AIMS Lecture Notes 2006

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5. Inner Products and Norms

The norm of a vector is a measure of its size. Besides the familiar Euclidean norm based on the dot product, there are a number of other important norms that are used in numerical analysis. In this section, we review the basic properties of inner products and norms.

5.1. Inner Products.

Some, but not all, norms are based on inner products. The most basic example is the familiar *dot product*

$$\langle \mathbf{v}\,, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{i=1}^{n} v_i w_i, \tag{5.1}$$

between (column) vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$, $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$, lying in the Euclidean space $\mathbb{R}^n$. A key observation is that the dot product (5.1) is equal to the matrix product

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \tag{5.2}$$

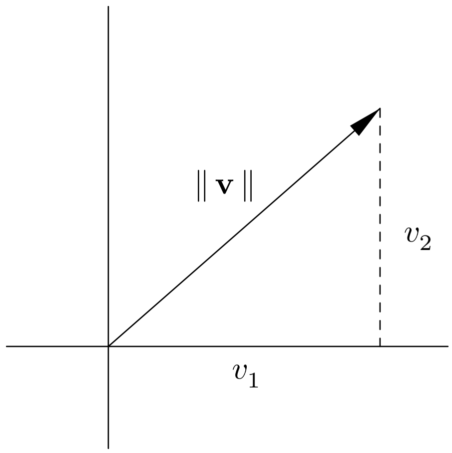
between the row vector $\mathbf{v}^T$ and the column vector $\mathbf{w}$. The key fact is that the dot product of a vector with itself,

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2,$$

is the sum of the squares of its entries, and hence, by the classical Pythagorean Theorem, equals the square of its length; see Figure 5.1. Consequently, the *Euclidean norm* or *length* of a vector is found by taking the square root:

$$\| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\,. \tag{5.3}$$

Note that every nonzero vector $\mathbf{v} \neq \mathbf{0}$ has positive Euclidean norm, $\| \mathbf{v} \| > 0$, while only the zero vector has zero norm: $\| \mathbf{v} \| = 0$ if and only if $\mathbf{v} = \mathbf{0}$. The elementary properties of dot product and Euclidean norm serve to inspire the abstract definition of more general inner products.



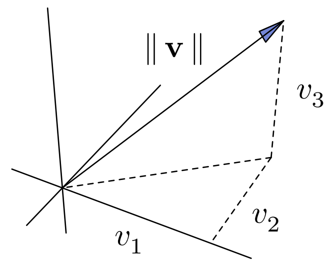


Figure 5.1. The Euclidean Norm in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 5.1. An *inner product* on the vector space $\mathbb{R}^n$ is a pairing that takes two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and produces a real number $\langle \mathbf{v}, \mathbf{w} \rangle \in \mathbb{R}$. The inner product is required to satisfy the following three axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and scalars $c, d \in \mathbb{R}$.

(*i*) *Bilinearity*:

$$\begin{aligned} \langle c\,\mathbf{u} + d\,\mathbf{v}, \mathbf{w} \rangle &= c\,\langle \mathbf{u}, \mathbf{w} \rangle + d\,\langle \mathbf{v}, \mathbf{w} \rangle, \\ \langle \mathbf{u}, c\,\mathbf{v} + d\,\mathbf{w} \rangle &= c\,\langle \mathbf{u}, \mathbf{v} \rangle + d\,\langle \mathbf{u}, \mathbf{w} \rangle. \end{aligned} \tag{5.4}$$

(*ii*) *Symmetry*:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle. \tag{5.5}$$

(*iii*) *Positivity*:

$$\langle \mathbf{v}, \mathbf{v} \rangle > 0 \qquad \text{whenever} \qquad \mathbf{v} \neq \mathbf{0}, \qquad \text{while} \qquad \langle \mathbf{0}, \mathbf{0} \rangle = 0. \tag{5.6}$$

Given an inner product, the associated *norm* of a vector $\mathbf{v} \in V$ is defined as the positive square root of the inner product of the vector with itself:

$$\| \mathbf{v} \| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}\,. \tag{5.7}$$

The positivity axiom implies that $\| \mathbf{v} \| \geq 0$ is real and non-negative, and equals 0 if and only if $\mathbf{v} = \mathbf{0}$ is the zero vector.

