{array}
$$

Notice that the axioms of cluster algebras require all exchange polynomials to be binomials, and accordingly, the exchange polynomials associated with the legs for Somos-4 and Somos-5 are binomials. On the other hand, for Somos-6 and

Somos-7, the exchange polynomials for the edges of the spine are not binomials (they are trinomials) and the number of terms in the exchange polynomials of the legs is not bounded by three.

Unlike the previous Somos sequences, Somos-8, $\tilde{S}_n$, is not a Laurent sequence. If one attempts to use the caterpillar lemma, one finds that R is such a large polynomial that it would be cumbersome to include it in this exposition. Thus $R \neq P_8$. This alone does not suffice to show that Somos-8 is not Laurent. However using the initial conditions $\tilde{S}_1 = \cdots = \tilde{S}_8 = 1$ and applying the recurrence $\tilde{S}_n\tilde{S}_{n+8} = \tilde{S}_{n+1}\tilde{S}_{n+7} + \tilde{S}_{n+2}\tilde{S}_{n+6} + \tilde{S}_{n+3}\tilde{S}_{n+5} + \tilde{S}_{n+4}^2$, one finds the first 18 terms are $1, 1, 1, 1, 1, 1, 1, 1, 4, 7, 13, 25, 61, 187, 775, 5827, 14815, 420514/7$. Thus Somos-8 is not a Laurent sequence.

3 Exchange Graphs

Another beautiful application of cluster algebra theory is the construction of exchange graphs. In the initial definition of cluster algebras, we are given a collection of clusters and exchange relations between the variables in the form of exchange binomials. We saw that one could construct an n-regular tree $\mathbb{T}_n$ where the clusters are the vertices. In practice this graph $\mathbb{T}_n$ need not be a tree, it could have cycles or it could even be finite. Any exchange graph must be n-regular [7, pg. 27]. Before delving into the theory of exchange graphs, we will discuss some background material concerning semisimple Lie algebras and root systems.

Lie algebras and root systems are significant to the theory of cluster algebras because these structures appear to help classify cluster algebras. One can classify a special class of Lie algebras, known as semisimple Lie algebras, according to their associated root systems and reflection groups. As we will see later on, the machinery of root systems will allow us to classify the semisimple Lie algebras according to Cartan matrices. This famous classification is known as the Cartan-Killing classification. We will see that to each Cartan matrix, we can associate an exchange matrix, and lastly each exchange matrix will uniquely determine a cluster algebra. We will explicitly use the theory of semisimple Lie algebras and root systems to classify the exchange graphs of some low-rank cluster algebras of *finite type*. By finite type, I refer to a cluster algebra whose exchange graph has a finite number of vertices.

3.1 Lie Algebras

The following background material is from Fulton and Harris' text, *Representation Theory* [9].

Definition 3 A Lie group is a group which is also a $\mathbb{C}_\infty$ smooth manifold. In this group, the composition operator $\circ : G \times G \to G$ and the inverse operator $^{-1} : G \to G$ are both differentiable [9, pg. 93].

A fundamental example of a Lie group is the general linear group $GL_n\mathbb{R}$, the group of invertible $n \times n$ real matrices [9, pg. 95].

Definition 4 A Lie algebra, $\mathfrak{g}$, is a vector space and a accompanying skew-symmetric bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying *Jacobi's identity* [9, pg. 108]

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Definition 5 The subspace of $\mathfrak{g}$ such that $Z(\mathfrak{g}) = \{X \in \mathfrak{g} : [X,Y] = 0]$ for all $Y \in \mathfrak{g}\}$ is defined to be the *center* of $\mathfrak{g}$ [9, pg. 121].

Remark 3 A Lie Algebra and Lie Group can be associated to each other via the exp and derivative maps [9, pgs. 104-120], however this explicit correspondence is not needed for the rest of this exposition.

3.1.1 The Classification of Semisimple Lie Algebras

Semisimple Lie algebras are those Lie algebras that are reducible as direct products of simple Lie algebras. There are four infinite families of simple Lie algebras, $A_n(n \geq 1), B_n(n \geq 2), C_n(n \geq 2)$, and $D_n(n \geq 3)$. There are also several exceptional Lie Algebras, E_6, E_7, E_8, F_4, and G_2. (Page 326 Fulton and Harris) The families have nice representations as fundamental matrix algebras, which correspond to important matrix Lie groups.

$$\begin{array}{rcl} (A_n) & \leftrightarrow & \mathfrak{sl}_{n+1}\mathbb{C} \\ (B_n) & \leftrightarrow & \mathfrak{so}_{2n+1}\mathbb{C} \\ (C_n) & \leftrightarrow & \mathfrak{sp}_{2n}\mathbb{C} \\ (D_n) & \leftrightarrow & \mathfrak{so}_{2n}\mathbb{C} \end{array}$$

It turns out these Lie algebras can be better understood using root systems. Before defining root systems, we first need to discuss reflection groups. That background material comes from Humphreys' text *Reflection Groups and Coxeter Groups* [12, pgs. 5-11, 39].

3.2 Reflection Groups and Root Systems

Given a real Euclidean Vector space V with a positive definite symmetric bilinear form $\langle\ ,\ \rangle$, a *reflection* is a linear map s_α on V that sends a nonzero vector α to its negative while fixing the hyperplane H_α orthogonal to α. We may write a reflection as the formula

$$s_\alpha(\beta) = \beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha.$$

A finite group generated by such maps is a *finite reflection group*. Such a group is in fact a subgroup of the orthogonal group $\mathbf{O}(V)$ since reflections preserve the length of elements. It is customary to denote a reflection group by W.

We define a *root system* Φ as a set of vectors that satisfy the conditions

$$\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\},\ \forall \alpha \in \Phi,$$

$$s_\alpha(\Phi) = \Phi,\ \forall \alpha \in \Phi.$$

The reflection group W associated to a root system is the group generated by the s_α for $\alpha \in \Phi$. Consequently the set of vectors Φ are fixed under the action of W.

A root system Φ is called *crystallographic* if $2\langle\alpha,\beta\rangle/\langle\beta,\beta\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. We will actually need such a condition for the group generated by the reflections $s_\alpha(\alpha \in \Phi)$ to be a Weyl group so that the root system will be associated to a semisimple Lie algebra.

