Moving Frames in Applications

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Moving Frames

Classical contributions:

M. Bartels (~1800), J. Serret, J. Frénet, G. Darboux, É. Cotton,

$\boxed{\text{Élie Cartan}}$

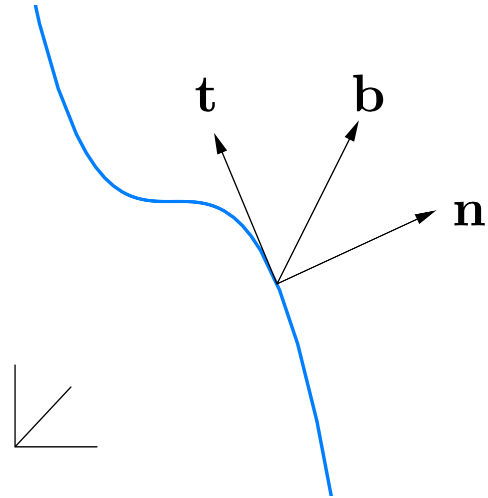
Modern developments: (1970's)

S.S. Chern, M. Green, P. Griffiths, G. Jensen, . . .

The equivariant approach: (1997 –)

PJO, M. Fels, G. Marí–Beffa, I. Kogan, J. Cheh, J. Pohjanpelto, P. Kim, M. Boutin, D. Lewis, E. Mansfield, E. Hubert, O. Morozov, R. McLenaghan, R. Smirnov, J. Yue, A. Nikitin, J. Patera, F. Valiquette, R. Thompson, . . .

Moving Frame — Space Curves



tangent normal binormal

$$\mathbf{t} = \frac{dz}{ds} \qquad \mathbf{n} = \frac{z_{ss}}{\|\, z_{ss} \,\|} \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}$$

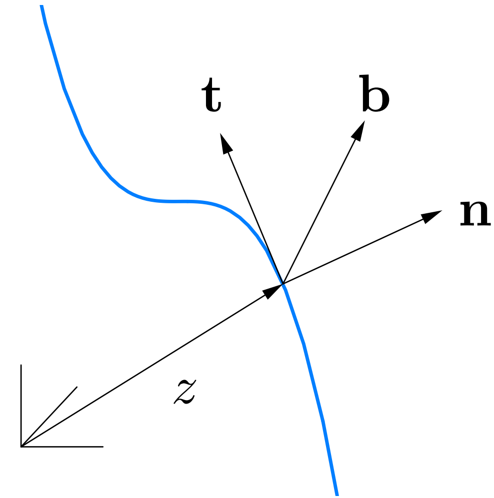
s — arc length

Frénet–Serret equations

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n} \qquad \frac{d\mathbf{n}}{ds} = -\,\kappa\,\mathbf{t} + \tau\,\mathbf{b} \qquad \frac{d\mathbf{b}}{ds} = -\,\tau\,\mathbf{n}$$

κ — curvature $\qquad \tau$ — torsion

Moving Frame — Space Curves



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Frénet–Serret equations

$$\frac{d\mathbf{t}}{ds} = \kappa \, \mathbf{n} \qquad \frac{d\mathbf{n}}{ds} = - \kappa \, \mathbf{t} + \tau \, \mathbf{b} \qquad \frac{d\mathbf{b}}{ds} = - \tau \, \mathbf{n}$$

κ — curvature $\qquad$ τ — torsion

“I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear.”

“Nevertheless, I must admit I found the book, like most of Cartan’s papers, hard reading.”

