Chapter 2

Lie Groups and Lie Algebras

The symmetry groups that arise most often in the applications to geometry and differential equations are Lie groups of transformations acting on a finite-dimensional manifold. Since Lie groups will be one of the cornerstones of our investigations, it is essential that we gain a basic familiarity with these fundamental mathematical objects. The present chapter is devoted to a survey of a number of fundamental facts concerning Lie groups and Lie algebras, and their actions on manifolds. More detailed presentations can be found in a variety of references, including [**43**, **48**, **54**].

Recall first that a *group* is a set G that has an associative (but not necessarily commutative) multiplication operation, denoted $g \cdot h$ for group elements $g, h \in G$. The group must also contain a (necessarily unique) identity element, denoted e, and each group element g has an inverse g^{-1} satisfying $g \cdot g^{-1} = g^{-1} \cdot g = e$.

The continuous nature of Lie groups is formalized by the requirement that, in addition to satisfying the basic group axioms, they are also endowed with the structure of a smooth manifold.

Definition 2.1. A *Lie group* G is a smooth manifold which is also a group, such that the group multiplication $(g, h) \mapsto g \cdot h$ and inversion $g \mapsto g^{-1}$ define smooth maps.

Analytic Lie groups are defined by analytic manifolds, with analytic multiplication and inversion maps. Most of our examples are, in fact, analytic; indeed, any smooth Lie group can be endowed with an analytic structure, [**48**]. Often, an r-dimensional Lie group is referred to as an r parameter group, the "group parameters" referring to a choice of local coordinates on the group manifold.

Example 2.2. The simplest example of an r parameter Lie group is the abelian (meaning commutative) Lie group $\mathbb{R}^r$. The group operation is given by vector addition. The identity element is the zero vector, and the inverse of a vector x is the vector $-x$.

Example 2.3. The prototypical example of a real Lie group is the general linear group $\mathrm{GL}(n, \mathbb{R})$ consisting of all invertible $n \times n$ real matrices, with matrix multiplication defining the group multiplication, and matrix inversion defining the inverse. Equivalently, $\mathrm{GL}(n, \mathbb{R})$ can be regarded as the group of all invertible linear transformations on $\mathbb{R}^n$, where composition serves to define the group operation. Note that $\mathrm{GL}(n, \mathbb{R})$ is an n^2-dimensional manifold, simply because it is an open subset (namely, where the determinant is nonzero) of the space of all $n \times n$ matrices, which is itself isomorphic to $\mathbb{R}^{n^2}$. The group operations are clearly analytic. Similarly, the prototypical complex Lie group is the group $\mathrm{GL}(n, \mathbb{C})$ of invertible $n \times n$ complex matrices. We will often employ the notation $\mathrm{GL}(n)$ to mean either the real or complex general linear group — in such usage, the precise version will either be irrelevant or clear from the context.

A subset $H \subset G$ of a group is a *subgroup* provided the multiplication and inversion operations restrict to it. The subgroup H is called a *Lie subgroup* if it is also an (immersed) submanifold parametrized by a smooth group homomorphism $F\colon \widetilde{H} \to H \subset G$. The term "homomorphism" means that F respects the group operations: $F(g \cdot h) = F(g) \cdot F(h)$, $F(e) = e$, $F(g^{-1}) = F(g)^{-1}$, so the parameter space $\widetilde{H}$ is a Lie group isomorphic to H. Most (but not all!) Lie groups can be realized as Lie subgroups of the general linear group $\mathrm{GL}(n)$; these are the so-called "matrix Lie groups". The following result is useful for analyzing matrix (and other) subgroups; see [**54**; Theorem 3.42] for a proof.

Proposition 2.4. *If $H \subset G$ is a subgroup of a Lie group G, which is also a (topologically) closed subset, then H is a Lie subgroup of G.*

In particular, if G is a Lie group, then the connected component of G containing the identity element, denoted G^+, is itself a Lie group. For example, the real general linear group consists of two disconnected components, indexed by the sign of the determinant. The subgroup $\mathrm{GL}(n,\mathbb{R})^+ = \{\, A \in \mathrm{GL}(n,\mathbb{R}) \mid \det A > 0 \,\}$, consisting of orientation-preserving linear transformations, is the connected component containing the identity matrix. On the other hand, the complex general linear group $\mathrm{GL}(n,\mathbb{C})$ is connected. The other connected components of a general Lie group are recovered by multiplying the connected component G^+ by a single group element lying therein. For example, every orientation-reversing matrix in $\mathrm{GL}(n,\mathbb{R})$ can be obtained by multiplying an orientation-preserving matrix by a fixed matrix with negative determinant, e.g., the diagonal matrix with entries $-1, +1, \ldots, +1$.

Example 2.5. The most important Lie groups are the three families of "classical groups". The *special linear* or *unimodular group* is $\mathrm{SL}(n) = \{\, A \in \mathrm{GL}(n) \mid \det A = 1 \,\}$ consisting of all volume-preserving linear transformations. In other words, $\mathrm{SL}(n)$ is the group of linear symmetries of the standard volume form $dx = dx^1 \wedge \cdots \wedge dx^n$. Both the real and complex versions are connected, and have dimension $n^2 - 1$. The *orthogonal group* $\mathrm{O}(n) = \{\, A \in \mathrm{GL}(n) \mid A^T A = 1\!\!1 \,\}$ is the group of norm-preserving linear transformations — rotations and reflections — and forms the group of linear symmetries of the Euclidean metric $ds^2 = (dx^1)^2 + \cdots + (dx^n)^2$ on $\mathbb{R}^n$. The component containing the identity forms the *special orthogonal group* $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n)$, consisting of just the rotations. As we shall see, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ have dimension $\frac{1}{2}n(n-1)$. The *symplectic group* is the $r(2r+1)$-dimensional Lie group

$$\mathrm{Sp}(2r) = \left\{\, A \in \mathrm{GL}(2r,\mathbb{R}) \;\middle|\; A^T \mathrm{J} A = \mathrm{J} \,\right\}, \tag{2.1}$$

consisting of linear transformations which preserve the canonical nonsingular skew-symmetric matrix $\mathrm{J} = \begin{pmatrix} 0 & -1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}$. In particular, $\mathrm{Sp}(2) \simeq \mathrm{SL}(2)$. We remark that a linear transformation $L\colon x \mapsto Ax$ lies in $\mathrm{Sp}(2r)$ if and only if it is a canonical transformation, in the sense of Hamiltonian mechanics, hence $\mathrm{Sp}(2r)$ forms the group of linear symmetries of the canonical symplectic form $\Omega = dx^1 \wedge dx^2 + \cdots + dx^{2r-1} \wedge dx^{2r}$.

Exercise 2.6. Let G be a group. If $H \subset G$ is a subgroup, then the *quotient space* G/H is defined as the set of all left cosets $g \cdot H = \{ g \cdot h \mid h \in H \}$, for $g \in G$. Show that if H is a *normal* subgroup, meaning that it equals its conjugate subgroups: $gHg^{-1} = H$ for all $g \in G$, then G/H can be given the structure of a group. If H is an s-dimensional closed subgroup of an r-dimensional Lie group G, then G/H can be endowed with the structure of a smooth manifold of dimension $r - s$, which is a Lie group provided H is a normal closed subgroup.

At the other extreme, a *discrete subgroup* $\Gamma \subset G$ of a Lie group is a subgroup whose intersection with some neighborhood $\{e\} \subset U \subset G$ of the identity element consists only of the identity: $\Gamma \cap U = \{e\}$. Examples include the integer lattices $\mathbb{Z}^r \subset \mathbb{R}^r$, and the group $\mathrm{SL}(n, \mathbb{Z})$ of integer matrices of determinant 1. Although discrete groups can be regarded as zero-dimensional Lie groups, they are totally disconnected, and so cannot be handled by any of the wonderful tools associated with connected Lie groups. The quotient group G/Γ by a discrete normal subgroup, then, is a Lie group which is locally isomorphic to G itself; see Theorem 2.52 below for more details.

Transformation Groups

In most instances, groups are not given to us in the abstract, but, rather, concretely as a family of transformations acting on a space. In the case of Lie groups, the most natural setting is as groups of transformations acting smoothly on a manifold.

Definition 2.7. A *transformation group* acting on a smooth manifold M is determined by a Lie group G and smooth map $\Phi\colon G \times M \to M$, denoted by $\Phi(g, x) = g \cdot x$, which satisfies

$$e \cdot x = x, \qquad g \cdot (h \cdot x) = (g \cdot h) \cdot x, \qquad \text{for all } \ x \in M,\ g \in G. \tag{2.2}$$

Condition (2.2) implies that the inverse group element g^{-1} determines the inverse to the transformation defined by the group element g, so that each group element g induces a diffeomorphism from M to itself. Definition 2.7 assumes that the group action is *global*, meaning that $g \cdot x$ is defined for every $g \in G$ and every $x \in M$. In applications, though, a group action may only be defined "locally", meaning that, for a given $x \in M$, the transformation $g \cdot x$ is only defined for group elements g sufficiently near the identity. Thus, for a *local transformation group*, the map Φ is defined on an open subset $\{e\} \times M \subset \mathcal{V} \subset G \times M$, and the conditions (2.2) are imposed wherever they make sense.

Example 2.8. An obvious example is provided by the usual linear action of the general linear group $\mathrm{GL}(n, \mathbb{R})$, acting by matrix multiplication on column vectors $x \in \mathbb{R}^n$. This action clearly induces a linear action of any subgroup of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$. Since linear transformations map lines to lines, there is an induced action of $\mathrm{GL}(n, \mathbb{R})$, and its subgroups, on the projective space $\mathbb{RP}^{n-1}$. Of particular importance is the planar case, $n = 2$, so we discuss this in some detail. The linear action of $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$ is

$$(x, y) \longmapsto (\alpha x + \beta y, \gamma x + \delta y), \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2). \tag{2.3}$$

Now, as in Example 1.3, we can identify the projective line $\mathbb{RP}^1$ with a circle S^1. If we use the projective coordinate $p = x/y$, the induced action is given by the *linear fractional* or *Möbius transformations*

$$p \longmapsto \frac{\alpha p + \beta}{\gamma p + \delta}, \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2). \tag{2.4}$$

In this coordinate chart, the x–axis in $\mathbb{R}^2$ is identified with the point $p = \infty$ in $\mathbb{RP}^1$, and the linear fractional transformations (2.4) have a well-defined extension to include the point at infinity. Alternatively, we can regard (2.4) as defining a local action of $\mathrm{GL}(2,\mathbb{R})$ on the real line $\mathbb{R}$, defined on the subset $\mathcal{V} = \{ (A,p) \mid \gamma p + \delta \neq 0 \} \subset \mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}$.

Similarly, the complex general linear group $\mathrm{GL}(n,\mathbb{C})$ acts linearly on $\mathbb{C}^n$, and there is an induced action on the complex projective space $\mathbb{CP}^{n-1}$. In particular, the action (2.3) of $\mathrm{GL}(2,\mathbb{C})$ on $\mathbb{C}^2$ induces an action on complex projective space $\mathbb{CP}^1 \simeq S^2$, given by complex linear fractional transformations (2.4). As in the real case, it restricts to define a local action of $\mathrm{GL}(2,\mathbb{C})$ on the complex plane $\mathbb{C}$.