So far, it is unclear which root systems are more natural than others. To define specific types of root systems that are more canonical, we need a total ordering on the vectors of V. A total ordering on a real vector space V is a transitive relation $<$ such that the following additional conditions hold:

$$\text{for every distinct pair } a \neq b \in V, \text{ either } a < b \text{ or } b < a \text{ but not both;}$$

$$a < b \Rightarrow a + c < b + c;$$

$$a < b \in \mathbb{R}, c \in \mathbb{R} - 0 \Rightarrow ca < cb \text{ if } c > 0 \text{ and } cb < ca \text{ if } c < 0.$$

One can construct a total ordering on V in many ways, the easiest example is lexicographical ordering: suppose $v_1, \ldots v_n$ is a basis for V then $a_1v_1 + \cdots a_nv_n < b_1v_1 + \cdots b_nv_n$ if and only if $a_1 = b_1, a_2 = b_2, \ldots, a_k = b_k$, and $a_{k+1} < b_{k+1}$ where k can be zero and $a_i, b_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$.

We can call a vector λ *positive* if it is larger than the zero vector under the chosen total ordering of V. A *positive system* is a subset Π of a root system Φ where all of the constituent vectors are positive. A reflection of α is also a reflection of $-\alpha$ so one can also construct a *negative system* $-\Pi$ where all of the roots are negative. Φ is the disjoint union of Π and $-\Pi$.

Definition 6 A subset Δ of Φ is a *simple system* with *simple roots* as elements if Δ is a basis for the $\mathbb{R}$-span of Φ in V, and each $\alpha \in \Phi$ can be written as a $\mathbb{R}_{\geq 0}$- or $\mathbb{R}_{\leq 0}$-linear combination of elements of Δ.

Proposition 2 *For every root system Φ, there is a unique positive system Π that contains a unique simple system Δ*

For a proof, see [12, pgs. 8-9].

Definition 7 The *rank* of a reflection group W is the cardinality of the simple system contained in a root system associated with W. Note that even though the choice of the root system is not unique, the cardinality of Δ is invariant of the choice.

Simple root systems are so fundamental because W is actually generated by the reflections s_α for $\alpha \in \Delta$ for any simple system Δ. These reflections are called *simple* reflections.

Definition 8 A *reduced word* for $w \in W$ is a product of simple reflections equal to w so that the number of constituent reflections is minimal.

3.2.1 Simple Root Systems for Simple Lie Algebras

The following is from [12, pgs. 41-42] and outlines part of the classification of semisimple Lie algebras according to their root system.

($A_n, n \geq 1$) Let V be the hyperplane in $\mathbb{R}^{n+1}$ where all the coordinates add up to 0. Let Φ be the set of vectors $v \in V \cap \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_1 + \cdots + \mathbb{Z}\epsilon_{n+1}$ such that $|v| = \sqrt{2}$. Here $\mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_1 + \cdots + \mathbb{Z}\epsilon_{n+1}$ is the standard unit integer lattice of $\mathbb{R}^{n+1}$. More explicitly, $\Phi = \{\epsilon_i - \epsilon_j : 1 \leq i \neq j \leq n+1\}$. The associated simple system is

$$\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n\}.$$

The associated reflection group W is the symmetric group on $n+1$ letters, S_{n+1} that permutes the ϵ_i.

($B_n, n \geq 2$) Let $V = \mathbb{R}^n$, Φ be the set of vectors $v \in \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_1 + \cdots + \mathbb{Z}\epsilon_n$ such that $|v| = 1$ or $\sqrt{2}$. $\Phi = \{\pm\epsilon_i\} \cup \{\pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq n\}$. The associated simple system is

$$\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n-1\} \cup \{\epsilon_n\}.$$

The associated reflection group W is the semidirect product of S_n with $(\mathbb{Z}/2\mathbb{Z})^n$.

($C_n, n \geq 2$) is B_n's dual and its simple system is

$$\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n-1\} \cup \{2\epsilon_n\}.$$

($D_n, n \geq 4$) Let $V = \mathbb{R}^n$, Φ be the set of vectors $v \in \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_1 + \cdots + \mathbb{Z}\epsilon_n$ such that $|v| = \sqrt{2}$. $\Phi = \{\pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq n\}$. The associated simple system is

$$\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n-1\} \cup \{\epsilon_{n-1} + \epsilon_n\}.$$

The associated reflection group W is the semidirect product of S_n with $(\mathbb{Z}/2\mathbb{Z})^{n-1}$.

(G_2) Let V be the hyperplane in $\mathbb{R}^3$ where the coordinates sum to 0. Let Φ be the set of vectors $v \in V \cap \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_1 + \cdots + \mathbb{Z}\epsilon_n$ such that $|v| = \sqrt{2}$ or $\sqrt{6}$. $\Phi = \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i \neq j \leq 3\} \cup \{\pm(2\epsilon_i - \epsilon_j - \epsilon_k)\}$ where (i, j, k) is a permutation of $(1, 2, 3)$. The associated simple system is

$$\Delta = \{\epsilon_1 - \epsilon_2, -2\epsilon_1 + \epsilon_2 + \epsilon_3\}.$$ [4]

Letting the simple roots associated with a given simple Lie algebra as $\{\alpha_i\}$, to each of these simple Lie algebras we can associate a Cartan matrix [14, pg. 111].

[4] Since they are not relevant to our later discussion of cluster algebras, I will omit a description of the simple root systems for the other simple Lie algebras.

Definition 9 A *Cartan Matrix* is a matrix where the (i,j)th entry is $\langle \alpha_i, \alpha_j \rangle$.

One can show that $\langle \alpha, \beta \rangle = 2\frac{||\beta||}{||\alpha||}\cos\theta_{\alpha,\beta}$ where $\theta_{\alpha,\beta}$ is the angle between α and β. Thus all of the diagonal entries of a Cartan matrix will be 2 [14, pg. 114]. Furthermore, the off-diagonal entries will be less than or equal to zero [21, pg. 34]. Since semisimple Lie algebras are reducible, one can also associate a Cartan matrix to them. If a semisimple Lie algebra S is the direct product of $S_1 \times \cdots \times S_n$ where S_i is a simple Lie algebra with associated Cartan matrix M_i, then the Cartan matrix associated to S is the direct sum $M_1 \oplus \cdots \oplus M_n$.