Example 5.2. While certainly the most common inner product on $\mathbb{R}^2$, the dot product

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

is by no means the only possibility. A simple example is provided by the *weighted inner product*

$$\langle \mathbf{v}, \mathbf{w} \rangle = 2 v_1 w_1 + 5 v_2 w_2, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{5.8}$$

Let us verify that this formula does indeed define an inner product. The symmetry axiom (5.5) is immediate. Moreover,

$$\begin{aligned} \langle c\,\mathbf{u} + d\,\mathbf{v}, \mathbf{w} \rangle &= 2(c u_1 + d v_1) w_1 + 5(c u_2 + d v_2) w_2 \\ &= c(2 u_1 w_1 + 5 u_2 w_2) + d(2 v_1 w_1 + 5 v_2 w_2) = c\,\langle \mathbf{u}, \mathbf{w} \rangle + d\,\langle \mathbf{v}, \mathbf{w} \rangle, \end{aligned}$$

which verifies the first bilinearity condition; the second follows by a very similar computation. Moreover, $\langle \mathbf{0}, \mathbf{0} \rangle = 0$, while

$$\langle \mathbf{v}, \mathbf{v} \rangle = 2v_1^2 + 5v_2^2 > 0 \qquad \text{whenever} \qquad \mathbf{v} \neq \mathbf{0},$$

since at least one of the summands is strictly positive. This establishes (5.8) as a legitimate inner product on $\mathbb{R}^2$. The associated *weighted norm* $\| \mathbf{v} \| = \sqrt{2v_1^2 + 5v_2^2}$ defines an alternative, "non-Pythagorean" notion of length of vectors and distance between points in the plane.

A less evident example of an inner product on $\mathbb{R}^2$ is provided by the expression

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + 4 v_2 w_2. \tag{5.9}$$

Bilinearity is verified in the same manner as before, and symmetry is immediate. Positivity is ensured by noticing that

$$\langle \mathbf{v}, \mathbf{v} \rangle = v_1^2 - 2v_1 v_2 + 4v_2^2 = (v_1 - v_2)^2 + 3v_2^2 \geq 0$$

is always non-negative, and, moreover, is equal to zero if and only if $v_1 - v_2 = 0, v_2 = 0$, i.e., only when $\mathbf{v} = \mathbf{0}$. We conclude that (5.9) defines yet another inner product on $\mathbb{R}^2$, with associated norm

$$\| \mathbf{v} \| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 - 2v_1 v_2 + 4v_2^2}\,.$$

The second example (5.8) is a particular case of a general class of inner products.

Example 5.3. Let $c_1, \ldots, c_n > 0$ be a set of *positive* numbers. The corresponding *weighted inner product* and *weighted norm* on $\mathbb{R}^n$ are defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} c_i v_i w_i, \qquad \| \mathbf{v} \| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\sum_{i=1}^{n} c_i v_i^2}\,. \tag{5.10}$$

The numbers c_i are the *weights*. Observe that the larger the weight c_i, the more the i^{th} coordinate of $\mathbf{v}$ contributes to the norm. We can rewrite the weighted inner product in the useful vector form

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T C \mathbf{w}, \qquad \text{where} \qquad C = \begin{pmatrix} c_1 & 0 & 0 & \ldots & 0 \\ 0 & c_2 & 0 & \ldots & 0 \\ 0 & 0 & c_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & c_n \end{pmatrix} \tag{5.11}$$

is the diagonal *weight matrix*. Weighted norms are particularly relevant in statistics and data fitting, [**12**], where one wants to emphasize certain quantities and de-emphasize others; this is done by assigning appropriate weights to the different components of the data vector $\mathbf{v}$.