— Hermann Weyl

“Cartan on groups and differential geometry”
Bull. Amer. Math. Soc. **44** (1938) 598–601

Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Joint invariants and semi-differential invariants
- Invariant differential forms and tensors
- Identities and syzygies
- Classical invariant theory

- Computer vision — object recognition & symmetry detection
- Invariant numerical methods
- Invariant variational problems
- Invariant submanifold flows
- Poisson geometry & solitons
- Killing tensors in relativity
- Invariants of Lie algebras in quantum mechanics
- Lie pseudo-groups

The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Equivalence:

Determine when two p-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \subset M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Find all symmetries,

i.e., self-equivalences or *self-congruences*:

$$N = g \cdot N$$

Classical Geometry — *F. Klein*

- **Euclidean group:** $$G = \begin{cases} \mathrm{SE}(m) = \mathrm{SO}(m) \ltimes \mathbb{R}^m \\ \mathrm{E}(m) = \mathrm{O}(m) \ltimes \mathbb{R}^m \end{cases}$$

  $\boxed{z \longmapsto A \cdot z + b}$ $\qquad A \in \mathrm{SO}(m)$ or $\mathrm{O}(m)$, $\quad b \in \mathbb{R}^m$, $\quad z \in \mathbb{R}^m$

  $\Rightarrow$ isometries: rotations, translations , (reflections)

- **Equi-affine group:** $\qquad G = \mathrm{SA}(m) = \mathrm{SL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{SL}(m)$ — volume-preserving

- **Affine group:** $\qquad G = \mathrm{A}(m) = \mathrm{GL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{GL}(m)$

- **Projective group:** $\qquad G = \mathrm{PSL}(m+1)$

  acting on $\mathbb{R}^m \subset \mathbb{RP}^m$

$\Longrightarrow$ Applications in computer vision

Tennis, Anyone?

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

- multiplier representation of GL(2)
- modular forms

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)$$

Transformation group:

$$g : (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}\,,\ \frac{u}{(\gamma x + \delta)^n}\right)$$

Equivalence of functions $\iff$ equivalence of graphs

$$\Gamma_Q = \{\, (x, u) = (x, Q(x)) \,\} \subset \mathbb{C}^2$$

Moving Frames

Definition.

A moving frame is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Isotropy & Freeness

Isotropy subgroup: $G_z = \{\, g \mid g \cdot z = z \,\}$ for $z \in M$

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity

  $$\Longrightarrow \quad G_z = \{e\} \text{ for all } z \in M$$

- locally free — the orbits all have the same dimension as G

  $$\Longrightarrow \quad G_z \subset G \text{ is discrete for all } z \in M$$

- regular — the orbits form a regular foliation

  $$\not\approx \quad \text{irrational flow on the torus}$$

- effective — the only group element which fixes *every* point in M is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$:

  $$G_M^* = \bigcap_{z \in M} G_z = \{e\}$$

Proof of the Main Theorem

Necessity: Let $\rho : M \to G$ be a left moving frame.

Freeness: If $g \in G_z$, so $g \cdot z = z$, then by left equivariance:

$$\rho(z) = \rho(g \cdot z) = g \cdot \rho(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$.

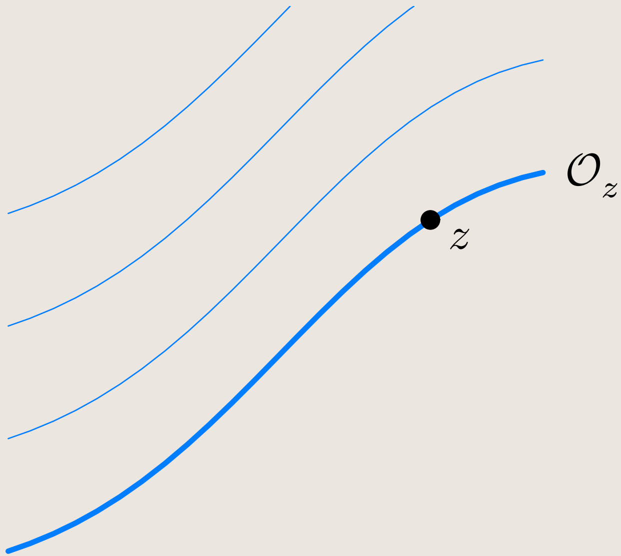
Regularity: Suppose $z_n = g_n \cdot z \longrightarrow z$ as $n \to \infty$.