3.3 Cluster Algebras and Root Systems

This section, as well as the next (3.4) comes from [7]. Root systems can be used to analyze Cluster Algebras. Specifically, let the exchange binomial associated to edge j between vertices t and t' be $M_j(t) + M_j(t')$. Then we let b_{ij} be the exponent of x_i in the expression $\frac{M_j(t)}{M_j(t')}$. It follows that

$$M_j(t) = \prod_{i:b_{ij}(t)>0} x_i^{b_{ij}(t)} \tag{11}$$

$$M_j(t') = \prod_{i:b_{ij}(t)<0} x_i^{-b_{ij}(t)}. \tag{12}$$

and $B(t) = (b_{ij}(t)$ will be a $n \times n$ integer matrix associated to vertex t. We will call such a matrix an *exchange matrix* associated to vertex t of cluster algebra $\mathcal{A}$.

In other words, $B(t)$ encodes the exponents of the exchange binomials for all of the edges stemming from vertex t. All of the exchange binomials have positive exponents but in an effort to differentiate between the binomials $x_1^2x_2 + x_3^3x_4$ and $x_1^2x_3^3 + x_2x_4$, the encoding of the exponents which appear in the second monomial are given a negative sign. Note, by axiom (5), x_i cannot divide both monomials. Axiom (6) implies that B is forced to be *sign-skew symmetric*, i.e. $b_{ij} = b_{ji} = 0$ or b_{ij} and b_{ji} have opposite signs.

Fomin and Zelevinsky also define a family of *matrix mutation* functions $\{\mu_i\}$, so that $\mu_k(B) = B' = (b'_{ij})$, where

$$\begin{aligned} b'_{ij} &= -b_{ij} && \text{if } i = k \text{ or } j = k && (13)\\ &= b_{ij} && \text{if } b_{ik}b_{kj} \le 0 && (14)\\ &= b_{ij} + b_{ik}b_{kj} && \text{if } b_{ik}, b_{kj} > 0 && (15)\\ &= b_{ij} - b_{ik}b_{kj} && \text{if } b_{ik}, b_{kj} < 0. && (16) \end{aligned}$$

Proposition 3 *A family of $n \times n$ integer matrices $(B(t))_{t \in \mathbb{T}_n}$ corresponds to an exchange pattern if and only if*
- *$B(t)$ is sign-skew-symmetric for all $t \in \mathbb{T}_n$.*
- *If there is an edge labeled k connecting vertices t and t', then $B(t') = \mu_k(B(t))$.*

Proof. First, let us assume that the family of matrices $(B(t))$ corresponds to an exchange pattern. Then by axiom (4), $b_{jj} = 0$ otherwise $x_j | M_j(t)$. Likewise, axiom (6) implies that $B(t)$ will be sign-skew symmetric.

The equality $b_{ik} = b'_{ik}$ stems from definitions (11) and (12) which is a formal way of saying that the ordering of the monomials in the exchange binomial depends on whether you are traveling from t to t' or t' to t.

If $j \neq k$, applying axiom (7) to the edge labeled k between vertices t and t' along with the two edges emanating from t and t' labeled with j, we see that

$$\prod_i x_i^{b'_{ij}} = \prod_i x_i^{b_{ij}} |_{x_k \leftarrow M/x_k} \tag{17}$$

for $M = \prod_{i: b_{ik} b_{jk} < 0} x_i^{|b_{ik}|}$. Considering the exponents on the left-hand-side and right-hand-side of x_k in (17), we see that $b'_{kj} = -b_{kj}$. Comparing the exponents on the left-hand-side and right-hand-side of x_i in (17) for arbitrary i completes the proof.

Assuming that we have a family of sign-skew-symmetric matrices subject to matrix mutation as above, it is clear that the corresponding exchange binomials will obey the axioms of an exchange pattern, axioms (2-7). □

Once an exchange matrix B is defined for a given vertex (cluster) of the exchange graph, axiom (7) will uniquely define matrix mutation μ and all the exchange matrices associated to each vertex of the exchange graph.[5] To each of these exchange matrices we can associate a (generalized) Cartan matrix $A = A(B) = (a_{ij})$ of the same size where

$$\begin{aligned} a_{ij} &= 2 \text{ if } i = j \text{ and} \\ &-|b_{ij}| \text{ if } i \neq j. \end{aligned} \tag{18}$$

These generalized Cartan matrices appear in the theory of Kac-Moody algebras. Fomin and Zelevinsky note that there seems to be a relation between cluster algebra with exchange matrix M and a Kac-Moody algebra with generalized Cartan matrix M' when M and M' are associated as in (18) [7, pgs. 15-16]. In general it is hard to prove that a given choice of B will force the whole family of $B(t)$ to be sign-skew-symmetric. However, the following condition implies that the whole family will in fact be sign-skew-symmetric. For more details on the proof, see [7, pg. 15].

Definition 10 A matrix B is called *skew-symmetrizable* if there exists a diagonal matrix D s.t. DB is skew-wymmetric.

There are other kinds of matrices that will work, but for the purposes of this exposition, we will restrict our attention to exchange matrices that are skew-symmetrizable. In fact, many of the following examples will only require the matrices to be skew-symmetric.

[5] The exchanges are uniquely defined since we have assumed all of the monomial coefficients are 1.

3.3.1 The Rank 2 Case

Let $\mathbb{T}_n$ be a 2-regular tree whose vertices are labeled t_m for $m \in \mathbb{Z}$ where an edge labeled $m \mod 2$ joins vertices t_m and t_{m+1}. Let the cluster associated with vertex t_m be $\{x_m, x_{m+1}\}$. By theorem 1, all of the succeeding cluster variables can be rewritten in terms of a Laurent polynomial of two initial cluster variables (x_1, x_2) after completing a series of exchanges according to the exchange binomials. Let

$$x_m = \frac{P_m(x_1, x_2)}{x_1^{d_1(m)} x_2^{d_2(m)}},$$

where P_m is a polynomial with coefficients in $\mathbb{Z}$ not divisible by x_1 or x_2 and $d_1, d_2 \in \mathbb{Z}$.