Example 2.9. An important, but almost tautological example, is provided by the action of a Lie group on itself by multiplication. In this case, the manifold M coincides with G itself, and the map $\Phi: G \times G \mapsto G$ is given by left multiplication: $\Phi(g,h) = g \cdot h$. Alternatively, we can let G act on itself by right multiplication via $\widetilde{\Phi}(g,h) = h \cdot g^{-1}$; in this case, the inverse ensures that the composition laws (2.2) remain valid.

Example 2.10. Let $\mathbf{v}$ be a vector field on a manifold M. The properties (1.6) imply that the flow $\exp(t\mathbf{v})$ defines a (local) action of the one-parameter group $\mathbb{R}$, parametrized by the "time" t, on the manifold M. For example, if $M = \mathbb{R}$, with coordinate x, then the vector fields ∂_x, $x\partial_x$ and $x^2\partial_x$ generate flows, given explicitly in Example 1.22, which form one-parameter subgroups of the projective group (2.4) (identifying p with x). In fact, these three particular subgroups — translations, scalings, and inversions — serve to generate the full projective group, a fact that will become clear from the Lie algebra methods discussed below.

Example 2.11. The (real) *affine group* $\mathrm{A}(n)$ is defined as the group of affine transformations $x \mapsto Ax + a$ in $\mathbb{R}^n$. Thus, the affine group is parametrized by a pair (A,a) consisting of an invertible matrix $A \in \mathrm{GL}(n)$ and a vector $a \in \mathbb{R}^n$. The affine group $\mathrm{A}(n)$ has dimension $n(n+1)$, being isomorphic, as a manifold, to the Cartesian product space $\mathrm{GL}(n) \times \mathbb{R}^n$. However, $\mathrm{A}(n)$ is *not* the Cartesian product of the groups $\mathrm{GL}(n)$ and $\mathbb{R}^n$ since the group multiplication law is given by $(A,a) \cdot (B,b) = (AB, a + Ab)$. The affine group can be realized as a subgroup of $\mathrm{GL}(n+1)$ by identifying the group element (A,a) with the $(n+1) \times (n+1)$ matrix $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$.

The affine group provides an example of a general construction known as the "semi-direct product". In general, if G and H are, respectively, r- and s-dimensional Lie groups, their Cartesian product $G \times H$ is an $(r+s)$-dimensional Lie group with the group operation being defined in the obvious manner: $(g,h) \cdot (\tilde{g},\tilde{h}) = (g \cdot \tilde{g}, h \cdot \tilde{h})$. If, in addition, G acts as a group of transformations on the Lie group H, satisfying $g \cdot (h \cdot \tilde{h}) = (g \cdot h) \cdot$

$(g \cdot \tilde{h})$, then the *semi-direct product* $G \ltimes H$ is the $(r+s)$-dimensional Lie group which, as a manifold just looks like the Cartesian product $G \times H$, but whose multiplication is given by $(g,h) \cdot (\tilde{g},\tilde{h}) = (g \cdot \tilde{g}, h \cdot (g \cdot \tilde{h}))$. In particular, the affine group is the semi-direct product of the general linear group $\mathrm{GL}(n)$ acting on the abelian group $\mathbb{R}^n$, written $\mathrm{A}(n) = \mathrm{GL}(n) \ltimes \mathbb{R}^n$. Another important example is provided by the *Euclidean group* $\mathrm{E}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$, which is generated by the groups of orthogonal transformations and translations, and thus forms a subgroup of the full affine group. Its connected component is $\mathrm{SE}(n) = \mathrm{SO}(n) \ltimes \mathbb{R}^n$. The Euclidean group has as alternative characterization as the group of isometries, meaning norm-preserving transformations, of Euclidean space, and thus, according to Klein's characterization of geometry based on groups, lies at the foundation of Euclidean geometry. We also define the *equi-affine subgroup* $\mathrm{SA}(n) = \mathrm{SL}(n) \ltimes \mathbb{R}^n$, which consists of volume-preserving affine transformations, and forms the basis of affine geometry, cf. [**20**].

Given an action of the Lie group G on a manifold M, the *isotropy* subgroup of a point $x \in M$ is $G_x = \{\, g \mid g \cdot x = x \,\} \subset G$ consisting of all group elements g which fix x. Proposition 2.4 demonstrates that G_x is a Lie subgroup, and can be viewed as the symmetry subgroup for the point x. For example, the isotropy subgroup of the projective group (2.4) fixing the origin $p = 0$ is the group of invertible lower triangular matrices. In particular, $G_x = G$ if and only if x is a *fixed point* of the group action. If $g \cdot x = y$, then the isotropy group at y is conjugate to that at x, meaning $G_y = g \cdot G_x \cdot g^{-1}$. A transformation group acts *freely* if the isotropy subgroups are all trivial, $G_x = \{e\}$ for all $x \in M$, which means that, for $e \neq g \in G$, we have $g \cdot x \neq x$ for any $x \in M$. The action is *locally free* if this holds for all $g \neq e$ in a neighborhood of the identity; equivalently, the isotropy subgroups are discrete subgroups of G. A transformation group acts *effectively* if different group elements have different actions, so that $g \cdot x = h \cdot x$ for all $x \in M$ if and only if $g = h$; this is equivalent to the statement that the only group element acting as the identity transformation is the identity element of G. The effectiveness of a group action is measured by its *global isotropy subgroup* $G_M = \bigcap_{x \in M} G_x = \{\, g \mid g \cdot x = x \text{ for all } x \in M \,\}$, so that G acts effectively if and only if $G_M = \{e\}$. Clearly, a free group action is effective, although the converse is certainly not true. Slightly more generally, a Lie group G is said to act *locally effectively* if G_M is a discrete subgroup of G, which is equivalent to the existence of a neighborhood U of the identity e such that $G_M \cap U = \{e\}$.

Proposition 2.12. *Suppose G is a transformation group acting on a manifold M. Then the global isotropy subgroup G_M is a* normal *Lie subgroup of G. Moreover, there is a well-defined effective action of the* quotient group *G/G_M on M, which "coincides" with that of G in the sense that two group elements g and $\tilde{g}$ have the same action on M, so $g \cdot x = \tilde{g} \cdot x$ for all $x \in M$, if and only if they have the same image in $\widehat{G}$, so $\tilde{g} = g \cdot h$ for some $h \in G_M$.*

Thus, if a transformation group G does not act effectively, we can, without any significant loss of information or generality, replace it by the quotient group G/G_M, which does act effectively, and in the same manner as G does. For a locally effective action, the quotient group G/G_M is a Lie group having the same dimension, and the same local structure, as G itself. A group acts *effectively freely* if and only if G/G_M acts freely; this is equivalent to the statement that every local isotropy subgroup equals the global isotropy subgroup: $G_x = G_M$, $x \in M$.

Example 2.13. The general linear group $\mathrm{GL}(n, \mathbb{R})$ acts effectively on $\mathbb{R}^n$. The isotropy group of a nonzero point $0 \neq x \in \mathbb{R}^n$ can be identified with the affine group $\mathrm{A}(n-1)$ of Example 2.11; the isotropy group of 0 is all of $\mathrm{GL}(n, \mathbb{R})$. The induced action on projective space $\mathbb{RP}^{n-1}$ is no longer effective since the diagonal matrices $\lambda \mathbb{1}$ act trivially on lines through the origin. The quotient subgroup $\mathrm{PGL}(n, \mathbb{R}) = \mathrm{GL}(n, \mathbb{R}) / \{\lambda \mathbb{1}\}$ is called the *projective group*. Note that if n is odd, we can identify $\mathrm{PGL}(n, \mathbb{R}) \simeq \mathrm{SL}(n, \mathbb{R})$ with the special linear group. However, if n is even, $\mathrm{SL}(n, \mathbb{R})$ only acts locally effectively on $\mathbb{RP}^{n-1}$ since $-\mathbb{1} \in \mathrm{SL}(n, \mathbb{R})$, and hence $\mathrm{PSL}(n, \mathbb{R}) = \mathrm{SL}(n, \mathbb{R}) / \{\pm \mathbb{1}\}$ is equal to the connected component of $\mathrm{PGL}(n, \mathbb{R})$ containing the identity.

Exercise 2.14. Discuss the corresponding action of $\mathrm{GL}(n, \mathbb{C})$ on complex projective space $\mathbb{CP}^{n-1}$.

Example 2.15. Consider the action of $\mathrm{GL}(n, \mathbb{R})$ on the space of all real $n \times n$ matrices given by $X \mapsto A X A^T$ for $A \in \mathrm{GL}(n)$. The orthogonal Lie group is the isotropy group for the identity matrix $\mathbb{1}$. Similarly, if $n = 2r$ is even, the isotropy subgroup of the canonical nondegenerate skew-symmetric matrix J given in (2.1) is the symplectic group $\mathrm{Sp}(2r)$. The isotropy group of the diagonal matrix with p entries equaling $+1$ and $m-p$ entries equaling -1 is the pseudo-orthogonal group $\mathrm{O}(p, m-p)$; for example $\mathrm{O}(1,1)$ is the one-dimensional group of hyperbolic rotations $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$.

Invariant Subsets and Equations

Let G be a (local) group of transformations acting on the manifold M. A subset $S \subset M$ is called *G–invariant* if it is unchanged by the group transformations, meaning $g \cdot x \in S$ whenever $g \in G$ and $x \in S$ (provided $g \cdot x$ is defined if the action is only local). The most important classes of invariant subsets are the varieties defined by the vanishing of one or more functions. A group G is called a *symmetry group* of a system of equations

$$F_1(x) = \cdots = F_k(x) = 0, \tag{2.5}$$

if and only if the variety $\mathcal{S}_F = \{\, x \mid F_1(x) = \cdots = F_k(x) = 0 \,\}$ is a G-invariant subset of M. Thus, a symmetry group of a system of equations maps solutions to other solutions: If $x \in M$ satisfies (2.5) and $g \in G$ is any group element such that $g \cdot x$ is defined, then the transformed point $g \cdot x$ is also a solution to the system. Knowledge of a symmetry group of a system of equations allows us to construct new solutions from old ones, a fact that will be particularly useful when we apply these methods to systems of differential equations.

An *orbit* of a transformation group is a minimal (nonempty) invariant subset. For a global action, the orbit through a point $x \in M$ is just the set of all images of x under arbitrary group transformations: $\mathcal{O}_x = \{\, g \cdot x \mid g \in G \,\}$. More generally, for a local group action, the orbit is the set of all images of x under arbitrary finite sequences of group transformations: $\mathcal{O}_x = \{\, g_1 \cdot g_2 \cdots g_n \cdot x \mid g_i \in G, n \geq 0 \,\}$. If G is connected, its orbits are connected. Clearly, a subset $S \subset M$ is G-invariant if and only if it is the union of orbits.

The group action is called *transitive* if there is only one orbit, so (assuming the group acts globally), for every $x, y \in M$ there exists at least one $g \in G$ such that $g \cdot x = y$. At the other extreme, a fixed point is a zero-dimensional orbit; for connected group actions, the converse holds: Any zero-dimensional orbit is a fixed point.