By continuity, $\rho(z_n) = \rho(g_n \cdot z) = g_n \cdot \rho(z) \longrightarrow \rho(z)$.
Hence $g_n \longrightarrow e$ in G.

Sufficiency: By direct construction — "normalization".

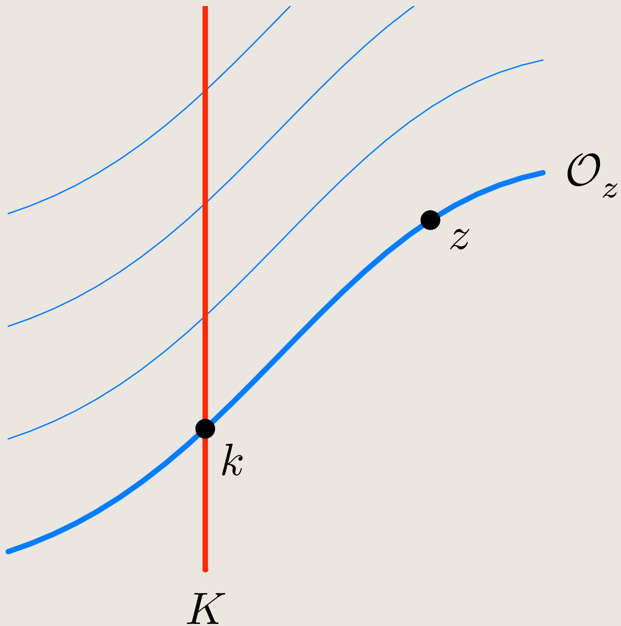
Q.E.D.

Geometric Construction



$$\boxed{\text{Normalization} = \text{choice of cross-section to the group orbits}}$$

Geometric Construction



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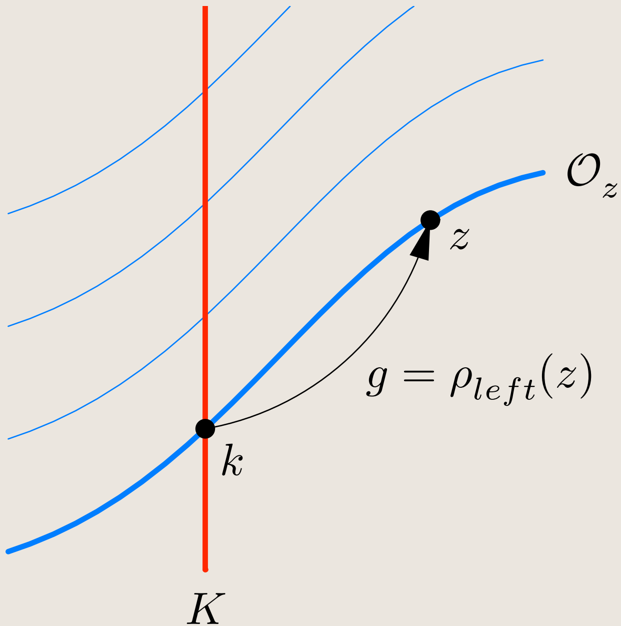

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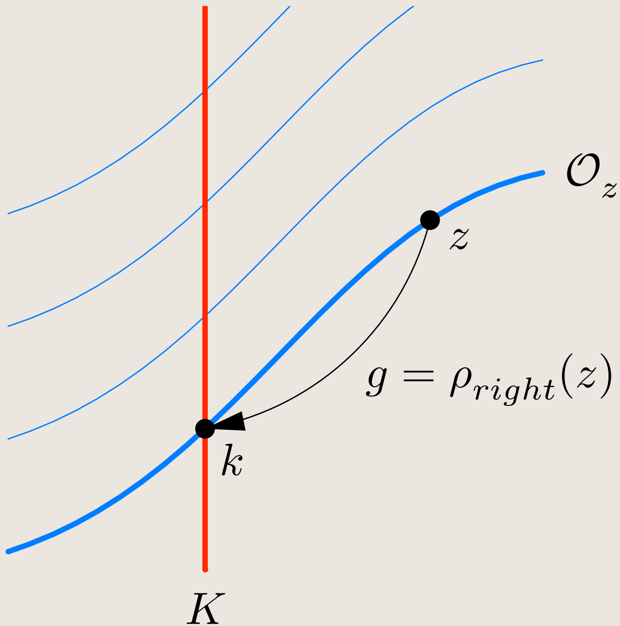