Corollary 1 *The only possible exchange patterns for a rank 2 cluster algebra correspond to a family of matrices with the form*

$$B(t_m) = (-1)^m \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$$

for integers b and c of like sign.

Furthermore, these exchange matrices will correspond to an exchange pattern that alternates between the two binomials $x^b + 1$ and $1 + x^c$.

Proof. This is a corollary of Proposition 3 restricted to the rank 2 case.
The corresponding generalized Cartan matrix is

$$A(B(t)) = \begin{bmatrix} 2 & -b \\ -c & 2 \end{bmatrix}.$$

A root system with basis of simple roots $\{\alpha_1, \alpha_2\}$ corresponds to this Cartan matrix. Let $W(A)$ be the reflection group generated by the two simple roots

$$s_1 = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}, \; s_2 = \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}$$

$s_1^2 = s_2^2 = 1$ so the possible reduced words $w \in W$ are

$$w_1(m) = s_1 s_2 s_1 \cdots s_{(m \bmod 2)} \text{ or } w_2(m) = s_2 s_1 s_2 \cdots s_{(m+1 \bmod 2)}.$$

After perusing our list of rank 2 semisimple Lie algebras and their corresponding root systems, we see that W finite $\Leftrightarrow bc \leq 3$. Further study reveals that the rank 2 cluster algebras of finite type can be summarized in the following table.

Lie Algebra	Cartan Matrix	Representative Exchange Matrix	# Clusters
$A_1 \times A_1$	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	4
A_2	$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	5
B_2	$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$	6
C_2	$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$	6
G_2	$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$	8
$G_2^\vee$	$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$	8

3.4 The Formalism Behind Exchange Graphs

Fomin and Zelevinsky define two clusters t and t' to be $\mathcal{M}$*-equivalent* if there is a permutation $\sigma \in S_n$ such that $x_i(t') = x_{\sigma(i)}(t)$ for all $i \in \{1, \ldots . n\}$ and if $E_{\sigma(j)}(t, t_1)$ along with $E_j(t', t_1')$ implies $M_j(t') = M_{\sigma(j)}(t)$ and $M_j(t_1') = M_{\sigma(j)}(t_1)$. In other words, the two clusters are composed of a permutation of the same variables.

Let one follow a path on the tree $\mathbb{T}_n$ associated with a particular exchange pattern where the edges alternate $i, j, i, j, \ldots$. If $t \equiv_{\mathcal{M}} t'$ after a sequence of such steps, then we see that $\mathbb{T}_n$ has a cycle. By the analysis of the rank 2 case, the only cycles will be of length $4, 5, 6$ or 8. All other paths of alternating edges will be infinite.

The type A_2 case where the exchange graph is a pentagon is exceptional since the number of clusters is odd. Starting at cluster $\{x_1, y_1\}$, the polynomial exchanges $x_i x_{i+1} = y_i + 1$, $y_i y_{i+1} = x_{i+1} + 1$ leads to the clusters

$$\{x_1, y_1\} \to \{\frac{y_1+1}{x_1}, y_1\} \to \{\frac{y_1+1}{x_1}, \frac{x_1+y_1+1}{x_1 y_1}\} \to \{\frac{x_1+1}{y_1}, \frac{x_1+y_1+1}{x_1 y_1}\}$$

$$\to \{\frac{x_1+1}{y_1}, x_1\} \to \{y_1, x_1\}.$$

Each edge will not have a precise edge label since $x_6 = y_1$ and $y_6 = x_1$ implies that changing the 2nd variable of the cluster $\{x_6, y_6\}$ is equivalent to changing the 1st variable of $\{x_1, y_1\}$ even though these two clusters are $\mathcal{M}$-equivalent to each other.

Considering the example from section 2, $\{g_n\}$, one can append the caterpillar to get the full exchange graph for this cluster algebra. It turns out not to be an infinite tree, but instead two infinite rows of pentagons. We will call this graph $\mathcal{G}_{5,2}$. From earlier analysis, we know that the edges emanating from a vertex on the spine is associated with a cyclic transformation of the exchange polynomials $1 + x_2x_3$, $x_1 + x_3$, and $x_1x_2 + 1$. We encode these exchanges as the exchange matrix

$$B(t_0)) = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Applying the matrix mutation functions $\{\mu_i\}$ to this starting matrix, we obtain

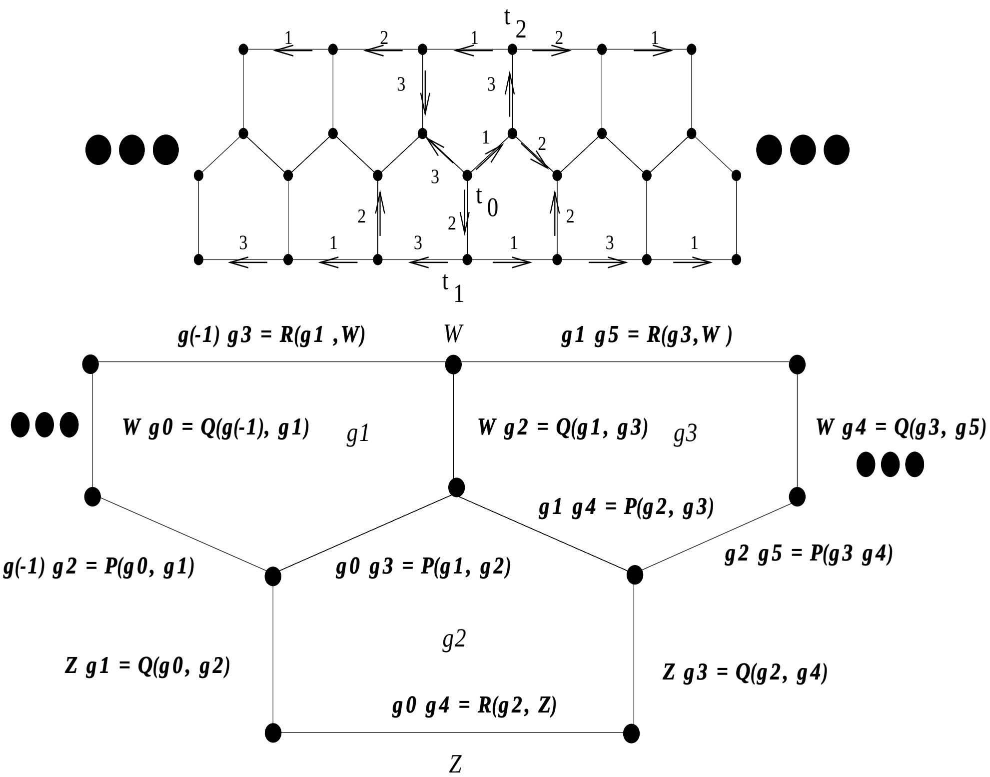
$$\mu_1(B(t_0)) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad (19)$$

$$B(t_1) = \mu_2(B(t_0)) = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} \quad (20)$$

$$\mu_3(B(t_0)) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad (21)$$

$$B(t_2) = \mu_3\mu_1(B(t_0)) = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \quad (22)$$