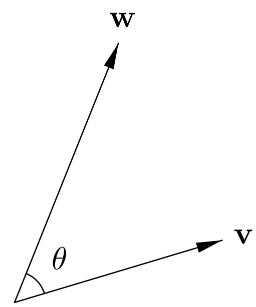


Figure 5.2. Angle Between Two Vectors.

5.2. Inequalities.

There are two absolutely fundamental inequalities that are valid for *any* inner product on any vector space. The first is inspired by the geometric interpretation of the dot product on Euclidean space in terms of the angle between vectors. It is named after two of the founders of modern analysis, Augustin Cauchy and Herman Schwarz, who established it in the case of the L^2 inner product on function space[†]. The more familiar triangle inequality, that the length of any side of a triangle is bounded by the sum of the lengths of the other two sides is, in fact, an immediate consequence of the Cauchy–Schwarz inequality, and hence also valid for any norm based on an inner product.

The Cauchy–Schwarz Inequality

In Euclidean geometry, the dot product between two vectors can be geometrically characterized by the equation

$$\mathbf{v} \cdot \mathbf{w} = \| \mathbf{v} \| \, \| \mathbf{w} \| \, \cos\theta, \tag{5.12}$$

where θ measures the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$, as drawn in Figure 5.2. Since

$$| \cos\theta | \leq 1,$$

the absolute value of the dot product is bounded by the product of the lengths of the vectors:

$$| \mathbf{v} \cdot \mathbf{w} | \ \leq \ \| \mathbf{v} \| \, \| \mathbf{w} \|.$$

This is the simplest form of the general *Cauchy–Schwarz inequality*. We present a simple, algebraic proof that does not rely on the geometrical notions of length and angle and thus demonstrates its universal validity for *any* inner product.

Theorem 5.4. *Every inner product satisfies the Cauchy–Schwarz inequality*

$$| \langle \mathbf{v}, \mathbf{w} \rangle | \ \leq \ \| \mathbf{v} \| \, \| \mathbf{w} \|, \qquad \textit{for all} \qquad \mathbf{v}, \mathbf{w} \in V. \tag{5.13}$$

Here, $\| \mathbf{v} \|$ is the associated norm, while $| \cdot |$ denotes absolute value of real numbers. Equality holds if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel vectors.

† Russians also give credit for its discovery to their compatriot Viktor Bunyakovskii, and, indeed, some authors append his name to the inequality.

Proof: The case when $\mathbf{w} = \mathbf{0}$ is trivial, since both sides of (5.13) are equal to 0. Thus, we may suppose $\mathbf{w} \neq \mathbf{0}$. Let $t \in \mathbb{R}$ be an arbitrary scalar. Using the three inner product axioms, we have

$$0 \leq \| \mathbf{v} + t\,\mathbf{w} \|^2 = \langle \mathbf{v} + t\,\mathbf{w}\,, \mathbf{v} + t\,\mathbf{w} \rangle = \| \mathbf{v} \|^2 + 2t \,\langle \mathbf{v}\,, \mathbf{w} \rangle + t^2 \,\| \mathbf{w} \|^2, \tag{5.14}$$

with equality holding if and only if $\mathbf{v} = -\,t\,\mathbf{w}$ — which requires $\mathbf{v}$ and $\mathbf{w}$ to be parallel vectors. We fix $\mathbf{v}$ and $\mathbf{w}$, and consider the right hand side of (5.14) as a quadratic function,

$$0 \leq p(t) = a\,t^2 + 2\,b\,t + c, \qquad \text{where} \qquad a = \| \mathbf{w} \|^2, \qquad b = \langle \mathbf{v}\,, \mathbf{w} \rangle, \qquad c = \| \mathbf{v} \|^2,$$

of the scalar variable t. To get the maximum mileage out of the fact that $p(t) \geq 0$, let us look at where it assumes its minimum, which occurs when its derivative is zero:

$$p'(t) = 2\,a\,t + 2\,b = 0, \qquad \text{and so} \qquad t \;=\; -\,\frac{b}{a} \;=\; -\,\frac{\langle \mathbf{v}\,, \mathbf{w} \rangle}{\| \mathbf{w} \|^2}\,.$$