$$\boxed{\text{Normalization} = \text{choice of cross-section to the group orbits}}$$

K — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

$\Longrightarrow$ freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Algebraic Construction

$$r = \dim G \;\leq\; m = \dim M$$

Coordinate cross-section

$$K = \{ z_1 = c_1, \;\ldots\;, z_r = c_r \}$$

left	right
$w(g, z) = g^{-1} \cdot z$	$w(g, z) = g \cdot z$

$g = (g_1, \ldots, g_r)$ — group parameters

$z = (z_1, \ldots, z_m)$ — coordinates on M

Choose $r = \dim G$ components to *normalize*:

$$w_1(g, z) = c_1 \qquad \ldots \qquad w_r(g, z) = c_r$$

Solve for the group parameters $g = (g_1, \ldots, g_r)$

$\Longrightarrow$ Implicit Function Theorem

The solution

$$g = \rho(z)$$

is a (local) moving frame.

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Completeness of Invariants

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ \ldots \ , I_{m-r}(z))$$

Invariantization

Definition. The *invariantization* of a function $F : M \to \mathbb{R}$ with respect to a right moving frame $g = \rho(z)$ is the the invariant function $I = \iota(F)$ defined by

$$I(z) = F(\rho(z) \cdot z).$$

$$\iota(z_1) = c_1, \quad \ldots \quad \iota(z_r) = c_r, \qquad \iota(z_{r+1}) = I_1(z), \quad \ldots \quad \iota(z_m) = I_{m-r}(z).$$

cross-section variables
"phantom invariants"

fundamental invariants

$$\boxed{\iota\,[\,F(z_1, \ldots, z_m)\,] \;=\; F(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z))}$$

Invariantization amounts to restricting F to the cross-section

$$I \mid K = F \mid K$$

and then requiring that $I = \iota(F)$ be constant along the orbits.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Invariantization defines a canonical projection

$$\iota : \text{functions} \quad \longmapsto \quad \text{invariants}$$

The Rotation Group

$$G = \mathrm{SO}(2) \qquad \text{acting on} \qquad \mathbb{R}^2$$

$$z = (x, u) \longmapsto g \cdot z = (\, x \cos\phi - u \sin\phi \,, \; x \sin\phi + u \cos\phi \,)$$

$$\Longrightarrow \quad \text{Free on } M = \mathbb{R}^2 \setminus \{0\}$$

Left moving frame:

$$w(g, z) = g^{-1} \cdot z = (y, v)$$

$$y = x \cos\phi + u \sin\phi \qquad v = -\, x \sin\phi + u \cos\phi$$

Cross-section

$$K = \{\, u = 0, \;\; x > 0 \,\}$$

Normalization equation

$$v = -x\sin\phi + u\cos\phi = 0$$

Left moving frame:

$$\phi = \tan^{-1}\frac{u}{x} \quad \Longrightarrow \quad \phi = \rho(x,u) \in \mathrm{SO}(2)$$

Fundamental invariant

$$r = \iota(x) = \sqrt{x^2 + u^2}$$

Invariantization

$$\iota[\,F(x,u)\,] = F(r,0)$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

Freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $m < r = \dim G$.

Thus, to make the action free, we must increase the dimension of the space via some natural prolongation procedure.