Notice that $\mu_1(B(t_0))$ and $\mu_3(B(t_0))$ are just cyclic transformations of $B(t_0)$. This symmetry arises since the exchange polynomials associated with every vertex on the spine are cyclic transformations of each other. Edge 2, on the other hand, will lead one to a vertex off the spine, $B(t_1)$. One notices that $B(t_2)$, the result of traveling along edge 1 on the spine, followed by edge 3 leads one to another vertex off the spine, one whose exchange polynomials are cyclic transformations of $B(t_1)$'s exchange polynomials. Since $B(t_0)$'s three principal 2×2 submatrices are the exchange graph for a cluster algebra of type A_2, this implies that a walk along a sequence of adjoining edges alternatively labeled (either $1, 2, 1, 2, \ldots$ or $1, 3, 1, 3, \ldots$ or $2, 3, 2, 3, \ldots$) will be a cycle of length 5. Thus the associated exchange graph will have three adjoining pentagons at each vertex. Furthermore $B(t_1)$ and $B(t_2)$ contain exactly one principal 2×2 submatrix of the form $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ thus the vertices t_1 and t_2 sit on infinite lines disjoint from the original spine. Based on the cyclic symmetry between $B(t_1)$ and $B(t_2)$, we are able to deduce that the pentagons must interlock in such a pattern to allow travel in both directions to be cyclically symmetric. Thus we find that the exchange graph consists of three spines, where the original spine associated with g_n is the middle spine. We use axiom (7) and the exchange matrices to compute the exchange binomials associated with the additional edges.



Exchange graph $\mathcal{G}_{5,2}$ for the sequence g_n with a close-up.

Each vertex is bordered by three bounded (or unbounded) regions. The cluster variables of each vertex are represented by these three regions. $P(x,y) = xy+1$, $Q(x,y) = x+y$ and $R(x,y) = x^2+y$. The edge labels assume one is starting from vertex t_0 and traveling outward. Since subgraphs are pentagons, each edge does not have a precise edge label.

3.5 New Recurrences for Old Sequences

Notice that if we start with the recurrence $g_n g_{n-3} = g_{n-1}g_{n-2}+1$, and build the exchange graph which has the corresponding exchange binomials on it spine, we get an exchange graph with two additional spines which satisfy the recurrences $g_{2n+2}g_{2n-2} = g_{2n}^2 + z$ and $g_{2n+1}g_{2n-3} = g_{2n-1}^2 + w$ respectively. If $g_0 = g_1 = g_2 = 1$, then $z = 2$ and $w = 3$. Our combinatorial interpretation of g_n as the number of perfect matchings for a family of graphs had previously revealed the extra recurrence $g_{2n+2}g_{2n-2} = g_{2n}^2 + 2$.

Based on numerical evidence and these exchange graphs, the author conjectures that there is a one-to-many surjective map between perfect matchings in $2 \times 2(n-1)$ grid graphs and perfect matchings in mutilated $3 \times 2(n-1)$ grid graphs where a matching with m pairs of horizontal edges map to 3^m perfect matchings of a mutilated $3 \times 2(n-1)$ grid graph. Thus, the sequence g_n can be split into two alternating subsequences where the terms $g_{2n} = 1, 3, 11, 41, \ldots$ satisfy the recurrence $g_{2n}g_{2n-4} = g_{2n-2}^2 + 2$ and the terms $g_{2n+1} = 1, 2, 7, 26, 97, \ldots$ satisfy the recurrence $g_{2n+1}g_{2n-3} = g_{2n-1}^2 + 3$. The cluster algebra method has given an alternate way to uncover and prove these recurrences without presupposing knowledge of the combinatorial objects the integer sequence counts, and without explicit bijections. Thus the exchange graph method for discovering new recurrences provides a method for discovering new recurrences for a sequence even where the combinatorial interpretation is unknown.

3.6 Three-dimensional Exchange Graphs

We noticed for cluster algebras of rank 2 that the only possible exchange graphs are an infinite line, a square, a pentagon, a hexagon, or an octagon. Likewise, cluster algebras of rank 3 will either be of infinite type or finite type. The graph $\mathcal{G}_{5,2}$ is a nice example of an exchange graph for a cluster algebra of infinite type. A 3-degree tree is another possibility for a rank 3 cluster algebra of infinite type.

As illustrated in section 3.3, Fomin and Zelevinsky [7] illustrate that it appears possible to classify cluster algebras in terms of corresponding semisimple Lie algebras. However, it is unclear whether or not all cluster algebras of finite-type correspond to semisimple Lie algebras. If they do not correspond to semisimple Lie algebras, perhaps they correspond to Kac-Moody algebras [7, pg. 16]. For the case of rank 2 cluster algebras of finite type, the classification can be completed only using semisimple Lie algebras, as explained in section 3.3.1 and explained more thoroughly in [7]. In the following pages, some cluster algebras of higher ranks will be classified, though this exposition will only hint at some patterns since a complete classification is still an open problem. We will refer to a cluster algebra as type S if one of the clusters (vertices) has an exchange matrix associated to the Cartan matrix for the semisimple Lie algebra S. The cluster algebras of infinite type are hard to classify, but for rank 3 cluster algebras of finite type, some possible exchange graphs will correspond to the Lie algebras $A_1 \times A_1 \times A_1$, $A_2 \times A_1$, $B_2 \times A_1$, $G_2 \times A_1$, A_3 or B_3.