Example 2.16. For the usual linear action of $\mathrm{GL}(n)$ on $\mathbb{R}^n$, there are two orbits: The origin $\{0\}$ and the remainder $\mathbb{R}^n \setminus \{0\}$. The same holds for $\mathrm{SL}(n)$ since we can still map any nonzero vector in $\mathbb{R}^n$ to any other nonzero vector by a matrix of determinant 1. The orthogonal group $\mathrm{O}(n)$ is a bit different: The orbits are spheres $\{|x| = \text{constant}\}$ (and the origin), and any other invariant subset is the union of spheres. The induced projective actions of each of these three groups on $\mathbb{RP}^{n-1}$ are all transitive.

A particularly important class of transitive group actions is provided by the *homogeneous spaces*, defined as the quotient space G/H of a Lie group G by a closed subgroup; see Exercise 2.6. The left multiplication of G induces a corresponding globally defined, transitive action of the group G on the homogeneous space G/H. For example, the left multiplication action of the three-dimensional rotation group $\mathrm{SO}(3)$ on itself induces the standard action on the two-dimensional sphere $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$ realized as the quotient space by any of the (non-normal) two-dimensional rotation subgroups. In fact, every global transitive group action can be identified with a homogeneous space.

Theorem 2.17. *A Lie group G acts globally and transitively on a manifold M if and only if $M \simeq G/H$ is isomorphic to the homogeneous space obtained by quotienting G by the* isotropy subgroup *$H = G_x$ of any designated point $x \in M$.*

Example 2.18. Consider the action of $\mathrm{GL}(n)$ on $M = \mathbb{R}^n \setminus \{0\}$. The isotropy group of the point $e_1 = (1, 0, \ldots, 0)$ is the subgroup of matrices whose first column equals e_1, which can be identified with the affine group $\mathrm{A}(n-1)$, as in Example 2.11. Theorem 2.17 implies that we can identify $\mathbb{R}^n \setminus \{0\}$ with the homogeneous space $\mathrm{GL}(n)/\mathrm{A}(n-1)$.

Exercise 2.19. Determine the homogeneous space structures of the sphere S^{n-1}, and of the projective space $\mathbb{RP}^{n-1}$, induced by the transitive group actions of $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, and $\mathrm{SO}(n)$.

In general, the orbits of a Lie group of transformations are all submanifolds of the manifold M. A group action is called *semi-regular* if all its orbits have the same dimension. The action is called *regular* if, in addition, each point $x \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset thereof. The condition that each orbit be a regular submanifold is necessary, but not sufficient, for the regularity of the group action.

Example 2.20. Let $T = S^1 \times S^1$ be the two-dimensional torus, with angular coordinates (θ, φ), $0 \le \theta, \varphi < 2\pi$. Consider the one-parameter group action $(\theta, \varphi) \longmapsto (\theta + t, \varphi + \kappa t) \mod 2\pi$, $t \in \mathbb{R}$, where $0 \neq \kappa \in \mathbb{R}$. If κ/π is a rational number, then the orbits of this action are closed curves, diffeomorphic to the circle S^1, and the action is regular. On the other hand, if κ/π is irrational, then the orbits never close, and, in fact, each orbit is a dense subset of the torus. The action in the latter case is semi-regular, but not regular.

Example 2.21. Consider the one-parameter group action,

$$(r, \theta) \longmapsto \left(\frac{r e^t}{1 + r(e^t - 1)}, \theta + t \right),$$

defined in terms of polar coordinates, on the punctured plane $M = \mathbb{R}^2 \setminus \{0\}$. The orbits of this group action are all regular one-dimensional submanifolds of M — they consist of the unit circle $r = 1$ and two families of spirals. However, the group action is *not* regular — indeed, all the spiral orbits intersect any small neighborhood of each point on the unit circle in infinitely many disconnected components.

Proposition 2.22. *An r-dimensional Lie group G acts* locally freely *on a manifold M if and only if its orbits have the same dimension r as G itself. The group acts* effectively freely *if and only if its orbits have dimension $s = \dim G - \dim G_M$.*

The orbits of regular and semi-regular group actions have a particularly simple local canonical form, generalizing the rectifying coordinates for a one-parameter group of transformations in Theorem 1.21.

Theorem 2.23. *Let G be a Lie group acting* regular*ly on a manifold M with s-dimensional orbits. Then, near every point of M, there exist* rectifying *local coordinates $(y, z) = (y^1, \ldots, y^s, z^1, \ldots, z^{m-s})$ having the property that any orbit intersects the coordinate chart in at most one slice $N_c = \{z^1 = c_1, \ldots, z^{m-s} = c_{m-s}\}$, for constants $c = (c_1, \ldots, c_{m-s})$. If the action is* semi-regular, *the same statement holds except that an orbit may intersect the chart in more than one such slice.*

This fundamental result is a direct consequence of Frobenius' Theorem 1.31. Theorem 2.23 demonstrates that the orbits form a *foliation* of the underlying manifold. In practice, the rectifying coordinates are most readily constructed using the infinitesimal methods to be discussed below.

Example 2.24. For the orthogonal group $\mathrm{O}(3)$ acting on $\mathbb{R}^3 \setminus \{0\}$, the orbits are the two-dimensional spheres, and rectifying coordinates are given by spherical coordinates, with the angular coordinates θ, φ playing the role of y, and the radius r the role of z in Theorem 2.23.

Invariant Functions

An invariant of a transformation group is defined as a real-valued function whose values are unaffected by the group transformations. The determination of a complete set of invariants of a given group action is a problem of supreme importance for the study of equivalence and canonical forms. In the regular case, the orbits, and hence the canonical forms, for a group action are completely characterized by its invariants.

Definition 2.25. Let G be a transformation group acting on a manifold M. An *invariant* of G is a real-valued function $I\colon M \to \mathbb{R}$ which satisfies $I(g \cdot x) = I(x)$ for all transformations $g \in G$.

Proposition 2.26. *Let $I: M \to \mathbb{R}$. The following conditions are equivalent:*

- *I is a G-invariant function.*
- *I is constant on the* orbits *of G.*
- *The* level sets *$\{I(x) = c\}$ of I are G-*invariant subsets *of M.*

For example, in the case of the orthogonal group $\mathrm{O}(n)$ acting on $\mathbb{R}^n$, the orbits are spheres $r = |x| =$ constant, and hence any orthogonal invariant is a function of the radius: $I = F(r)$. If G acts transitively on the manifold M, then there are no nonconstant invariants. If G acts transitively on a dense subset $M_0 \subset M$, then the only *continuous* invariants are the constants. For instance, the only continuous global invariants of the irrational flow on the torus, cf. Example 2.20, are the constants, since every orbit is dense in this case. Similarly, the only continuous invariants of the standard action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ are the constant functions, since the group acts transitively on $\mathbb{R}^n \setminus \{0\}$.

Note that the canonical form x_0 of any element $x \in M$ must have the same invariants: $I(x_0) = I(x)$; this condition is also sufficient if there are enough invariants to distinguish the orbits, i.e., x and y lie in the same orbit if and only if $I(x) = I(y)$ for every invariant I, which, according to Theorem 2.30 below, is the case for regular group actions. However, singular orbits are often not completely distinguished by invariants alone, and more sophisticated algebraic features must be utilized.

Example 2.27. Consider the action $X \mapsto AXA^T$ of $\mathrm{GL}(n, \mathbb{R})$ on the space of symmetric matrices. Since the group orbits are discrete, the only invariant functions are the matrix signatures, which serve to characterize the canonical forms. Restricting to the orthogonal subgroup $\mathrm{O}(n) \subset \mathrm{GL}(n)$, we see that any symmetric function of the eigenvalues of the matrix provides an invariant; again these are sufficient to completely characterize its canonical diagonal form. (An alternative system of invariants is provided by the traces of the powers of the matrix: $\operatorname{tr} A^k$, $k = 1, \ldots, n$.) On the other hand, consider the conjugation action $X \mapsto AXA^{-1}$ of $\mathrm{GL}(n, \mathbb{C})$ on the space of $n \times n$ matrices. Again, the symmetric functions of the eigenvalues provide invariants of the action, but, if the eigenvalues are repeated, these are not sufficient to distinguish the different canonical forms, and one must introduce additional discrete invariants to properly characterize the Jordan block structure.

A fundamental problem is the determination of *all* the invariants of a group of transformations. Note that if $I_1(x), \ldots, I_k(x)$ are invariants, and $H(y_1, \ldots, y_k)$ is any function, then $I(x) = H(I_1(x), \ldots, I_k(x))$ is also invariant. Therefore, we need only find a complete set of functionally independent invariants, cf. Definition 1.9, having the property that any other invariant can be written as a function of these fundamental invariants. In many cases, globally defined invariants are not so readily available or easy to find, and so we will often have to be content with the description of locally defined invariants.

Definition 2.28. Let G be a Lie group acting on a manifold M. A function $I: M \to \mathbb{R}$ defined on an open subset $U \subset M$ is called a *local invariant* of G if $I(g \cdot x) = I(x)$ for all $x \in U$ and all group transformations $g \in V_x$ in some neighborhood $V_x \subset G$ (possibly depending on x) of the identity element. If $I(g \cdot x) = I(x)$ for all $x \in U$ and all $g \in G$

such that $g \cdot x \in U$, then I is called a *global invariant* (even though it is only defined on an open subset of M).

Example 2.29. Consider the one-parameter group acting on the torus as discussed in Example 2.20. If κ is irrational, then, on the open subset $0 < \theta, \varphi < 2\pi$, the difference $\varphi - \kappa\,\theta$ is a local invariant of the group action that is clearly not globally invariant.

According to Theorem 2.23, the number of independent local invariants of a regular transformation group is completely determined by the orbit dimension. Indeed, in terms of the rectifying coordinates (y, z), the coordinate functions $z^1, \ldots, z^{m-s}$, and any function thereof, provide a complete set of local invariants for the group action. Thus, Theorem 2.23 implies the basic classification result for the invariants of regular group actions.

Theorem 2.30. *Let G be a Lie group acting* semi-regular*ly on the m-dimensional manifold M with s-dimensional orbits. At each $x \in M$, there exist $m - s$ functionally independent local invariants $I_1, \ldots, I_{m-s}$, defined on a neighborhood U of x, with the property that any other local invariant I defined on U can be written as a function of the* fundamental invariants*: $I = H(I_1, \ldots, I_{m-s})$. If G acts regularly, then we can choose the I_ν's to be global invariants on U. Moreover, in the regular case, two points $y, z \in U$ lie in the same* orbit *of G if and only if the invariants all have the same value, $I_\nu(y) = I_\nu(x)$, $\nu = 1, \ldots, m - s$.*

Theorem 2.30 provides a complete answer to the question of local invariants of group actions. Global considerations are more delicate. For example, consider the elementary one-parameter scaling group $(x, y) \mapsto (\lambda x, \lambda y)$, $\lambda \in \mathbb{R}^+$. Locally, away from the origin, the ratio x/y, or y/x, or any function thereof (e.g., $\theta = \tan^{-1}(y/x)$) provides the only independent invariant. However, if we include the origin, then there are no nonconstant continuous invariants. (A discontinuous invariant is provided by the function which is 1 at the origin and 0 elsewhere.) On the other hand, the scaling group $(x, y) \mapsto (\lambda x, \lambda^{-1} y)$ does have a global invariant: xy.