- An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

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Euclidean Plane Curves

Special Euclidean group: $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

acts on $M = \mathbb{R}^2$ via rigid motions: $w = R\,z + b$

To obtain the classical (left) moving frame we invert the group transformations:

$$\left.\begin{aligned} y &= \quad \cos\phi\,(x-a) + \sin\phi\,(u-b) \\ v &= -\sin\phi\,(x-a) + \cos\phi\,(u-b) \end{aligned}\right\} \qquad w = R^{-1}(z-b)$$

Assume for simplicity the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

$\Longrightarrow$ extensions to parametrized curves are straightforward

Prolong the action to J^n via implicit differentiation:

$$\begin{aligned}
y &= \cos\phi\,(x-a) + \sin\phi\,(u-b)\\
v &= -\sin\phi\,(x-a) + \cos\phi\,(u-b)\\
v_y &= \frac{-\sin\phi + u_x\cos\phi}{\cos\phi + u_x\sin\phi}\\
v_{yy} &= \frac{u_{xx}}{(\cos\phi + u_x\sin\phi)^3}\\
v_{yyy} &= \frac{(\cos\phi + u_x\sin\phi)u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi + u_x\sin\phi)^5}\\
&\vdots
\end{aligned}$$

Prolong the action to J^n via implicit differentiation:

$$y = \cos\phi\,(x - a) + \sin\phi\,(u - b)$$

$$v = -\sin\phi\,(x - a) + \cos\phi\,(u - b)$$

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$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x \sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi + u_x \sin\phi)u_{xxx} - 3u_{xx}^2 \sin\phi}{(\cos\phi + u_x \sin\phi)^5}$$

$$\vdots$$

Normalization: $\qquad r = \dim G = 3$

$$y = \cos\phi\,(x - a) + \sin\phi\,(u - b) = 0$$

$$v = -\sin\phi\,(x - a) + \cos\phi\,(u - b) = 0$$

$$v_y = \frac{-\sin\phi + u_x \cos\phi}{\cos\phi + u_x \sin\phi} = 0$$

$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x \sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi + u_x \sin\phi)u_{xxx} - 3u_{xx}^2 \sin\phi}{(\cos\phi + u_x \sin\phi)^5}$$

$$\vdots$$

Solve for the group parameters:

$$y = \quad \cos\phi\,(x - a) + \sin\phi\,(u - b) = 0$$

$$v = -\sin\phi\,(x - a) + \cos\phi\,(u - b) = 0$$

$$v_y = \frac{-\sin\phi + u_x \cos\phi}{\cos\phi + u_x \sin\phi} = 0$$

$\Longrightarrow$ Left moving frame $\qquad \rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x \qquad b = u \qquad \phi = \tan^{-1} u_x$$

$$a = x \qquad b = u \qquad \phi = \tan^{-1} u_x$$

Differential invariants

$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x \sin\phi)^3} \longmapsto \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$v_{yyy} = \cdots \qquad \longmapsto \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}$$

$$v_{yyyy} = \cdots \qquad \longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 = \cdots$$

$\Longrightarrow$ recurrence formulae

Contact invariant one-form — arc length

$$dy = (\cos\phi + u_x \sin\phi)\, dx \qquad \longmapsto \qquad ds = \sqrt{1+u_x^2}\; dx$$

Dual invariant differential operator

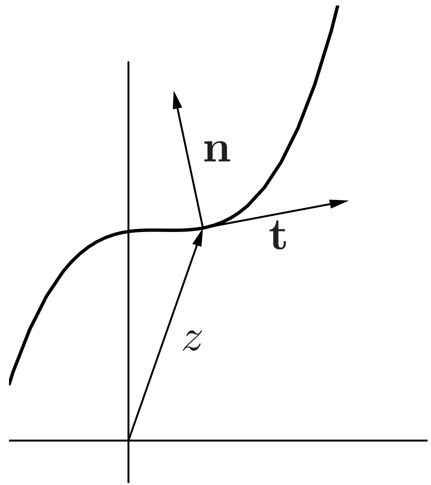
— arc length derivative

$$\frac{d}{dy} = \frac{1}{\cos\phi + u_x \sin\phi}\,\frac{d}{dx} \qquad \longmapsto \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa, \qquad \frac{d\kappa}{ds}, \qquad \frac{d^2\kappa}{ds^2}, \qquad \cdots$$