The one-dimensional exchange graph corresponding to A_1 is a line between two points ($\mathbb{Z}/2\mathbb{Z}$), the one for $A_1 \times A_1$ is a square, and $A_1 \times A_1 \times A_1$ has a cube as its exchange graph. Such evidence motivates the following result which seems not to have appeared in the literature before.

Proposition 4 *In general, the cluster algebra of type A_1^n has an n-cube as its exchange graph.*

Proof. The semisimple Lie algebra A_1^n has the diagonal matrix $2I_n$ as its Cartan matrix. Consequently, the associated exchange matrix is N, the matrix of all zeros. So the cluster algebra of type A_1^n (which will be of rank n) contains at least one cluster $X = \{x_1, \ldots, x_n\}$ with the binomial exchange relations:

$$\begin{aligned} x_1 y_1 &= 1 \\ x_2 y_2 &= 1 \\ &\cdots \\ x_n y_n &= 1 \end{aligned}$$

which means all the adjacent clusters must look like

$$Y_i = \{x_1, \ldots, x_{i-1}, \frac{1}{x_i}, x_{i+1}, \ldots, x_n\}.$$

Furthermore, $\mu_i(N) = N$ for all i so each cluster of $\mathcal{A}$ will look be of the form

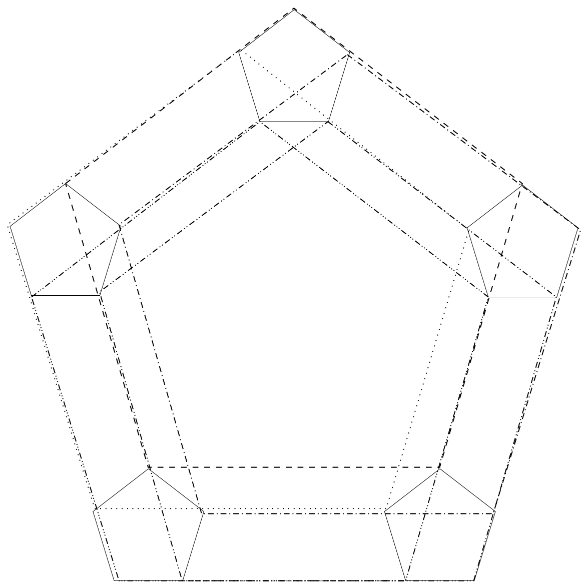
$$\{x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}\}$$

where $\epsilon_i = \pm 1$, and each exchange changes the sign of exactly one ϵ_i. The corresponding exchange graph is an n-cube. □

Proposition 5 *We can generalize this result. Let G_X be the exchange graph for a cluster algebra of type X. Then a cluster algebra of type $X \times A_1$ has $G_X \times \mathbb{Z}/2\mathbb{Z}$ as its exchange graph. This is a graph consisting of two copies of G_X where vertex $(v, 0)$ is connected to vertex $(v', 1)$ if and only if $v = v'$.*

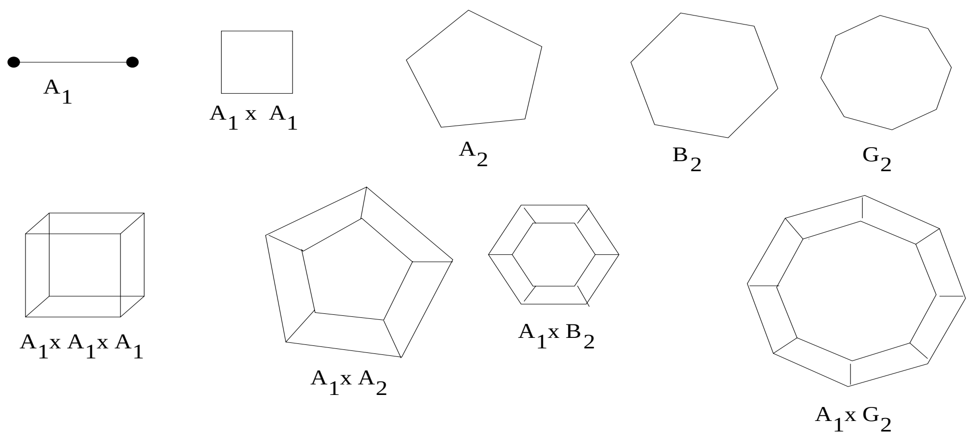
Proof. The proof is analogous. The exchange matrix associated with $X \times A_1$ will be M_X, the exchange matrix associated with X, with an extra row and column of zeros. This corresponds to adding an edge corresponding to the exchange $xy = 1$ to one of the vertices of G_X. Let $M_X \oplus 0$ be the corresponding exchange matrix for a cluster algebra of type $X \times A_1$. For $i \neq n$, $\mu_i(M_X \oplus 0) = \mu_i(M_X) \oplus 0$ and $\mu_n(M_X \oplus 0) = M_X \oplus 0$ which means we have added an nth variable to each cluster, and an edge at every vertex which sends x_n to its reciprocal. This addition will force the exchange graph to be $G_X \times \mathbb{Z}/2\mathbb{Z}$ where each vertex of G_X has been replaced by two vertices $\{(v, x_n), (v, \frac{1}{x_n})\}$ connected by an edge. □

Similarly, we conjecture that the rank 4 cluster algebra $A_2 \times A_2$ would have an exchange graph $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, a pentagonal graph for each vertex is blown up to a pentagon.



The conjectured exchange graph for the rank 4 cluster algebra of type $A_2 \times A_2$.

Getting back to the low rank cases, the following are all of the exchange graphs for rank 1 or rank 2 cluster algebras of finite type. Several rank 3 cluster algebras have also been included.



Exchange graphs for some low rank cluster algebras of finite type.

Representative exchange matrices associated with each of these exchange graphs are:

$$
\begin{aligned}
A_1 &\leftrightarrow \begin{bmatrix} 0 \end{bmatrix} \\
A_1 \times A_1 &\leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
A_2 &\leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
B_2 &\leftrightarrow \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \\
G_2 &\leftrightarrow \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} \\
A_1 \times A_1 \times A_1 &\leftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
A_2 \times A_1 &\leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
B_2 \times A_1 &\leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
G_2 \times A_1 &\leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

A cluster algebra of type A_3 is the simplest rank 3 cluster algebra of finite type whose exchange graph cannot be described as the direct product of lower rank graphs. To construct its exchange graph we complete the following procedure.