Example 2.31. Consider the projective action of $\mathrm{SL}(2, \mathbb{C})$ on the n-fold Cartesian product $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$

$$(p_1, \ldots, p_n) \mapsto \left(\frac{\alpha p_1 + \beta}{\gamma p_1 + \delta}, \ldots, \frac{\alpha p_n + \beta}{\gamma p_n + \delta} \right), \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

Let us concentrate on the open subset $M = \{ p_i \neq p_j, i \neq j \}$ consisting of distinct n-tuples of points. For $n \leq 3$, the action is transitive on M, and there are no nonconstant invariants. Indeed, we can map any three distinct points (p_1, p_2, p_3) on the Riemann sphere to any desired canonical form, e.g., $(0, 1, \infty)$, by a suitable linear fractional transformation. For $n = 4$, the *cross-ratio*

$$[\, p_1, p_2, p_3, p_4 \,] = \frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)}, \tag{2.6}$$

is invariant, as can be verified directly. (If one of the points is infinite, (2.6) is computed in a consistent way.) Now only the first three points can be fixed, so a canonical form could be $(0, 1, \infty, z)$ where z can be any other point, whose value is fixed by the cross-ratio

$1/(z-1)$. An alternative choice is $(s, -s, s^{-1}, -s^{-1})$, which is unique if we restrict $s > 1$; the cross-ratio is now $-4/(s-s^{-1})^2$. For $n > 4$, it can be proved that every invariant can be written as a function of the cross-ratios of the points p_j, taken four at a time. According to Theorem 2.30, only $n-3$ of these cross-ratios are functionally independent, and we can clearly take $[\,p_1, p_2, p_3, p_k\,]$, $k = 4, \ldots, n$, as our fundamental invariants.

Exercise 2.32. Determine how to express other cross-ratios, e.g., $[\,p_1, p_3, p_4, p_5\,]$, in terms of the fundamental cross-ratios.

Exercise 2.33. Discuss the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{RP}^1 \times \cdots \times \mathbb{RP}^1$, and on $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$, viewed as a real manifold of dimension $2n$.

The cross-ratios are a special case of the general concept of a joint invariant. If G acts simultaneously on the manifolds $M_1, \ldots, M_n$, then by a *joint invariant* we mean an ordinary invariant $I(x_1, \ldots, x_n)$ of the Cartesian product group action of G on $M_1 \times \cdots \times M_n$; in other words, $I(g \cdot x_1, \ldots, g \cdot x_n) = I(x_1, \ldots, x_n)$ for all $g \in G$. In most cases of interest, the manifolds $M_\kappa = M$ are identical, with identical actions of G.

The invariants of a regular group action can be used to completely characterize the invariant submanifolds. First, if G acts on M, and $I_1, \ldots, I_k$ are any invariants, then Proposition 2.26 implies that their common level set $\{I_1(x) = c_1, \ldots, I_k(x) = c_k\}$ is an invariant subset of M. Conversely, if the group action is regular, Theorem 2.23 implies that any invariant submanifold can be expressed (locally) as the vanishing set for some collection of invariant functions.

Theorem 2.34. *Let G act regularly on an m-dimensional manifold M. A regular n-dimensional submanifold $N \subset M$ is G-invariant if and only if at each point $x \in N$ there exists a neighborhood U and invariants $I_1, \ldots, I_{m-n}$ such that $N \cap U = \{I_1(x) = \cdots = I_{m-n}(x) = 0\}$.*

Proof: Choose rectifying local coordinates (y, z) as in Theorem 2.23. Since any invariant submanifold must be a collection of orbits, which, in the given coordinate chart, are just the slices where the invariant coordinates z are constant, we can characterize its intersection with the coordinate chart by the vanishing of one or more functions depending on $z^1, \ldots, z^{m-s}$. But any function of the z's is a (local) invariant of the group action, so the result follows immediately. *Q.E.D.*

Lie Algebras

Besides invariant functions, there are many other important invariant objects associated with a transformation group, including vector fields, differential forms, differential operators, etc. The most important of these are the invariant vector fields, since they serve as the "infinitesimal generators" of the group action.

Definition 2.35. Let G be a group acting on the manifold M. A vector field $\mathbf{v}$ on M is called *G-invariant* if it is unchanged by the action of any group element, meaning that $dg(\mathbf{v}|_x) = \mathbf{v}|_{g \cdot x}$ for all $g \in G$, and all $x \in M$ such that $g \cdot x$ is defined.

The most important example is provided by the action of a Lie group G on itself by left or right multiplication, as described in Example 2.9. Here, the invariant vector fields determine the Lie algebra or "infinitesimal" Lie group, which plays an absolutely crucial role in both the general theory of Lie groups and its many applications. Given $g \in G$, we let $L_g : h \mapsto g \cdot h$ and $R_g : h \mapsto h \cdot g$ denote the associated left and right multiplication maps. A vector field $\mathbf{v}$ on G is called *left-invariant* if $dL_g(\mathbf{v}) = \mathbf{v}$, and *right-invariant* if $dR_g(\mathbf{v}) = \mathbf{v}$, for all $g \in G$.

Definition 2.36. The *left* (respectively *right*) *Lie algebra* of a Lie group G is the space of all left-invariant (respectively right-invariant) vector fields on G.

Thus, associated to any Lie group, there are two different Lie algebras, which we denote by $\mathfrak{g}_L$ and $\mathfrak{g}_R$ respectively. Traditionally, one refers to "the" Lie algebra associated with a Lie group, and denotes it by $\mathfrak{g}$, but this requires the adoption of either a left or a right convention. The more common is to use left-invariant vector fields for the Lie algebra, although some authors do prefer the right-invariant ones. Both Lie algebras play a role, although the right-invariant one is by far the more useful of the two, due, perhaps surprisingly, to our convention that Lie groups act on the left on manifolds; the reason for this apparent switch will become clear below. Therefore, in the sequel, when we talk about the Lie algebra associated with a Lie group, we shall mean the right-invariant Lie algebra, and use $\mathfrak{g} = \mathfrak{g}_R$ to denote it.

Every right-invariant vector field $\mathbf{v}$ is uniquely determined by its value at the identity e, because $\mathbf{v}|_g = dR_g(\mathbf{v}|_e)$. Thus we can identify the right Lie algebra with the tangent space to G at the identity element, $\mathfrak{g}_R \simeq TG|_e$, so $\mathfrak{g}_R$ is a finite-dimensional vector space having the same dimension as G. A similar statement holds for left-invariant vector fields, providing a similar isomorphism $\mathfrak{g}_L \simeq TG|_e$. Thus, a given tangent vector $\mathbf{v}|_e \in TG|_e$ determines both a left- and a right-invariant vector field on the Lie group, which are (usually) different.

Each Lie algebra associated with a Lie group comes equipped with a natural bracket, induced by the Lie bracket of vector fields. This follows immediately from the invariance (1.17) of the Lie bracket under diffeomorphisms, which implies that if both $\mathbf{v}$ and $\mathbf{w}$ are right-invariant vector fields, so is their Lie bracket $[\mathbf{v}, \mathbf{w}]$. The basic properties of the Lie bracket translate into the defining properties of an (abstract) Lie algebra.

Definition 2.37. A *Lie algebra* $\mathfrak{g}$ is a vector space equipped with a bracket operation $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is bilinear, anti-symmetric, $[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}]$, and satisfies the Jacobi identity

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] = 0. \tag{2.7}$$

Example 2.38. The Lie algebra $\mathfrak{gl}(n)$ of the general linear group $\mathrm{GL}(n)$ can be identified with the space of all $n \times n$ matrices. In terms of the coordinates provided by the matrix entries $X = (x_{ij}) \in \mathrm{GL}(n)$, the left-invariant vector field associated with a matrix $A = (a_{ij}) \in \mathfrak{gl}(n)$ has the explicit formula

$$\widehat{\mathbf{v}}_A = \sum_{i,j,k=1}^{n} x_{ik} a_{kj} \, \frac{\partial}{\partial x_{ij}} \,. \tag{2.8}$$

The Lie bracket of two such vector fields is $[\widehat{\mathbf{v}}_A, \widehat{\mathbf{v}}_B] = \widehat{\mathbf{v}}_C$, where $C = AB - BA$, so the left-invariant Lie bracket on $\mathrm{GL}(n)$ can be identified with the standard matrix commutator $[A, B] = AB - BA$. On the other hand, the right-invariant vector field associated with a matrix $A \in \mathfrak{gl}(n)$ is given by

$$\mathbf{v}_A = \sum_{i,j,k=1}^{n} a_{ik} x_{kj} \frac{\partial}{\partial x_{ij}}\,. \tag{2.9}$$

Now the Lie bracket is $[\mathbf{v}_A, \mathbf{v}_B] = \mathbf{v}_{\widehat{C}}$, where $\widehat{C} = -C = BA - AB$ is the *negative* of the matrix commutator. Thus, the matrix formula for the Lie algebra bracket on $\mathfrak{gl}(n)$ depends on whether we are dealing with its left-invariant or right-invariant version!

Exercise 2.39. Prove that the right and left Lie algebras for the abelian Lie group $G = \mathbb{R}^r$ are identical, each isomorphic to the abelian Lie algebra $\mathfrak{g} = \mathbb{R}^r$ with trivial Lie bracket: $[\mathbf{v}, \mathbf{w}] = 0$ for all $\mathbf{v}$, $\mathbf{w}$.

The right and left Lie algebras associated with a Lie group are, in fact, isomorphic as abstract Lie algebras. The inversion map $\iota(g) = g^{-1}$ provides the explicit isomorphism since it interchanges the roles of left and right. Thus, its differential $d\iota$ maps right-invariant vector fields to left-invariant ones, and vice versa, while preserving the Lie bracket.

Proposition 2.40. *If G is a Lie group, then the differential of the inversion map defines a Lie algebra isomorphism between its associated left and right Lie algebras $d\iota\colon \mathfrak{g}_L \simeq \mathfrak{g}_R$.*

As above, we identify an invariant vector field $\mathbf{v}_R$ or $\mathbf{v}_L$ with its value $\mathbf{v} = \mathbf{v}_R|_e = \mathbf{v}_L|_e$ at the identity. Since $d\iota|_e = -\mathbb{1}$, the differential $d\iota$ maps the right-invariant vector field $\mathbf{v}_R$ to *minus* its left-invariant counterpart $\mathbf{v}_L = -d\iota(\mathbf{v}_R)$. Consequently, the left and right Lie brackets on a Lie group always have *opposite* signs: $[\mathbf{v}, \mathbf{w}]_L = -[\mathbf{v}, \mathbf{w}]_R$. This explains the observation in Example 2.38.

The operations of right and left multiplication commute: $L_g \circ R_h = R_h \circ L_g$ for all $g, h \in G$. Therefore, according to Theorem 1.25, the corresponding infinitesimal generators commute: $[\mathbf{v}_L, \mathbf{w}_R] = 0$ for all $\mathbf{v}_L \in \mathfrak{g}_L$, $\mathbf{w}_R \in \mathfrak{g}_R$. In fact, it is not hard to see that the two Lie algebras are uniquely characterized in terms of each other in this manner.

Proposition 2.41. *The right Lie algebra $\mathfrak{g}_R$ of a Lie group G can be characterized as the set of vector fields on G which commute with all left-invariant vector fields:*

$$\mathfrak{g}_R = \{\, \mathbf{w} \mid\ [\mathbf{v}_L, \mathbf{w}] = 0 \text{ for all } \mathbf{v}_L \in \mathfrak{g}_L \,\}\,.$$

The same result holds with left and right interchanged.