The Classical Picture:



Moving frame $\qquad \rho : \; (x, u, u_x) \longmapsto (R, \mathbf{a}) \in \mathrm{SE}(2)$

$$R = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\,\mathbf{t},\;\; \mathbf{n}\,) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix}, \qquad \mathbf{n} = \mathbf{t}^{\perp} = \begin{pmatrix} -\,y_s \\ x_s \end{pmatrix}.$$

Frenet equations = Pulled-back Maurer–Cartan forms:

$$\frac{d\mathbf{x}}{ds} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -\,\kappa\,\mathbf{t}.$$

Equi-affine Curves $\qquad G = \mathrm{SA}(2)$

$$z \longmapsto A\, z + \mathbf{b} \qquad\qquad A \in \mathrm{SL}(2), \qquad \mathbf{b} \in \mathbb{R}^2$$

Invert for left moving frame:

$$\left.\begin{aligned} y &= \delta\,(x-a) - \beta\,(u-b) \\ v &= -\,\gamma\,(x-a) + \alpha\,(u-b) \end{aligned}\right\} \qquad w = A^{-1}(z-b)$$

$$\alpha\,\delta - \beta\,\gamma = 1$$

Prolong to J^3 via implicit differentiation

$$dy = (\delta - \beta\, u_x)\; dx \qquad\qquad D_y = \frac{1}{\delta - \beta\, u_x}\, D_x$$

Prolongation:

$$y = \delta\,(x - a) - \beta\,(u - b)$$

$$v = -\,\gamma\,(x - a) + \alpha\,(u - b)$$

$$v_y = -\,\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x}$$

$$v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta\,u_x)^3}$$

$$v_{yyy} = -\,\frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5}$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \;\ldots$$

Normalization: $\qquad r = \dim G = 5$

$$y = \delta\,(x - a) - \beta\,(u - b) = 0$$

$$v = -\,\gamma\,(x - a) + \alpha\,(u - b) = 0$$

$$v_y = -\,\frac{\gamma - \alpha\, u_x}{\delta - \beta\, u_x} = 0$$

$$v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta\, u_x)^3} = 1$$

$$v_{yyy} = -\,\frac{(\delta - \beta\, u_x)\, u_{xxx} + 3\,\beta\, u_{xx}^2}{(\delta - \beta\, u_x)^5} = 0$$

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$$v_{yyyyy} = \ \ldots$$

Equi-affine Moving Frame

$$\rho : \; (x, u, u_x, u_{xx}, u_{xxx}) \;\longmapsto\; (A, \mathbf{b}) \in \mathrm{SA}(2)$$

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Nondegeneracy condition: $\qquad u_{xx} \neq 0.$

Totally Singular Submanifolds

Definition. A p-dimensional submanifold $N \subset M$ is totally singular if $G^{(n)}$ does *not* act freely on $\mathrm{j}_n N$ for any $n \geq 0$.

Theorem. N is totally singular if and only if its symmetry group $G_N = \{\, g \mid g \cdot N \subset N \,\}$ has dimension $> p$, and so G_N does not act freely on N itself.

Thus, the totally singular submanifolds are the only ones that do not admit a moving frame of any order.

In equi-affine geometry, only the straight lines ($u_{xx} \equiv 0$) are totally singular since they admit a three-dimensional equi-affine symmetry group.

