

# AIMS Exercise Set # 4

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1. Find the explicit formula for the solution to the following linear iterative system:

$$u^{(k+1)} = u^{(k)} - 2v^{(k)}, \quad v^{(k+1)} = -2u^{(k)} + v^{(k)}, \quad u^{(0)} = 1, \quad v^{(0)} = 0.$$

2. Determine whether or not the following matrices are convergent:

$$(a) \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}, \quad (b) \frac{1}{5} \begin{pmatrix} 5 & -3 & -2 \\ 1 & -2 & 1 \\ 1 & -5 & 4 \end{pmatrix}.$$

3. (a) Find the spectral radius of the matrix  $T = \begin{pmatrix} 1 & 1 \\ -1 & -\frac{7}{6} \end{pmatrix}$ . (b) Predict the long term behavior of the iterative system  $\mathbf{u}^{(k+1)} = T\mathbf{u}^{(k)} + \mathbf{b}$ , where  $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ , in as much detail as you can.

4. Consider the linear system  $A\mathbf{x} = \mathbf{b}$ , where  $A = \begin{pmatrix} 4 & 1 & -2 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$ .

- (a) First, solve the equation directly by Gaussian Elimination. (b) Using the initial approximation  $\mathbf{x}^{(0)} = \mathbf{0}$ , carry out three iterations of the Jacobi algorithm to compute  $\mathbf{x}^{(1)}$ ,  $\mathbf{x}^{(2)}$  and  $\mathbf{x}^{(3)}$ . How close are you to the exact solution? (c) Write the Jacobi iteration in the form  $\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{c}$ . Find the  $3 \times 3$  matrix  $T$  and the vector  $\mathbf{c}$  explicitly. (d) Using the initial approximation  $\mathbf{x}^{(0)} = \mathbf{0}$ , carry out three iterations of the Gauss–Seidel algorithm. Which is a better approximation to the solution — Jacobi or Gauss–Seidel? (e) Write the Gauss–Seidel iteration in the form  $\mathbf{x}^{(k+1)} = \tilde{T}\mathbf{x}^{(k)} + \tilde{\mathbf{c}}$ . Find the  $3 \times 3$  matrix  $\tilde{T}$  and the vector  $\tilde{\mathbf{c}}$  explicitly. (f) Determine the spectral radius of the Jacobi matrix  $T$ , and use this to prove that the Jacobi method iteration will converge to the solution of  $A\mathbf{x} = \mathbf{b}$  for any choice of the initial approximation  $\mathbf{x}^{(0)}$ . (g) Determine the spectral radius of the Gauss–Seidel matrix  $\tilde{T}$ . Which method converges faster? (h) For the faster method, how many iterations would you expect to need to obtain 5 decimal place accuracy? (i) Test your prediction by computing the solution to the desired accuracy.

5. The matrix  $A = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}$  arises in the

finite difference (and finite element) discretization of the Poisson equation on a nine point square grid. (a) Is  $A$  diagonally dominant? (b) Find the spectral radius of the Jacobi and Gauss–Seidel iteration matrices. (c) Use formula (7.69) to fix the optimal value of the SOR parameter. Verify that the spectral radius of the resulting iteration matrix agrees with the second formula in (7.69). (d) For each iterative scheme, predict how many iterations are needed to solve the linear system  $A\mathbf{x} = \mathbf{e}_1$  to 3 decimal places, and then verify your predictions by direct computation.

6. The generalization of Exercise 5 to the Poisson equation on an  $n \times n$  grid results

in an  $n^2 \times n^2$  matrix in block tridiagonal form  $A = \begin{pmatrix} K & -I & & & \\ -I & K & -I & & \\ & -I & K & -I & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$ ,

in which  $K$  is the tridiagonal  $n \times n$  matrix with 4's on the main diagonal and  $-1$ 's on the sub- and super-diagonal, while  $I$  denotes an  $n \times n$  identity matrix. Use the known value of the Jacobi spectral radius  $\rho_J = \cos \frac{\pi}{n+1}$ , [47], to design an SOR method to solve the linear system  $A\mathbf{u} = \mathbf{f}$ . Run the Jacobi, Gauss–Seidel, and SOR methods for the cases  $n = 5$  and  $\mathbf{f} = \mathbf{e}_{13}$  and  $n = 25$  and  $\mathbf{f} = \mathbf{e}_{313}$  corresponding to a unit force at the center of the grid.