3.7 A_3's Exchange Graph

First we note that A_3 has the associated Cartan matrix

$$
\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.
$$

We will consider the cluster algebra where one of the clusters has

$$
M = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}
$$

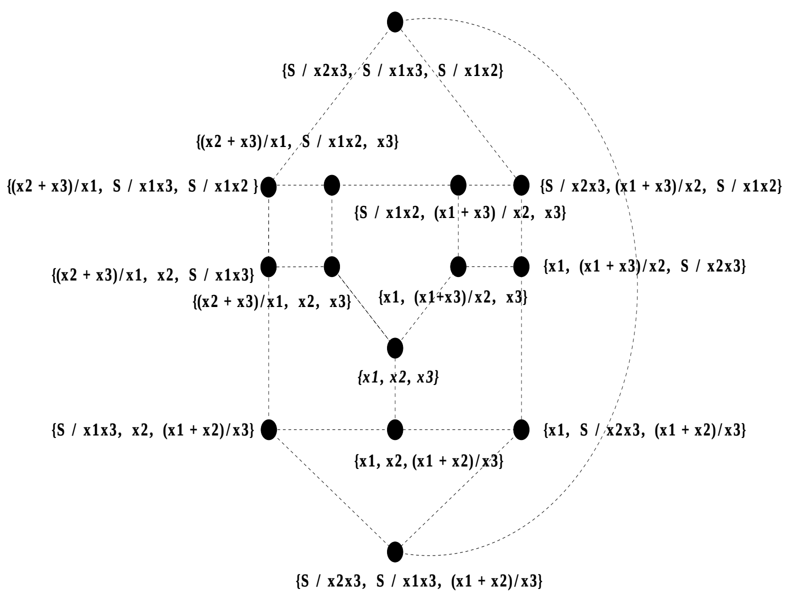
as its exchange matrix. Mutating M as defined in section 3.3, we find

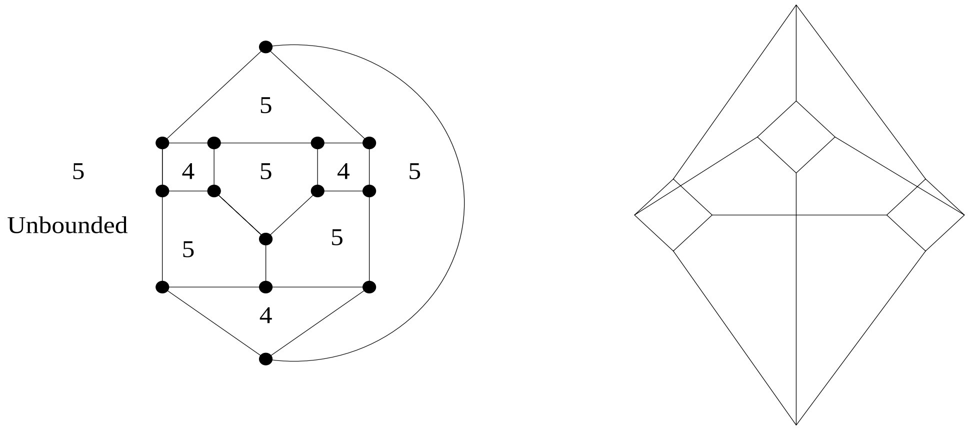
$$M_1 = \mu_1(M) = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$M_2 = \mu_2(M) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_3 = \mu_3(M) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

M_1, M_2 and M_3 each are cyclic transformations of each other, and they each have two principal submatrices that are exchange graphs of type A_2 and one principal submatrix that is of type $A_1 \times A_1$. Furthermore, all of M's principal submatrices were of type A_2. From this, we deduce that the vertex associated with M lies at the junction of three pentagons, and the vertices associated with M_1, M_2 and M_3 each are at the junction of two pentagons and a rectangle. We continue to apply the matrix mutation functions μ_1, μ_2 and μ_3 to M_1, M_2, M_3 and beyond, and at the same time apply the corresponding exchange relations to the initial cluster $\{x_1, x_2, x_3\}$ (which is associated with exchange matrix M). We find that the following graph characterizes all of the clusters of this cluster algebra where $S = x_1 + x_2 + x_3$.





Two representations of the exchange graph for the cluster algebra of type A_3.

This exchange graph can be pictured in three dimensions as two tetrahedra glued together where all of the corners at the adjoining faces have been rubbed down to make square faces. An easy way to see this is that if one shrinks the faces of size 4 in the planar version of the graph (left-hand side) down to points, one is left with six faces of size 3 which forms two adjoined tetrahedra. I am grateful to Curtis T. McMullen for noticing this three dimensional characterization of this graph.

This graph also turns out to be the three dimensional *associahedron* [3]. Notice it has 14 vertices, the A_2 graph had 5 vertices, and the A_1 graph had 2 vertices. These are the Catalan numbers C_2, C_3, C_4 where $C_n = \frac{1}{n+1}\binom{2n}{n}$ and this is not a coincidence. In fact, each of these vertices correspond to a triangulation of a hexagon and there are $C_4 = 14$ ways of doing this (there are C_n triangulations of the $(n+2)$-gon). Another interpretation is that the polytope's vertices correspond to the ways you can associatively write a product, thus the name associahedron [18].

Fomin and Zelevinsky have a more general result that all exchange graphs for an A_n-type cluster algebra are n-dimensional associahedra. Similarly they have a result that all exchange graphs for a B_n-type (C_n-type) cluster algebra are n-dimensional cyclohedra. See [2] or [24] for details about the cyclohedron. In fact, they define the families of exchange graphs for cluster algebras of finite type associated with simple Lie algebras to be polytopes that they call *generalized associahedra* [3]. It appears that this would extend to a classification of many cluster algebras of finite type, namely a cluster algebra is determined by its exchange graph, and some possible exchange graphs would be direct products (as graphs) of the generalized associahedra.[6]

[6]As mentioned earlier, it is unclear if all cluster algebras would correspond to semisimple Lie algebras. Furthermore, we have restricted the definition of cluster algebra in this exposition by not allowing coefficients other than 1. If one allows coefficients from $\mathbb{P}$, an abelian group without torsion, the classification would be even more complicated. Additionally, we have not mentioned anything about the classification of cluster algebras of infinite type, such as the cluster algebra defined by the sequence g_n with $\mathcal{G}_{5,2}$ as its exchagne graph.