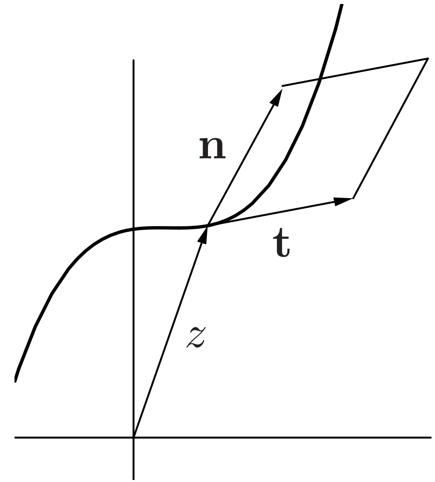
Equi-affine arc length

$$dy = (\delta - \beta\, u_x)\ dx \qquad \longmapsto \qquad ds = \sqrt[3]{u_{xx}}\ dx$$

Equi-affine curvature

$$\begin{aligned}
v_{yyyy} &\qquad \longmapsto \qquad \kappa \ = \ \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}} \\
v_{yyyyy} &\qquad \longmapsto \qquad \frac{d\kappa}{ds} \\
v_{yyyyyy} &\qquad \longmapsto \qquad \frac{d^2\kappa}{ds^2} - 5\kappa^2
\end{aligned}$$

The Classical Picture:



$$A = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix} = (\,\mathbf{t},\ \mathbf{n}\,) \qquad \mathbf{b} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{t} = \frac{dz}{ds}\,, \qquad\qquad \mathbf{n} = \frac{d^2 z}{ds^2}\,.$$

Frenet equations = Pulled-back Maurer–Cartan forms:

$$\frac{dz}{ds} = \mathbf{t}, \qquad\qquad \frac{d\mathbf{t}}{ds} = \mathbf{n}, \qquad\qquad \frac{d\mathbf{n}}{ds} = \kappa\,\mathbf{t}.$$

Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$

  must have the same invariants: $I = \bar{I}$.

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Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Equivalence & Invariants

- Equivalent submanifolds $\quad N \approx \overline{N}$

  must have the same invariants: $\;I = \bar{I}$.

Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

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- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

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- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Theorem. (Cartan) Two submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \tag{$*$}$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or $\ldots$) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

The Signature Map

The generating syzygies are encoded by the signature map

$$\Sigma : N \longrightarrow \mathcal{S}$$

of the submanifold N, which is parametrized by the fundamental differential invariants:

$$\Sigma(x) = (I_1(x), \ldots, I_m(x))$$

The image

$$\mathcal{S} = \operatorname{Im} \Sigma$$

is the signature subset (or submanifold) of N.

Equivalence & Signature

Theorem. Two regular submanifolds are equivalent

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Other Signatures

Euclidean space curves: $\mathcal{C} \subset \mathbb{R}^3$

$$\mathcal{S} = \{\, (\, \kappa \,,\, \kappa_s \,,\, \tau \,) \,\} \quad \subset \quad \mathbb{R}^3$$

- κ — curvature, τ — torsion

Euclidean surfaces: $\mathcal{S} \subset \mathbb{R}^3$ (generic)

$$\mathcal{S} = \left\{ \left(H \,,\, K \,,\, H_{,1} \,,\, H_{,2} \,,\, K_{,1} \,,\, K_{,2} \right) \right\} \quad \subset \quad \mathbb{R}^3$$

- H — mean curvature, K — Gauss curvature

Equi–affine surfaces: $\mathcal{S} \subset \mathbb{R}^3$ (generic)

$$\mathcal{S} = \left\{ \left(P \,,\, P_{,1} \,,\, P_{,2} ,\, P_{,11} \right) \right\} \quad \subset \quad \mathbb{R}^3$$

- P — Pick invariant

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$

Theorem. Two regular curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\mathcal{S}} = \mathcal{S}$$

Symmetry and Signature

Theorem. The dimension of the symmetry group

$$G_N = \{ g \mid g \cdot N \subset N \}$$

of a nonsingular submanifold $N \subset M$ equals the codimension of its signature:

$$\dim G_N = \dim N - \dim \mathcal{S}$$

Corollary. For a nonsingular submanifold $N \subset M$,

$$0 \leq \dim G_N \leq \dim N$$

$\Longrightarrow$ Only totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature $\mathcal{S}$ degenerates to a point: $\dim \mathcal{S} = 0$
- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a p-dimensional subgroup $H \subset G$

$\Longrightarrow$ Euclidean geometry: circles, lines, helices, spheres, cylinders, planes, . .