Substituting this particular value of t into (5.14), we find

$$0 \;\leq\; \| \mathbf{v} \|^2 - 2\,\frac{\langle \mathbf{v}\,, \mathbf{w} \rangle^2}{\| \mathbf{w} \|^2} + \frac{\langle \mathbf{v}\,, \mathbf{w} \rangle^2}{\| \mathbf{w} \|^2} \;=\; \| \mathbf{v} \|^2 - \frac{\langle \mathbf{v}\,, \mathbf{w} \rangle^2}{\| \mathbf{w} \|^2}\,.$$

Rearranging this last inequality, we conclude that

$$\frac{\langle \mathbf{v}\,, \mathbf{w} \rangle^2}{\| \mathbf{w} \|^2} \;\leq\; \| \mathbf{v} \|^2, \qquad \text{or} \qquad \langle \mathbf{v}\,, \mathbf{w} \rangle^2 \;\leq\; \| \mathbf{v} \|^2 \, \| \mathbf{w} \|^2.$$

Also, as noted above, equality holds if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel. Taking the (positive) square root of both sides of the final inequality completes the proof of the Cauchy–Schwarz inequality (5.13). *Q.E.D.*

Given any inner product, we can use the quotient

$$\cos\theta = \frac{\langle \mathbf{v}\,, \mathbf{w} \rangle}{\| \mathbf{v} \| \; \| \mathbf{w} \|} \tag{5.15}$$

to define the "angle" between the vector space elements $\mathbf{v}, \mathbf{w} \in V$. The Cauchy–Schwarz inequality tells us that the ratio lies between -1 and $+1$, and hence the angle θ is well defined, and, in fact, unique if we restrict it to lie in the range $0 \leq \theta \leq \pi$.

For example, the vectors $\mathbf{v} = (\,1, 0, 1\,)^T$, $\mathbf{w} = (\,0, 1, 1\,)^T$ have dot product $\mathbf{v} \cdot \mathbf{w} = 1$ and norms $\| \mathbf{v} \| = \| \mathbf{w} \| = \sqrt{2}$. Hence the Euclidean angle between them is given by

$$\cos\theta = \frac{1}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}\,, \qquad \text{and so} \qquad \theta = \tfrac{1}{3}\,\pi = 1.0472\ldots\,.$$

On the other hand, if we adopt the weighted inner product $\langle \mathbf{v}\,, \mathbf{w} \rangle = v_1\,w_1 + 2\,v_2\,w_2 + 3\,v_3\,w_3$, then $\mathbf{v} \cdot \mathbf{w} = 3$, $\| \mathbf{v} \| = 2$, $\| \mathbf{w} \| = \sqrt{5}$, and hence their "weighted" angle becomes

$$\cos\theta = \frac{3}{2\,\sqrt{5}} = .67082\ldots\,, \qquad \text{with} \qquad \theta = .835482\ldots\,.$$

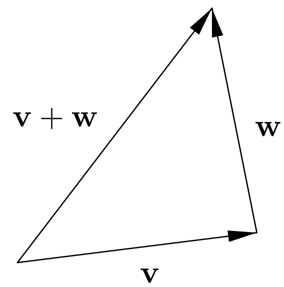


Figure 5.3. Triangle Inequality.

Thus, the measurement of angle (and length) is dependent upon the choice of an underlying inner product.

In Euclidean geometry, perpendicular vectors meet at a right angle, $\theta = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, with $\cos\theta = 0$. The angle formula (5.12) implies that the vectors $\mathbf{v}, \mathbf{w}$ are perpendicular if and only if their dot product vanishes: $\mathbf{v} \cdot \mathbf{w} = 0$. Perpendicularity is of interest in general inner product spaces, but, for historical reasons, has been given a more suggestive name.

