AIMS Exercise Set # 7

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1. In this exercise, you are asked to find "one-sided" finite difference formulas for derivatives. These are useful for approximating derivatives of functions at or near the boundary of their domain. (a) Construct a second order, one-sided finite difference formula that approximates the derivative $f'(x)$ using the values of $f(x)$ at the points $x, x+h$ and $x+2h$. (b) Find a finite difference formula for $f''(x)$ that involves the same values of f. What is the order of your formula? (c) Test your formulas by computing approximations to the first and second derivatives of $f(x) = e^{x^2}$ at $x = 1$ using step sizes $h = .1, .01$ and $.001$. What is the error in your numerical approximations? Are the errors compatible with the theoretical orders of the finite difference formulae? Discuss why or why not. (d) Answer part (c) at the point $x = 0$.

2. (a) Design an explicit numerical method for solving the initial-boundary value problem

$$u_t = \gamma\, u_{xx} + s(x), \qquad u(t,0) = u(t,1) = 0, \qquad u(0,x) = f(x), \qquad 0 \le x \le 1,$$

for the heat equation with a *source term* $s(x)$. (b) Test your scheme on the particular problem for

$$\gamma = \frac{1}{6}, \qquad s(x) = x(1-x)(10-22x), \qquad f(x) = \begin{cases} 2\,\left|\,x - \frac{1}{6}\,\right| - \frac{1}{3}, & 0 \le x \le \frac{1}{3}, \\ 0, & \frac{1}{3} \le x \le \frac{2}{3}, \\ \frac{1}{2} - 3\,\left|\,x - \frac{5}{6}\,\right|, & \frac{2}{3} \le x \le 1, \end{cases}$$

using space step sizes $h = .1$ and $.05$, and a suitably chosen time step k. (c) What is the long term behavior of your solution? Can you find a formula for its eventual profile? (d) Design an implicit scheme for the same problem. Does the behavior of your numerical solution change? What are the advantages of the implicit scheme?