AIMS Exercise Set # 6

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1. Prove that the *Midpoint Method* (10.58) is a second order method.

2. Consider the initial value problem

$$\frac{du}{dt} = u(1-u), \qquad u(0) = .1,$$

for the *logistic differential equation*.

(a) Find an explicit formula for the solution. Describe in words the behavior of the solution for $t > 0$.

(b) Use the Euler Method with step sizes $h = .2$ and $.1$ to numerically approximate the solution on the interval $[0, 10]$. Does your numerical solution behave as predicted from part (a)? What is the maximal error on this interval? Can you predict the error when $h = .05$? Test your prediction by running the method and computing the error. Estimate the step size needed to compute the solution accurately to 10 decimal places (assuming no round off error)? How many steps are required? (Just predict — no need to test it.)

(c) Answer part (b) for the Improved Euler Method.

(d) Answer part (b) for the fourth order Runge–Kutta Method.

(e) Discuss the behavior of the solution, both analytical and numerical, for the alternative initial condition $u(0) = -.1$.

3. The nonlinear second order ordinary differential equation

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0$$

describes the motion of a pendulum under gravity without friction, with $\theta(t)$ representing the angle from the vertical: $\theta = 0$ represents the stable equilibrium where the pendulum is hanging straight down, while $\theta = \pi$ corresponds to the unstable equilibrium where the pendulum is standing straight up.

(a) Write out an equivalent first order system of ordinary differential equations in $u(t) = \theta(t)$, $v(t) = \dfrac{d\theta}{dt}$.

(b) Prove that the total energy of the pendulum

$$E(u,v) = \tfrac{1}{2}v^2 + (1-\cos u) = \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 + (1-\cos\theta)$$

is constant on solutions. *Hint*: Show that $dE/dt = 0$. Explain why each solution moves along a single level curve $E(u,v) = c$ of the energy.

(c) Use either your physical intuition and/or part (b) to describe the motion of the pendulum for the following initial conditions:

(i) $u(0) = 0$, $v(0) = 1$; (ii) $u(0) = 0$, $v(0) = 1.95$; (iii) $u(0) = 0$, $v(0) = 2$.

(d) Use the Euler Method to integrate the initial value problems for $0 \le t \le 50$ with step sizes $h = .1$ and $.01$. How accurately do your numerical solutions preserve the energy? How accurately do your numerical solutions follow the behavior you predicted in part (b)?

(e) Answer part (d) using the fourth order Runge–Kutta Method.