Definition 5.5. Two elements $\mathbf{v}, \mathbf{w} \in V$ of an inner product space V are called *orthogonal* if their inner product vanishes: $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

In particular, the zero element is orthogonal to everything: $\langle \mathbf{0}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$. Orthogonality is a remarkably powerful tool in all applications of linear algebra, and often serves to dramatically simplify many computations.

The Triangle Inequality

The familiar triangle inequality states that the length of one side of a triangle is at most equal to the sum of the lengths of the other two sides. Referring to Figure 5.3, if the first two sides are represented by vectors $\mathbf{v}$ and $\mathbf{w}$, then the third corresponds to their sum $\mathbf{v} + \mathbf{w}$. The triangle inequality turns out to be an elementary consequence of the Cauchy–Schwarz inequality, and hence is valid in *any* inner product space.

Theorem 5.6. *The norm associated with an inner product satisfies the* triangle inequality

$$\| \mathbf{v} + \mathbf{w} \| \leq \| \mathbf{v} \| + \| \mathbf{w} \| \qquad \textit{for all} \qquad \mathbf{v}, \mathbf{w} \in V. \tag{5.16}$$

Equality holds if and only if $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel vectors.*

Proof: We compute

$$\begin{aligned} \| \mathbf{v} + \mathbf{w} \|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle = \| \mathbf{v} \|^2 + 2 \langle \mathbf{v}, \mathbf{w} \rangle + \| \mathbf{w} \|^2 \\ &\leq \| \mathbf{v} \|^2 + 2 \, | \langle \mathbf{v}, \mathbf{w} \rangle | + \| \mathbf{w} \|^2 \leq \| \mathbf{v} \|^2 + 2 \| \mathbf{v} \| \, \| \mathbf{w} \| + \| \mathbf{w} \|^2 \\ &= \big(\| \mathbf{v} \| + \| \mathbf{w} \| \big)^2, \end{aligned}$$

where the middle inequality follows from Cauchy–Schwarz. Taking square roots of both sides and using positivity completes the proof. *Q.E.D.*

Example 5.7. The vectors $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ sum to $\mathbf{v} + \mathbf{w} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$. Their Euclidean norms are $\| \mathbf{v} \| = \sqrt{6}$ and $\| \mathbf{w} \| = \sqrt{13}$, while $\| \mathbf{v} + \mathbf{w} \| = \sqrt{17}$. The triangle inequality (5.16) in this case says $\sqrt{17} \leq \sqrt{6} + \sqrt{13}$, which is valid.

5.3. Norms.

Every inner product gives rise to a norm that can be used to measure the magnitude or length of the elements of the underlying vector space. However, not every norm that is used in analysis and applications arises from an inner product. To define a general norm, we will extract those properties that do not directly rely on the inner product structure.

Definition 5.8. A *norm* on the vector space $\mathbb{R}^n$ assigns a real number $\| \mathbf{v} \|$ to each vector $\mathbf{v} \in V$, subject to the following axioms for every $\mathbf{v}, \mathbf{w} \in V$, and $c \in \mathbb{R}$.

(*i*) *Positivity*: $\| \mathbf{v} \| \geq 0$, with $\| \mathbf{v} \| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

(*ii*) *Homogeneity*: $\| c\, \mathbf{v} \| = | c | \, \| \mathbf{v} \|$.

(*iii*) *Triangle inequality*: $\| \mathbf{v} + \mathbf{w} \| \leq \| \mathbf{v} \| + \| \mathbf{w} \|$.

As we now know, every inner product gives rise to a norm. Indeed, positivity of the norm is one of the inner product axioms. The homogeneity property follows since

$$\| c\, \mathbf{v} \| \;=\; \sqrt{\langle c\, \mathbf{v}\, , c\, \mathbf{v} \rangle} \;=\; \sqrt{c^2 \, \langle \mathbf{v}\, , \mathbf{v} \rangle} \;=\; | c | \sqrt{\langle \mathbf{v}\, , \mathbf{v} \rangle} \;=\; | c | \, \| \mathbf{v} \|.$$

Finally, the triangle inequality for an inner product norm was established in Theorem 5.6. Let us introduce some of the principal examples of norms that do not come from inner products.