Example 2.42. Consider the two parameter group $\mathrm{A}(1)$ of affine transformations $x \mapsto ax + b$ on the line $x \in \mathbb{R}$, as in Example 2.11. The group multiplication law is given by $(a, b) \cdot (c, d) = (ac, ad + b)$, and the identity element is $e = (1, 0)$. The right and left multiplication maps are therefore given by

$$R_{(a,b)}(c, d) = (c, d) \cdot (a, b) = (ac, bc + d), \qquad L_{(a,b)}(c, d) = (a, b) \cdot (c, d) = (ac, ad + b).$$

A basis for the right Lie algebra $\mathfrak{a}(1)_R$, corresponding to the coordinate basis $\partial_a|_e$, $\partial_b|_e$ of $T\mathrm{A}(1)|_e$, is provided by the pair of right-invariant vector fields

$$\mathbf{v}_1 = dR_{(a,b)}\big[\partial_a|_e\big] = a\partial_a + b\partial_b, \qquad \mathbf{v}_2 = dR_{(a,b)}\big[\partial_b|_e\big] = \partial_b. \tag{2.10}$$

These satisfy the commutation relation $[\mathbf{v}_1, \mathbf{v}_2] = -\mathbf{v}_2$. A similar basis for the left Lie algebra $\mathfrak{a}(1)_L$ is provided by the pair of left-invariant vector fields

$$\widehat{\mathbf{v}}_1 = dL_{(a,b)}\big[\partial_a|_e\big] = a\partial_a, \qquad \widehat{\mathbf{v}}_2 = dL_{(a,b)}\big[\partial_b|_e\big] = a\partial_b, \tag{2.11}$$

satisfying the negative commutation relation $[\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2] = \widehat{\mathbf{v}}_2$. Note that the vector fields (2.10) commute with those in (2.11): $[\mathbf{v}_i, \widehat{\mathbf{v}}_j] = 0$, confirming Proposition 2.41. Finally, the inversion map is given by $\iota(a,b) = (a,b)^{-1} = (1/a, -b/a)$, and, in accordance with the above remarks, its action on the two Lie algebras is $d\iota(\mathbf{v}_1) = -\widehat{\mathbf{v}}_1$, $d\iota(\mathbf{v}_2) = -\widehat{\mathbf{v}}_2$, as can be checked by an explicit computation.

Exercise 2.43. Define the semi-direct product of Lie algebras, and show that $\mathfrak{a}(1) = \mathbb{R} \ltimes \mathbb{R}$ can be identified as a semi-direct product of two one-dimensional Lie algebras.

Exercise 2.44. Prove that every two-dimensional Lie algebra is isomorphic to either the abelian algebra $\mathbb{R}^2$ or the affine algebra $\mathfrak{a}(1)$.

Structure Constants

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis of a Lie algebra $\mathfrak{g}$. We define the associated *structure constants* C^k_{ij} by the bracket relations

$$[\mathbf{v}_i, \mathbf{v}_j] = \sum_{k=1}^{r} C^k_{ij}\mathbf{v}_k. \tag{2.12}$$

Anti-symmetry of the Lie bracket and the Jacobi identity imply the basic identities

$$C^k_{ji} = -C^k_{ij}, \qquad \sum_{l=1}^{r} \big(C^l_{ij}C^m_{lk} + C^l_{ki}C^m_{lj} + C^l_{jk}C^m_{li} \big) = 0, \tag{2.13}$$

which must be satisfied by the structure constants of any Lie algebra. Conversely, given $\frac{1}{2}r^2(r-1)$ constants satisfying the identities (2.13), we can reconstruct the Lie algebra $\mathfrak{g}$ by introducing a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$, and then imposing the bracket relations (2.12). In turn, as stated in Theorem 2.52 below, one can always construct an associated Lie group whose right (or left) Lie algebra coincides with $\mathfrak{g}$. Thus (from an admittedly reductionist standpoint) the theory of Lie groups can be essentially reduced to the study of its structure constants. This fact plays a key role in the detailed structure theory of Lie groups, cf. [**25**].

Exercise 2.45. Find a formula showing how the structure constants change under a change of basis of the Lie algebra $\mathfrak{g}$.

Exercise 2.46. Show that, relative to a fixed basis of the tangent space $TG|_e$, the structure constants for the left and right Lie algebras of a Lie group differ by an overall sign.

Given a right-invariant vector field $\mathbf{v} \in \mathfrak{g}_R$ on the Lie group G, we let $\exp(t\mathbf{v})\colon G \to G$ denote the associated flow. An easy continuation argument can be used to prove that this flow is globally defined for all $t \in \mathbb{R}$. Applying the flow to the identity element e serves to define the *one-parameter subgroup* $\exp(t\mathbf{v}) \equiv \exp(t\mathbf{v})e$; the vector field $\mathbf{v}$ is known as the *infinitesimal generator* of the subgroup. The notation is not ambiguous, since the flow through any $g \in G$ is the same as *left* multiplication by the elements of the subgroup, so $\exp(t\mathbf{v})g$ can be interpreted either as a flow or as a group multiplication. This fact is a consequence of the invariance of the flow, as in (1.16), under the right multiplication map R_h. Therefore, the right-invariant vector fields form the infinitesimal generators of the action of G on itself by *left* multiplication. Vice versa, the infinitesimal generators of the action of G on itself by right multiplication is the Lie algebra $\mathfrak{g}_L$ of left-invariant vector fields. This interchange of the role of infinitesimal generators and invariant vector fields is one of the interesting peculiarities of Lie group theory. Finally, note that although the left- and right-invariant vector fields associated with a given tangent vector $\mathbf{v} \in TG|_e$ are (usually) different, and have different flows, nevertheless the associated one-parameter groups coincide: $\exp(t\mathbf{v}_R) = \exp(t\mathbf{v}_L)$.

Example 2.47. Consider the general linear group $\mathrm{GL}(n)$ discussed in Example 2.38. The flow corresponding to the right-invariant vector field $\mathbf{v}_A$ given by (2.9) is given by left multiplication by the usual matrix exponential: $\exp(t\mathbf{v}_A)X = e^{tA}X$. Conversely, the flow corresponding to the left-invariant vector field $\widehat{\mathbf{v}}_A$ given by (2.8) is given by right multiplication $\exp(t\widehat{\mathbf{v}}_A)X = Xe^{tA}$. In either case, the one-parameter subgroup generated by the vector field associated with a matrix $A \in \mathfrak{gl}(n)$ is the matrix exponential $\exp(t\mathbf{v}_A) = \exp(t\widehat{\mathbf{v}}_A) = e^{tA}$.

From now on, in view of subsequent applications, we shall restrict our attention to the right-invariant vector fields, so that $\mathfrak{g}$ will always denote the right Lie algebra of the Lie group G. Evaluation of the flow $\exp(t\mathbf{v})$ at $t = 1$ for each $\mathbf{v} \in \mathfrak{g}$ serves to define the *exponential map* $\exp\colon \mathfrak{g} \to G$. Since $\exp(0) = e$, $d\exp(0) = 1\!\!1$, the exponential map defines a local diffeomorphism in a neighborhood of $0 \in \mathfrak{g}$. Consequently, all Lie groups having the same Lie algebra look locally the same in a neighborhood of the identity; only their global topological properties are different. (Indeed, in Lie's day, one only considered "local Lie groups", the global version being a more recent introduction, cf. [**12**].) Globally, the exponential map is not necessarily one-to-one nor onto. However, if a Lie group is connected, it can be completely recovered by successive exponentiations.

Proposition 2.48. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Every group element can be written as a product of exponentials: $g = \exp(\mathbf{v}_1)\exp(\mathbf{v}_2)\cdots\exp(\mathbf{v}_k)$, for $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathfrak{g}$.*

This result forms the basis of the infinitesimal or Lie algebraic approach to symmetry groups and invariants. It is proved by first noting that the set of all group elements $g \in G$ of the indicated form forms an open and closed subgroup of G, and then invoking connectivity, cf. [**54**].

By definition, a *subalgebra* of a Lie algebra is a subspace $\mathfrak{h} \subset \mathfrak{g}$ which is invariant under the Lie bracket. Every one-dimensional subspace forms a subalgebra, and generates an associated one-parameter subgroup of the associated Lie group via exponentiation. More generally, each subalgebra $\mathfrak{h} \subset \mathfrak{g}$ generates a unique connected Lie subgroup $H \subset G$, satisfying $\mathfrak{h} \simeq TH|_e \subset TG|_e \simeq \mathfrak{g}$.

Theorem 2.49. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. There is a one-to-one correspondence between connected s-dimensional* Lie subgroups *$H \subset G$ and s-dimensional* Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$.

Example 2.50. In particular, every matrix Lie group $G \subset \mathrm{GL}(n)$ will correspond to a subalgebra of $\mathfrak{g} \subset \mathfrak{gl}(n)$ the Lie algebra of $n \times n$ matrices, cf. Example 2.38. To determine the subalgebra, we need only find the tangent space to the subgroup at the identity matrix. For instance, the standard formula $\det \exp(tA) = \exp(t \operatorname{tr} A)$ implies that the Lie algebra $\mathfrak{sl}(n)$ of the unimodular subgroup $\mathrm{SL}(n)$ consists of all matrices with trace 0. The orthogonal groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ have the same Lie algebra, $\mathfrak{so}(n)$, consisting of all skew-symmetric $n \times n$ matrices. This result proves our earlier claim that $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ have dimension $\frac{1}{2} n(n-1)$. The reader should verify that the indicated subspaces are indeed subalgebras under the matrix commutator.

Exercise 2.51. Determine the Lie algebra $\mathfrak{sp}(2r)$ of the symplectic group $\mathrm{Sp}(2r)$.

Ado's Theorem, cf. [**28**], says that any finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{gl}(n)$ for some n. Consequently, $\mathfrak{g}$ is realized as the Lie algebra of the associated matrix Lie group $G \subset \mathrm{GL}(n)$. The full connection between Lie groups and Lie algebras is stated in the following fundamental theorem, whose proof can be found in [**48**].

Theorem 2.52. *Each finite-dimensional Lie algebra $\mathfrak{g}$ corresponds to a unique* connected, simply connected *Lie group $\widetilde{G}$. Moreover, any other connected Lie group G having the same Lie algebra $\mathfrak{g}$ is isomorphic to the quotient group of $\widetilde{G}$ by a* discrete normal subgroup Γ, *so that $G \simeq \widetilde{G}/\Gamma$.*

Note that the projection $\pi \colon \widetilde{G} \to G$ defined in Theorem 2.52 defines a (uniform) covering map, so that $\widetilde{G}$ is as a *covering group* for any other connected Lie group G having the same Lie algebra. Consequently, any two Lie groups having the same Lie algebra $\mathfrak{g}$ are not only locally isomorphic, but are in fact both covered by a common Lie group.

Exercise 2.53. The subgroup $C = \{\, g \in G \mid ghg^{-1} = h, \ \text{forall } h \in G \,\}$ consisting of all group elements which commute with every element in a group G is known as the *center* of G. Prove that the Lie algebra associated with the center of a Lie group G is the abelian subalgebra $\{\, \mathbf{v} \in \mathfrak{g} \mid [\mathbf{v}, \mathbf{w}] = 0 \text{ for all } \mathbf{w} \in \mathfrak{g} \,\}$, known as the *center* of $\mathfrak{g}$.