4 Open Problems

The caterpillar lemma is great for proving that certain sequences are Laurent sequences. However, the converse does not hold, and it cannot prove definitively that a sequence x_n is not Laurent. Is there a way to refine the condition to make this lemma an exact criterion? Or is there at least a way to determine for what kinds of sequences the caterpillar method will fail [4]?

The caterpillar lemma can prove that a sequence is Laurent, which in turn proves that the sequence is an integer sequence given that the first several terms are 1. However, this cannot determine whether the coefficients of the resulting Laurent polynomials are all nonnegative. It appears they are (Remark 1) and this would allow one to conclude the more powerful result that the sequence would be a sequence of nonnegative integers. Hence it could count combinatorial objects. Finding an explicit combinatorial interpretation for a sequence proves that sequence indeed consists of nonnegative integers. Is there a more universal way to assign such interpretations? The nonnegativity condition is also important because it appears that the dual canonical basis, the original motivation for the development of cluster algebras, should only involve nonnegative coefficients. This will be discussed briefly in the appendix.

Type A_n exchange graphs were identified as associahedra, and B_n (C_n) exchange graphs as cyclohedra in [3]. In this same article, Fomin and Zelevinsky ask about the structure of D_n exchange graphs. It would also be significant to classify exchange graphs of infinite type. Perhaps this can be done for rank 3 or at least for rank 3 cluster algebras which have exchange graphs that Zelevinsky refers to as tame, i.e. they are highly symmetrical like $\mathcal{G}_{5,2}$.

In particular, Fomin and Zelevinsky are pursuing a more complete classification of all cluster algebras (or at least tame ones) which would be analogous to the classification of semisimple Lie algebras or Kac-Moody algebras. Zelevinsky explained (personal communication) that seeing patterns and connections to Laurent sequences and associahedra help them develop insight as to patterns in the classification.

After a more explicit classification of cluster algebras has been formulated, one could explore the theory of Laurent polynomials and recurrence relations in more depth. In particular, we saw how the exchange graph $\mathcal{G}_{5,2}$ helped reveal secondary recurrences that the sequence $\{g_n\}$ satisfies. In practice one should be able to do this for other sequences, and the classification of cluster algebras and exchange patterns would correspond to a classification of families of recurrences where two recurrences R_1 and R_2 would be in the same family if any sequence that satisfies R_1 must also satisfy recurrence R_2.

5 Appendix: Fomin and Zelevinsky's Motivation for the Development of Cluster Algebras

The inspiration for the development of cluster algebras came from Fomin and Zelevinsky's study of the dual canonical basis of Quantum groups. In the following paragraphs, we will summarize the results and conjectures that led Fomin and Zelevinsky to create a new algebraic structure. First, we will recall some notation and results from [25]. We let $U^+ = U_q(\mathfrak{n}) \subseteq U_q(\mathfrak{g})$ be the subalgebra of the quantized universal enveloping algebra generated by elements E_i and let $R(w_0)$ be the set of *reduced words* for the permutation $w_0 = (n\ n-1\ \cdots 3\ 2\ 1)$. We will not give a more precise definition of E_i. For such a definition, see [25, pg. 8].

Remark 4 For every $\bar{i} \in R(w_0)$ and $t \in \mathbb{Z}^m_{\geq 0}$, there is a unique element $b = b_{\bar{i}}(t)$ of U^+ such that b and $b - p^{(t)}_{\bar{i}}$ is a linear combination of the elements of $\mathcal{B}_{\bar{i}}$, a basis with coefficients in $q^{-1}\mathbb{Z}[q^{-1}]$. It turns out $\mathcal{B}_{\bar{i}}$ is not dependent on the choice of $\bar{i}$ thus we let $\mathcal{B}$ be the *canonical basis.*

We will not define the elements $p^{(t)}_{\bar{i}}$, see [25, pg. 8] for a defintion. The importance of Remark 4 and the related notation is that it allows a definition of a *canonical basis* to make sense. Prior to their formulation of cluster algebras, the canonical basis, which is due to G. Lusztig [13], had been a main object of study for Fomin and Zelevinsky.

For example, Fomin and Zelevinsky describe a more explicit parameterization of the canonical basis $\mathcal{B}$ in [1] and [5]. Then to study more of the algebraic structure of $\mathcal{B}$, they investigated the *dual canonical basis* $\mathcal{B}^{dual}$ in the ring of regular functions $\mathbb{C}[N]$ where N is the maximal unipotent subgroup of the group under investigation. After many examples, a pattern hinting at an underlying algebraic structure emerged [25]. The properties of these algebraic structures were axiomatized as the theory of cluster algebras. Zelevinsky explains

> The dual canonical basis $\mathcal{B}^{dual}$ was constructed explicitly in several small rank cases. ... In all of these cases, $\mathcal{B}^{dual}$ consists of certain monomials in a distinguished family of generators. ... The monomials that constitute $\mathcal{B}^{dual}$ are defined by not allowing certain pairs of generators to appear together. In each case, the product of every two "incompatible" generators can be expressed as the sum of two allowed monomials [25, pg. 12].

In addition to the theory of dual canonical bases, Lusztig generalized the concept of totally positive[7] matrices to total positivity in any reductive group G.

Lusztig related the theory of total positivity back to the dual canonical basis. He showed that the elements of the dual canonical basis in $\mathbb{C}[G]$ take positive values. It is this connection that motivated remark 1, i.e. Fomin and Zelevinsky's

[7] A matrix is considered ***totally positive*** if all of its minors are positive. Such matrices are important in the study of differential equations and Polya frequency sequences [15].

conjecture that not only are the cluster variables Laurent polynomials in any other cluster but are Laurent polynomials with nonnegative coefficients. If such a conjecture was true, a more explicit correspondence between the dual canonical basis and cluster algebras might be possible. In fact Fomin and Zelevinsky conjecture that any coordinate ring $\mathbb{C}[G]$ or $\mathbb{C}[G/N]$ can be characterized as a cluster algebra assuming one is free to use coefficients in $\mathbb{P}$ other than 1 [7].

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