The 1*–norm* of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n$ is defined as the sum of the absolute values of its entries:

$$\| \mathbf{v} \|_1 = | v_1 | + | v_2 | + \;\cdots\; + | v_n |. \tag{5.17}$$

The *max* or ∞*–norm* is equal to its maximal entry (in absolute value):

$$\| \mathbf{v} \|_\infty = \max \{ \, | v_1 |, | v_2 |, \;\ldots\; , | v_n | \, \}. \tag{5.18}$$

Verification of the positivity and homogeneity properties for these two norms is straightforward; the triangle inequality is a direct consequence of the elementary inequality

$$| a + b | \leq | a | + | b |, \qquad a, b \in \mathbb{R},$$

for absolute values.

The Euclidean norm, 1–norm, and ∞–norm on $\mathbb{R}^n$ are just three representatives of the general *p–norm*

$$\| \mathbf{v} \|_p \;=\; \sqrt[p]{\sum_{i=1}^{n} | v_i |^p} \, . \tag{5.19}$$

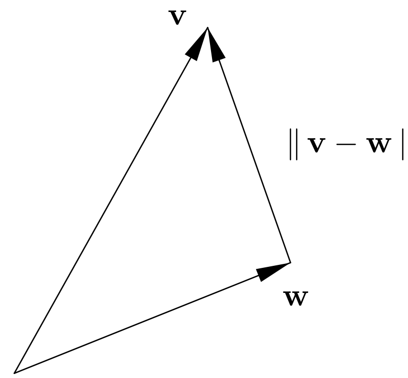


Figure 5.4. Distance Between Vectors.

This quantity defines a norm for any $1 \leq p < \infty$. The ∞–norm is a limiting case of (5.19) as $p \to \infty$. Note that the Euclidean norm (5.3) is the 2–norm, and is often designated as such; it is the only p–norm which comes from an inner product. The positivity and homogeneity properties of the p–norm are not hard to establish. The triangle inequality, however, is not trivial; in detail, it reads

$$\sqrt[p]{\sum_{i=1}^{n} |\, v_i + w_i \,|^p} \;\leq\; \sqrt[p]{\sum_{i=1}^{n} |\, v_i \,|^p} \;+\; \sqrt[p]{\sum_{i=1}^{n} |\, w_i \,|^p}\,, \tag{5.20}$$

and is known as *Minkowski's inequality*. A complete proof can be found in [**30**].

Every norm defines a *distance* between vector space elements, namely

$$d(\mathbf{v}, \mathbf{w}) = \| \, \mathbf{v} - \mathbf{w} \, \|. \tag{5.21}$$

For the standard dot product norm, we recover the usual notion of distance between points in Euclidean space. Other types of norms produce alternative (and sometimes quite useful) notions of distance that are, nevertheless, subject to all the familiar properties:

(*a*) *Symmetry*: $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{w}, \mathbf{v})$;
(*b*) *Positivity*: $d(\mathbf{v}, \mathbf{w}) = 0$ if and only if $\mathbf{v} = \mathbf{w}$;
(*c*) *Triangle Inequality*: $d(\mathbf{v}, \mathbf{w}) \leq d(\mathbf{v}, \mathbf{z}) + d(\mathbf{z}, \mathbf{w})$.

Equivalence of Norms

While there are many different types of norms on $\mathbb{R}^n$, in a certain sense, they are all more or less equivalent†. "Equivalence" does not mean that they assume the same value, but rather that they are always close to one another, and so, for many analytical purposes, may be used interchangeably. As a consequence, we may be able to simplify the analysis of a problem by choosing a suitably adapted norm.

† This statement remains valid in any finite-dimensional vector space, but is *not* correct in infinite-dimensional function spaces.