Exercise 2.54. Let G be a Lie group. Prove that a vector field $\mathbf{v}$ on G lies in the intersection $\mathfrak{g}_L \cap \mathfrak{g}_R$ of the left and right Lie algebras, and so is both left- and right-invariant, if and only if it lies in the center of both $\mathfrak{g}_L$ and $\mathfrak{g}_R$.

Exercise 2.55. Let $H \subset G$ be a Lie subgroup with Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Prove that the *normalizer subgroup* $G_H = \{\, g \mid gHg^{-1} \subset H \,\}$ is a normal subgroup with Lie algebra $\mathfrak{g}_H = \{\, \mathbf{v} \in \mathfrak{g} \mid [\mathbf{v}, \mathbf{w}] \in \mathfrak{h}, \text{ for all } \mathbf{w} \in \mathfrak{h} \,\}$.

Exercise 2.56. A subspace $\mathfrak{i} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is an *ideal* if $[\mathbf{v}, \mathfrak{i}] \subset \mathfrak{i}$ for every $\mathbf{v} \subset \mathfrak{g}$. Prove that an ideal is a subalgebra, but the converse is not necessarily true. Show that ideals in $\mathfrak{g}$ are in one-to-one correspondence with connected normal Lie subgroups of the Lie group G.

A Lie algebra of dimension greater than 1 is called *simple* if it contains no proper nonzero ideals. Each of the classical Lie algebras $\mathfrak{sl}(n)$, $n \geq 2$, $\mathfrak{so}(n)$, $3 \leq n \neq 4$, and $\mathfrak{sp}(2r)$, $r \geq 2$, is known to be simple, although $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is the direct sum of two simple algebras. In fact, besides the three infinite families of "classical" complex Lie groups, there exist just five additional "exceptional" simple Lie groups. See [**25**] for the precise classification theorem, due to Killing and Cartan.

Exercise 2.57. The *derived algebra* $\mathfrak{g}'$ of a Lie algebra $\mathfrak{g}$ is defined as the subalgebra spanned by all Lie brackets $[\mathbf{v}, \mathbf{w}]$ for all $\mathbf{v}, \mathbf{w} \in \mathfrak{g}$. Prove that $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$, hence, if $\mathfrak{g}$ is simple, then $\mathfrak{g}' = \mathfrak{g}$. Prove that the derived algebra of $\mathfrak{gl}(n)$ is $\mathfrak{sl}(n)$. Show that the corresponding subgroup of the associated Lie group G is the *derived subgroup* G', generated by all commutators $ghg^{-1}h^{-1}$ for $g, h \in G$. (*A subtle point*: G' is *not* equal to the set of all commutators, as this may not even be a subgroup.)

Infinitesimal Group Actions

Just as a one-parameter group of transformations is generated as the flow of a vector field, so a general Lie group of transformations G acting on a manifold M will be generated by a set of vector fields on M, known as the *infinitesimal generators* of the group action. Each infinitesimal generator's flow coincides with the action of the corresponding one-parameter subgroup of G. Specifically, if $\mathbf{v} \in \mathfrak{g}$ generates the one-parameter subgroup $\{\, \exp(t\mathbf{v}) \mid t \in \mathbb{R} \,\} \subset G$, then we identify $\mathbf{v}$ with the infinitesimal generator $\widehat{\mathbf{v}}$ of the one-parameter group of transformations or flow $x \mapsto \exp(t\mathbf{v}) \cdot x$. According to (1.13) the infinitesimal generators of the group action are found by differentiation:

$$\widehat{\mathbf{v}}|_x = \frac{d}{dt} \exp(t\mathbf{v})x \,\bigg|_{t=0}, \qquad x \in M, \quad \mathbf{v} \in \mathfrak{g}. \tag{2.14}$$

Consequently, $\widehat{\mathbf{v}}|_x = d\Phi_x(\mathbf{v}|_e)$, where $\Phi_x : G \to M$ is given by $\Phi_x(g) = g \cdot x$. Since $\Phi_x \circ R_h = \Phi_{h \cdot x}$, if $\mathbf{v} \in \mathfrak{g} = \mathfrak{g}_R$ is any *right-invariant* vector field on G, then $d\Phi_x(\mathbf{v}|_g) = \widehat{\mathbf{v}}|_{g \cdot x}$, where defined. The differential $d\Phi_x$ preserves the Lie bracket between vector fields; therefore the resulting vector fields form a finite-dimensional Lie algebra of vector fields on the manifold M, satisfying the same commutation relations as the right Lie algebra $\mathfrak{g}$ of G (and hence the negative of the commutation relations of the left Lie algebra — a fact that reflects our convention that group elements act on the left). The infinitesimal generators of the group action are not quite isomorphic to the Lie algebra $\mathfrak{g}$ since some of the nonzero Lie algebra elements may map to the trivial (zero) vector field on M. It is not hard to prove that this possibility is directly connected to the effectiveness of the group action, leading to an infinitesimal test for the local effectiveness of group actions.

Theorem 2.58. *Let G be a transformation group acting on a manifold M. The linear map σ taking an element $\mathbf{v} \in \mathfrak{g} = \mathfrak{g}_R$ to the corresponding vector field $\widehat{\mathbf{v}} = \sigma(\mathbf{v})$ on M defines a Lie algebra* homomorphism*: $\sigma([\mathbf{v}, \mathbf{w}]) = [\sigma(\mathbf{v}), \sigma(\mathbf{w})]$. Moreover, the image $\widehat{\mathfrak{g}} = \sigma(\mathfrak{g})$ forms a finite-dimensional Lie algebra of vector fields on M which is isomorphic to the Lie algebra of the effectively acting quotient group G/G_M, where G_M denotes the* global isotropy subgroup *of G. In particular, G acts* locally effectively *on M if and only if σ is injective, i.e.,* $\ker \sigma = \{0\}$.

Thus, if $\mathbf{v}_1, \ldots, \mathbf{v}_r$ forms a basis for $\mathfrak{g}$, the condition for local effectiveness is that the corresponding vector fields $\widehat{\mathbf{v}}_i = \sigma(\mathbf{v}_i)$ be linearly independent over $\mathbb{R}$, i.e., $\sum_i c_i \widehat{\mathbf{v}}_i = 0$ on M for constant $c_i \in \mathbb{R}$ if and only if $c_1 = \cdots = c_r = 0$. Usually, we will not distinguish between an element $\mathbf{v} \in \mathfrak{g}$ in the Lie algebra and the associated infinitesimal generator $\widehat{\mathbf{v}} = \sigma(\mathbf{v})$ of the group action of G, which we also denote as $\mathbf{v}$ from now on. When the action is locally effective, which, according to Proposition 2.12, can always be assumed, this identification does not result in any ambiguities.

Just as every Lie algebra generates a corresponding Lie group, given a finite-dimensional Lie algebra of vector fields on a manifold M, we can always reconstruct a (local) action of the corresponding Lie group via the exponentiation process. See [**48**; Theorem 98] for a proof.

Theorem 2.59. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra of vector fields on a manifold M. Let G denote a Lie group having Lie algebra $\mathfrak{g}$. Then there is a local action of G whose infinitesimal generators coincide with the given Lie algebra.*

Example 2.60. Consider the action of the group SL(2) acting by linear fractional transformations (2.4) on $\mathbb{RP}^1$. Its Lie algebra $\mathfrak{sl}(2)$ consists of all 2×2 matrices of trace 0, and hence is spanned by the three matrices

$$J^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.15}$$

having the commutation relations

$$[J^-, J^0] = -2J^-, \qquad [J^+, J^0] = 2J^+, \qquad [J^-, J^+] = J^0.$$

The corresponding one-parameter subgroups and their infinitesimal generators are

Translations:	$p \longmapsto p + t$	$\mathbf{v}_- = \partial_p$,
Scalings:	$p \longmapsto e^{2t} p$	$\mathbf{v}_0 = 2p\partial_p$,
Inversions:	$p \longmapsto \dfrac{p}{tp+1}$	$\mathbf{v}_+ = -p^2\partial_p$.

The vector fields $\mathbf{v}_-$, $\mathbf{v}_0$, $\mathbf{v}_+$, obey the same commutation relations as the matrices J^-, J^0, J^+, except for an overall sign:

$$[\mathbf{v}_-, \mathbf{v}_0] = 2\mathbf{v}_-, \qquad [\mathbf{v}_+, \mathbf{v}_0] = -2\mathbf{v}_+, \qquad [\mathbf{v}_-, \mathbf{v}_+] = -\mathbf{v}_0,$$

since the right Lie algebra bracket is the negative of the matrix commutator, cf. Example 2.38. Although the three infinitesimal generators $\mathbf{v}_-$, $\mathbf{v}_0$, $\mathbf{v}_+$ are pointwise linearly dependent (since the underlying manifold is merely one-dimensional), there is no non-trivial *constant coefficient* linear combination $c_- \mathbf{v}_- + c_0 \mathbf{v}_0 + c_+ \mathbf{v}_+ = 0$ which vanishes identically. Therefore, Theorem 2.58 reconfirms the fact that SL(2) acts locally effectively on $\mathbb{RP}^1$. On the other hand, if we extend the linear fractional action to all of GL(2), the generator $\mathbb{1} \in \mathfrak{gl}(2)$ of the scaling subgroup maps to the trivial (zero) vector field, reflecting the ineffectiveness of this action.

The infinitesimal generators also determine the tangent space to, and hence the dimension of, the orbits of a group action.

Proposition 2.61. *Let G be a Lie group acting on a manifold M with Lie algebra $\mathfrak{g}$ of G. Then, for each $x \in M$, the* tangent space *to the* orbit *through x is the subspace spanned by the infinitesimal generators: $\mathfrak{g}|_x = \{ \widehat{\mathbf{v}}|_x \mid \mathbf{v} \in \mathfrak{g} \} \subset TM|_x$. In particular, the dimension of the orbit equals the dimension of $\mathfrak{g}|_x$.*

Corollary 2.62. *A Lie group acts* transitively *on a connected manifold M if and only if $\mathfrak{g}|_x = TM|_x$ for all $x \in M$.*

There is an important connection between the dimension of the isotropy subgroup at a point and the dimension of the orbit through that point.

Proposition 2.63. *If G is an r-dimensional Lie group acting on M, then the isotropy subgroup G_x of any point $x \in M$ has dimension $r - s$, where s is the dimension of the orbit of G through x. In particular, G acts* semi-regularly *if and only if all its isotropy subgroups have the same dimension.*

Exercise 2.64. Prove that if $x \in M$, then the isotropy subgroup G_x has Lie algebra $\mathfrak{g}_x = \ker d\Phi_x \subset \mathfrak{g}$. Use this to prove Proposition 2.63.

Exercise 2.65. Prove that a group acts locally freely if and only if its infinitesimal generators are pointwise linearly independent: for each $x \in M$, $\sum_i c_i \widehat{\mathbf{v}}_i|_x = 0$ if and only if $c_1 = \cdots = c_r = 0$.

Infinitesimal Invariance

The fundamental feature of (connected) Lie groups is the ability to work infinitesimally, thereby effectively linearizing complicated invariance criteria. Indeed, the practical applications of Lie groups all ultimately rely on this basic method, and its importance cannot be overestimated. We begin by stating the infinitesimal criterion for the invariance of a real-valued function.

Theorem 2.66. *Let G be a connected group of transformations acting on a manifold M. A function $I \colon M \to \mathbb{R}$ is* invariant *under G if and only if*

$$\mathbf{v}[I] = 0, \tag{2.16}$$

for all $x \in M$ and every infinitesimal generator *$\mathbf{v} \in \mathfrak{g}$ of G.*

Proof: Let $\mathbf{v} \in \mathfrak{g}$ be fixed. We differentiate the invariance condition $I[\exp(t\mathbf{v})x] = I(x)$ with respect to t and set $t = 0$ to deduce the infinitesimal condition (2.16). Conversely, if (2.16) holds, then $d(I[\exp(t\mathbf{v})x])/dt = 0$ where defined, and hence $I[\exp(t\mathbf{v})x] = I(x) = c$ is constant for t in the connected interval containing 0 in $\{ t \in \mathbb{R} \mid \exp(t\mathbf{v}) \in G^x \}$, where $G^x \subset G$ is the set of group elements such that $g \cdot x$ is defined. (If the action is global, $G^x = G$.) Using the fact that the exponential map is a local diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of $e \in G^x$, we conclude that $I(g \cdot x) = c$ for all g in an open neighborhood of the identity in G^x. Now, set $\widetilde{G}^x = \{ g \in G^x \mid I(g \cdot x) = c \}$. Applying the preceding argument at the point $g \cdot x$ for any $g \in \widetilde{G}^x$ proves that $\widetilde{G}^x$ is open, while continuity proves that it is closed in G^x. Thus, by connectivity, $\widetilde{G}^x = G^x$, and the result follows. *Q.E.D.*

Thus, according to Theorem 2.66, the invariants $u = I(x)$ of a one-parameter group with infinitesimal generator $\mathbf{v} = \sum_i \xi^i(x)\partial_{x^i}$ satisfy the first order, linear, homogeneous partial differential equation

$$\sum_{i=1}^{m} \xi^i(x) \frac{\partial u}{\partial x^i} = 0. \tag{2.17}$$

The solutions of (2.17) are effectively found by the method of characteristics. We replace the partial differential equation by the characteristic system of ordinary differential equations

$$\frac{dx^1}{\xi^1(x)} = \frac{dx^2}{\xi^2(x)} = \cdots = \frac{dx^m}{\xi^m(x)}. \tag{2.18}$$

The general solution to (2.18) can be (locally) written in the form $I_1(x) = c_1, \ldots, I_{m-1}(x) = c_{m-1}$, where the c_i are constants of integration. It is not hard to prove that the resulting functions $I_1, \ldots, I_{m-1}$ form a complete set of functionally independent invariants of the one-parameter group generated by $\mathbf{v}$.

Example 2.67. Consider the (local) one-parameter group generated by the vector field

$$\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (1+z^2)\frac{\partial}{\partial z}\,.$$

The group transformations (or flow) are

$$(x,y,z) \longmapsto \left(x\cos t - y\sin t,\, x\sin t + y\cos t,\, \frac{\sin t + z\cos t}{\cos t - z\sin t} \right). \tag{2.19}$$

Note that if we fix a point (x,y,z), then (2.19) parametrizes the integral curve passing through the point. The characteristic system (2.18) for this vector field is

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{1+z^2}.$$

The first equation reduces to a simple separable ordinary differential equation $dy/dx = -x/y$, with general solution $x^2 + y^2 = c_1$, for c_1 a constant of integration; therefore the cylindrical radius $r = \sqrt{x^2+y^2}$ provides one invariant. To solve the second characteristic equation, we replace x by $\sqrt{r^2 - y^2}$, and treat r as constant. The solution is $\tan^{-1} z - \sin^{-1}(y/r) = \tan^{-1} z - \tan^{-1}(y/x) = c_2$, hence $\tan^{-1} z - \tan^{-1}(y/x)$ is a second invariant. A more convenient choice is provided by the tangent of this invariant. We deduce that, for $yz + x \neq 0$, the functions $r = \sqrt{x^2 + y^2}$, $w = (xz - y)/(yz + x)$ form a complete system of functionally independent invariants, whose common level sets describe the integral curves.

Exercise 2.68. Let $\mathbf{v}$ be a nonvanishing vector field on a manifold M. Prove that the rectifying coordinates $y = \eta(x)$ of Theorem 1.21 satisfy the partial differential equations $\mathbf{v}(\eta^1) = 1$, $\mathbf{v}(\eta^i) = 0$, $i > 1$. Thus the coordinates $y^i = \eta^i(x)$, $i = 2, \ldots, m$, are the functionally independent invariants of the one-parameter group generated by $\mathbf{v}$.

For multi-parameter groups, the invariants are simultaneous solutions to an overdetermined system of linear, homogeneous, first order partial differential equations. One solution method is to look for invariants of each generator in turn and try to re-express subsequent generators in terms of the invariants. However, this can become quite complicated to implement in practice. We illustrate the technique with a simple example.

Example 2.69. Consider the action of the unimodular group $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^3$ generated by the three vector fields

$$\mathbf{v}_- = 2y\partial_x + z\partial_y, \qquad \mathbf{v}_0 = -2x\partial_x + 2z\partial_z, \qquad \mathbf{v}_+ = x\partial_y + 2y\partial_z. \tag{2.20}$$

Away from the origin $x = y = z = 0$, the vector fields (2.20) span a two-dimensional space, and hence, according to Proposition 2.61, the orbits of the group action are (except for the origin) all two-dimensional. Therefore, we expect to find one independent invariant. Solving first the characteristic system for $\mathbf{v}_0$, which is $dx/(2x) = dy/0 = dz/(-2z)$, we derive invariants $w = xz$ and y. Thus, our desired invariant must have the form $I = F(w, y) = F(xz, y)$. Applying $\mathbf{v}_+$ to I, we find $2xyF_w + xF_y = 0$, hence the desired invariant is $I = y^2 - xz$. (Why don't we need to check the invariance of I under $\mathbf{v}_-$?)

Infinitesimal techniques are also effective for the determination of invariant submanifolds and invariant systems of equations. The starting point is the following basic result on the invariance of submanifolds.

Theorem 2.70. *A closed submanifold $N \subset M$ is G-invariant if and only if the space $\mathfrak{g}$ of infinitesimal generators is tangent to N everywhere, i.e., $\mathfrak{g}|_x \subset TN|_x$ for every $x \in N$.*

The proof of Theorem 2.70 follows immediately from the uniqueness of the flow, which implies that if $\mathbf{v}$ is tangent to N, then $\exp(t\mathbf{v})N \subset N$. If the submanifold $N \subset M$ is not closed, then the infinitesimal criterion of Theorem 2.70 implies that N is merely *locally G-invariant*, meaning that for every $x \in N$ and every $g \in G_x$ in a neighborhood of the identity, possibly depending on x, we have $g \cdot x \in N$. For example, if G is the group of translations $(x, y) \mapsto (x + t, y)$, then any horizontal line segment, e.g., $\{\,(x, 0) \mid 0 < x < 1\,\}$ is locally, but not globally, translationally invariant.

An important consequence of Theorem 2.70 is the following crucially important characterization of symmetry groups of regular systems of equations — see Theorem 1.19.

Theorem 2.71. *A connected Lie group G is a* symmetry group *of the regular system of equations $F_1(x) = \cdots = F_k(x) = 0$ if and only if*

$$\mathbf{v}[F_\nu(x)] = 0, \quad \nu = 1, \ldots, k, \quad \textit{whenever} \quad F_1(x) = \cdots = F_k(x) = 0, \tag{2.21}$$

for every infinitesimal generator $\mathbf{v} \in \mathfrak{g}$ *of G.*

Using the formula $\mathbf{v}(F) = \langle\, dF\,; \mathbf{v}\,\rangle$, we see that Theorem 2.71 follows directly from Theorem 2.70 and Exercise 1.35. The following example provides a simple illustration of how the infinitesimal criterion (2.21) is verified in practice. However, the real power of Theorem 2.71 must await our applications to symmetry groups of differential equations, which is the subject of Chapter 4.

Example 2.72. The equation $x^2 + y^2 = 1$ defines a circle, which is rotationally invariant. To check the infinitesimal condition, we apply the generator $\mathbf{v} = -y\partial_x + x\partial_y$ to the defining function $F(x, y) = x^2 + y^2 - 1$. We find $\mathbf{v}(F) = 0$ everywhere (since F is an invariant). Since dF is nonzero on the circle, the equation is regular, and hence its solution set is rotationally invariant. A more complicated example is provided by $H(x, y) = x^4 + x^2y^2 + y^2 - 1$. Now, $\mathbf{v}(H) = -2xy(x^2 + 1)^{-1}H$, hence $\mathbf{v}(H) = 0$ whenever $H = 0$, hence the set of solutions to $H(x, y) = 0$ is rotationally invariant. (What is this set?) To see the importance of the regularity condition, consider the function $K(x, y) = y^2$. The solution set is the x-axis, which is clearly not rotationally invariant, even though $\mathbf{v}(K) = 2xy = 0$ vanishes when $y = 0$, and so the infinitesimal condition (2.21) is satisfied.

Lie Derivatives and Invariant Differential Forms

A differential form Ω on M is called *G-invariant* if it is unchanged under the pull-back action of the group: $g^*\big(\Omega|_{g\cdot x}\big) = \Omega|_x$ for all $g \in G$, $x \in M$. In particular, an invariant 0-form is just an ordinary invariant function. Formulae (1.28) demonstrate that the sum and wedge product of invariant forms are also invariant; in particular, multiplying an invariant k-form by an invariant function produces another invariant k-form. Furthermore, the fundamental invariance property (1.33) of the exterior derivative proves that the differential $d\Omega$ of any invariant k-form is an invariant $(k + 1)$-form. In particular, we find:

Proposition 2.73. *If I is an invariant function, its differential dI is an invariant one-form.*

There is, of course, an infinitesimal criterion for the invariance of a differential form under a connected group of transformations. This condition is formalized by the definition of the Lie derivative of a differential form with respect to a vector field $\mathbf{v}$, which indicates how the form varies infinitesimally under the associated flow $\exp(t\mathbf{v})$.

Definition 2.74. Let $\mathbf{v}$ be a vector field on the manifold M with flow $\exp(t\mathbf{v})$. The *Lie derivative* $\mathbf{v}(\Omega)$ of a differential form Ω with respect to $\mathbf{v}$ is defined as

$$\mathbf{v}(\Omega)|_x = \frac{d}{dt} \exp(t\mathbf{v})^* \big(\Omega|_{\exp(t\mathbf{v})x}\big)\Big|_{t=0}. \tag{2.22}$$

Note that the pull-back $\exp(t\mathbf{v})^*$ moves the form at the point $\exp(t\mathbf{v})x$ back to the point x, enabling us to compute the derivative consistently. Thus, we find the series expansion

$$\exp(t\mathbf{v})^*\big(\Omega|_{\exp(t\mathbf{v})x}\big) = \Omega|_x + t\,\mathbf{v}(\Omega)|_x + \cdots, \tag{2.23}$$

the higher order terms being provided by higher order Lie derivatives of Ω. In particular, the Lie derivative $\mathbf{v}(f)$ of a function $f\colon M \to \mathbb{R}$ (or 0-form) coincides with the action of the vector field on f, and (2.23) reduces to the earlier expansion (1.10).