Theorem 5.9. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be any two norms on $\mathbb{R}^n$. Then there exist positive constants $c^\star, C^\star > 0$ such that*

$$c^\star \, \| \mathbf{v} \|_1 \leq \| \mathbf{v} \|_2 \leq C^\star \, \| \mathbf{v} \|_1 \qquad \textit{for every} \qquad \mathbf{v} \in \mathbb{R}^n. \tag{5.22}$$

Proof: We just sketch the basic idea, leaving the details to a more rigorous real analysis course, cf. [**11**; §7.6]. We begin by noting that a norm defines a continuous real-valued function $f(\mathbf{v}) = \| \mathbf{v} \|$ on $\mathbb{R}^n$. (Continuity is, in fact, a consequence of the triangle inequality.) Let $S_1 = \left\{ \| \mathbf{u} \|_1 = 1 \right\}$ denote the unit sphere of the first norm. Any continuous function defined on a compact set achieves both a maximum and a minimum value. Thus, restricting the second norm function to the unit sphere S_1 of the first norm, we can set

$$c^\star = \min \{ \, \| \mathbf{u} \|_2 \mid \mathbf{u} \in S_1 \, \}, \qquad C^\star = \max \{ \, \| \mathbf{u} \|_2 \mid \mathbf{u} \in S_1 \, \}. \tag{5.23}$$

Moreover, $0 < c^\star \leq C^\star < \infty$, with equality holding if and only if the norms are the same. The minimum and maximum (5.23) will serve as the constants in the desired inequalities (5.22). Indeed, by definition,

$$c^\star \leq \| \mathbf{u} \|_2 \leq C^\star \qquad \text{when} \qquad \| \mathbf{u} \|_1 = 1, \tag{5.24}$$

which proves that (5.22) is valid for all unit vectors $\mathbf{v} = \mathbf{u} \in S_1$. To prove the inequalities in general, assume $\mathbf{v} \neq \mathbf{0}$. (The case $\mathbf{v} = \mathbf{0}$ is trivial.) The homogeneity property of the norm implies that $\mathbf{u} = \mathbf{v} / \| \mathbf{v} \|_1 \in S_1$ is a unit vector in the first norm: $\| \mathbf{u} \|_1 = \| \mathbf{v} \| / \| \mathbf{v} \|_1 = 1$. Moreover, $\| \mathbf{u} \|_2 = \| \mathbf{v} \|_2 / \| \mathbf{v} \|_1$. Substituting into (5.24) and clearing denominators completes the proof of (5.22). *Q.E.D.*

Example 5.10. For example, consider the Euclidean norm $\|\cdot\|_2$ and the max norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$. The bounding constants are found by minimizing and maximizing $\| \mathbf{u} \|_\infty = \max\{ \, | u_1 |, \ldots, | u_n | \, \}$ over all unit vectors $\| \mathbf{u} \|_2 = 1$ on the (round) unit sphere. The maximal value is achieved at the poles $\pm \mathbf{e}_k$, with $\| \pm \mathbf{e}_k \|_\infty = C^\star = 1$ The minimal value is attained at the points $\left(\pm \frac{1}{\sqrt{n}}, \ldots, \pm \frac{1}{\sqrt{n}} \right)$, whereby $c^\star = \frac{1}{\sqrt{n}}$. Therefore,

$$\frac{1}{\sqrt{n}} \, \| \mathbf{v} \|_2 \leq \| \mathbf{v} \|_\infty \leq \| \mathbf{v} \|_2. \tag{5.25}$$

We can interpret these inequalities as follows. Suppose $\mathbf{v}$ is a vector lying on the unit sphere in the Euclidean norm, so $\| \mathbf{v} \|_2 = 1$. Then (5.25) tells us that its ∞ norm is bounded from above and below by $\frac{1}{\sqrt{n}} \leq \| \mathbf{v} \|_\infty \leq 1$. Therefore, the Euclidean unit sphere sits inside the ∞ norm unit sphere and outside the ∞ norm sphere of radius $\frac{1}{\sqrt{n}}$.