Exercise 2.75. Define the Lie derivative of a vector field $\mathbf{w}$ with respect to the vector field $\mathbf{v}$. Prove that the Lie derivative coincides with the Lie bracket $[\mathbf{v}, \mathbf{w}]$.

The explicit local coordinate formula for the Lie derivative are most readily deduced from its basic linearity, derivation, and commutation properties:

$$\begin{aligned} \mathbf{v}(\Omega + \Theta) &= \mathbf{v}(\Omega) + \mathbf{v}(\Theta), \\ \mathbf{v}(\Omega \wedge \Theta) &= \mathbf{v}(\Omega) \wedge \Theta + \Omega \wedge \mathbf{v}(\Theta), \\ \mathbf{v}(d\Omega) &= d\mathbf{v}(\Omega). \end{aligned} \tag{2.24}$$

These all follow directly from the basic definition (2.22) using the properties of the pull-back map. In fact, the properties (2.24) along with the action on functions serve to uniquely define the Lie derivative operation. For example, the Lie derivative of a one-form (1.25) with respect to the vector field (1.3) is given by

$$\mathbf{v}(\omega) = \sum_{i=1}^m \Big[\mathbf{v}(h_i)\,dx^i + h_i\,d\xi^i\Big], \qquad \mathbf{v} = \sum_{i=1}^m \xi^i(x)\,\frac{\partial}{\partial x^i}, \qquad \omega = \sum_{i=1}^m h_i(x)dx^i.$$

Exercise 2.76. Prove that the exterior derivative of a differential form can be written in local coordinates $(x^1, \ldots, x^m)$ as

$$d\Omega = \sum_{i=1}^m dx^i \wedge \frac{\partial \Omega}{\partial x^i}, \tag{2.25}$$

the latter term being the Lie derivative of Ω with respect to the coordinate frame vector fields $\partial/\partial x^i$. Can this formula be generalized to other frames?

Exercise 2.77. Given a k-form Ω, its *interior product* with a vector field $\mathbf{v}$ is the $(k-1)$-form $\mathbf{v} \,\lrcorner\, \Omega$ defined so that

$$\langle\, \mathbf{v} \,\lrcorner\, \Omega\,; \mathbf{w}_1, \ldots, \mathbf{w}_{k-1} \,\rangle = \langle\, \Omega\,; \mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_{k-1} \,\rangle, \tag{2.26}$$

for any set of $k-1$ vector fields $\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}$. Using this, prove the following important formula relating the Lie derivative and the exterior derivative of a differential form:

$$\mathbf{v}(\Omega) = \mathbf{v} \,\lrcorner\, (d\Omega) + d(\mathbf{v} \,\lrcorner\, \Omega). \tag{2.27}$$

Theorem 2.78. *A differential form Ω is invariant under a connected Lie group of transformations G if and only if its Lie derivative with respect to every infinitesimal generator $\mathbf{v} \in \mathfrak{g}$ vanishes: $\mathbf{v}(\Omega) = 0$.*

Example 2.79. Let $dx = dx^1 \wedge \cdots \wedge dx^m$ denote the volume form on $\mathbb{R}^m$. The Lie derivative of dx with respect to a vector field $\mathbf{v} = \sum_{i=1}^p \xi^i(x)\partial_{x^i}$ is given as $\mathbf{v}(dx) = (\operatorname{div}\xi)\,dx$, where $\operatorname{div}\xi = \sum_i \partial\xi^i/\partial x^i$ is the divergence of the coefficients of $\mathbf{v}$. Therefore, a transformation group is volume-preserving on $\mathbb{R}^m$ if and only if each of its infinitesimal generators is divergence free: $\operatorname{div}\xi = 0$. (See also Exercise 1.38.)

The existence of invariant forms for general transformation groups parallels that of invariant vector fields. Only when the group acts freely are we guaranteed a (complete) collection of invariant one-forms.

Theorem 2.80. *Let G be an r-dimensional Lie group acting* effectively freely *on the m-dimensional manifold M. Then, locally, there exist m pointwise linearly independent G-invariant one-forms $\omega^1, \ldots, \omega^m$.*

At each point $x \in M$, the one-forms $\omega^1, \ldots, \omega^m$ form a basis for the cotangent space $T^*M|_x$, and so form a *G-invariant coframe*. Their wedge product $\omega^1 \wedge \cdots \wedge \omega^m \neq 0$ defines a nonvanishing G-invariant volume form on M.

Example 2.81. Consider the action of the connected component of the Euclidean group SE(2) on $M = \mathbb{R}^3$, which is generated by the vector fields

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_y, \qquad \mathbf{v}_3 = -y\partial_x + x\partial_y + (1+z^2)\partial_z. \tag{2.28}$$

Therefore, the general invariant vector field is a constant coefficient linear combination of

$$\mathbf{w}_1 = \frac{\partial_x + z\partial_y}{\sqrt{1+z^2}}, \qquad \mathbf{w}_2 = \frac{z\partial_x - \partial_y}{\sqrt{1+z^2}}, \qquad \mathbf{w}_3 = (1+z^2)\partial_z. \tag{2.29}$$

The invariant one-forms are dual to the invariant vector fields (2.29), and hence are given by

$$\omega^1 = \frac{dx + z\,dy}{\sqrt{1+z^2}}, \qquad \omega^2 = \frac{dy - z\,dx}{\sqrt{1+z^2}}, \qquad \omega^3 = \frac{dz}{1+z^2}.$$

Alternatively, one could directly construct the invariant one-forms by analyzing the Lie derivative condition of Theorem 2.78.

The Maurer–Cartan Forms

Of particular importance in the theory of Lie groups are the invariant differential forms associated with the right (and left) actions of the Lie group on itself. By definition, a differential form Ω on a Lie group G is right-invariant if it is unaffected by right multiplication by group elements: $(R_g)^*\Omega = \Omega$ for all $g \in G$. In particular, the right-invariant one-forms on G are known as the (right-invariant) *Maurer–Cartan forms*. The space of Maurer–Cartan forms is naturally dual to the Lie algebra of G, and hence forms a vector space of the same dimension as the Lie group. If we choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of the Lie algebra $\mathfrak{g}$, then there is a dual basis (or coframe) $\alpha^1, \ldots, \alpha^r$, consisting of Maurer–Cartan forms, satisfying $\langle\, \alpha^i \,; \mathbf{v}_j \,\rangle = \delta^i_j$. As a direct consequence of formula (1.32), and duality, the Maurer–Cartan forms are seen to satisfy the fundamental *structure equations*

$$d\alpha^k = -\frac{1}{2} \sum_{i,j=1}^{r} C^k_{ij}\, \alpha^i \wedge \alpha^j = - \sum_{\substack{i,j=1 \\ i<j}}^{r} C^k_{ij}\, \alpha^i \wedge \alpha^j, \qquad k = 1, \ldots, r. \tag{2.30}$$

The coefficients C^k_{ij} are the *same* as the structure constants (2.12) corresponding to our choice of basis of the Lie algebra $\mathfrak{g}$.

If the group G is given as a parametrized matrix Lie group, then a basis for the space of Maurer–Cartan forms can be found among the entries of the matrix of one-forms

$$\boldsymbol{\gamma} = dg \cdot g^{-1}, \qquad \text{or} \qquad \gamma^i_j = \sum_{k=1}^{r} dg^i_k (g^{-1})^k_j. \tag{2.31}$$

Each entry γ^i_j is clearly a right-invariant one-form since if h is any fixed group element, then

$$(R_h)^* \boldsymbol{\gamma} = d(g \cdot h) \cdot (g \cdot h)^{-1} = dg \cdot g^{-1} = \boldsymbol{\gamma}.$$

It is also not hard to see that the number of linearly independent entries of $\boldsymbol{\gamma}$ is the same as the dimension of the group, and hence the entries provide a complete basis for the space of Maurer–Cartan forms.

Of course, one can also define left-invariant Maurer–Cartan forms on a Lie group, dual to the Lie algebra of left-invariant vector fields. The construction goes through word for word. Note that the structure constants appearing in the left-invariant structure equations are those corresponding to the left-invariant Lie algebra, and so have the opposite sign to the right-invariant structure constants. Replacing formula (2.31) is the matrix $\widehat{\boldsymbol{\gamma}} = g^{-1} \cdot dg$ whose entries provide a basis for the left-invariant Maurer–Cartan forms.

Example 2.82. As in Example 2.11, the two-dimensional affine group A(1) can be identified with the group of 2×2 matrices of the form $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. According to (2.31), the right-invariant Maurer–Cartan forms on A(1) are provided by the independent entries of the matrix

$$dg \cdot g^{-1} = \begin{pmatrix} da & db \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{-1}\, da & a^{-1}(a\, db - b\, da) \\ 0 & 0 \end{pmatrix}.$$

Therefore, the two Maurer–Cartan forms are

$$\alpha^1 = \frac{da}{a}, \qquad \alpha^2 = \frac{a\, db - b\, da}{a},$$

forming the dual basis to the Lie algebra basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathfrak{a}(1)$ found in (2.10) above. The Maurer–Cartan structure equations (2.30) for the group A(1) are then

$$d\alpha^1 = 0, \qquad d\alpha^2 = \frac{da \wedge\, db}{a} = \alpha^1 \wedge \alpha^2,$$

reconfirming the Lie algebra commutation relation $[\mathbf{v}_1, \mathbf{v}_2] = -\mathbf{v}_2$. Similarly, the left-invariant Maurer–Cartan forms are the matrix entries of

$$g^{-1} \cdot dg = \begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} da & db \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^{-1}\, da & a^{-1}\, db \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$\widehat{\alpha}^1 = \frac{da}{a}, \qquad \widehat{\alpha}^2 = \frac{db}{a},$$

form the dual basis to the left Lie algebra basis in (2.11). Note that the left Maurer–Cartan structure equations have the opposite sign: $d\widehat{\alpha}^1 = 0$, $d\widehat{\alpha}^2 = -\,\widehat{\alpha}^1 \wedge \widehat{\alpha}^2$, in accordance with the effect of left and right invariance on the structure constants of the Lie algebra.

Exercise 2.83. Find the right and left Lie algebras and the corresponding Maurer–Cartan forms for the three-dimensional Heisenberg group U(3) consisting of all 3×3 upper triangular matrices of the form

$$g = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 2.84. Let $\rho^i_j = -\rho^j_i$ be the standard basis for the Maurer–Cartan forms for the orthogonal group $\mathrm{SO}(n)$, given as the matrix entries of the skew-symmetric matrix of one-forms $\boldsymbol{\rho} = dR \cdot R^{-1}$, $R \in \mathrm{SO}(n)$. Prove that the associated structure equations are $d\rho^i_j = \sum_k \rho^i_k \wedge \rho^k_j$, often written in matrix form as $d\boldsymbol{\rho} = \boldsymbol{\rho} \wedge \boldsymbol{\rho}$.

Exercise 2.85. Prove that a one-form α on G is a right-invariant Maurer–Cartan form if and only if it satisfies $\widehat{\mathbf{v}}(\alpha) = 0$ for all *left*-invariant vector fields $\widehat{\mathbf{v}} \in \mathfrak{g}_L$.