$\Longrightarrow$ Equi-affine plane geometry: conic sections.

$\Longrightarrow$ Projective plane geometry: W curves (*Lie & Klein*)

Discrete Symmetries

Definition. The index of a submanifold N equals the number of points in N which map to a generic point of its signature:

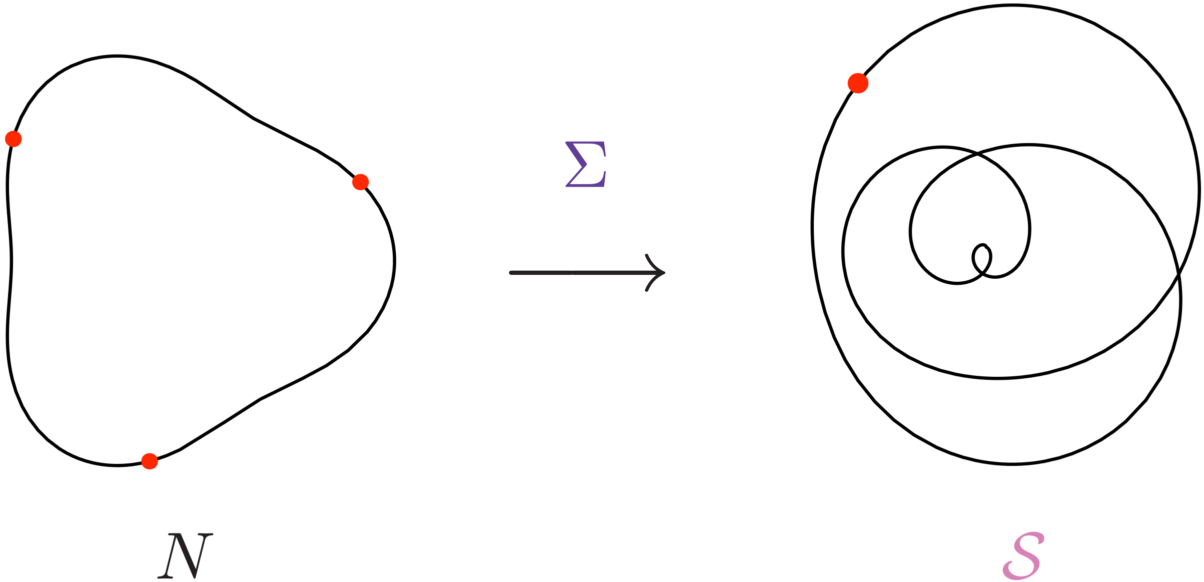
$$\iota_N = \min \left\{ \# \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self–intersections

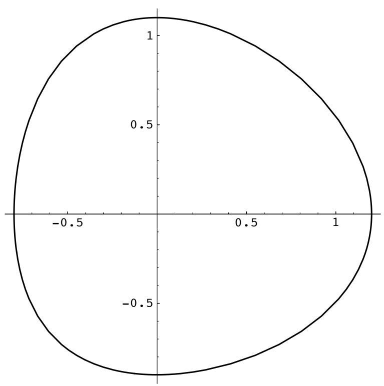
Theorem. The cardinality of the symmetry group of a submanifold N equals its index ι_N.

$\Longrightarrow$ Approximate symmetries

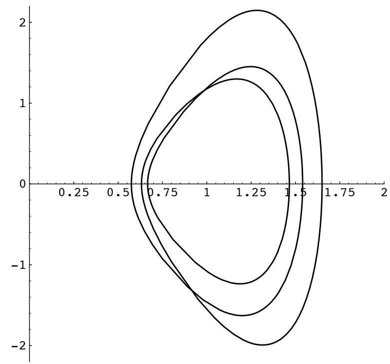
The Index



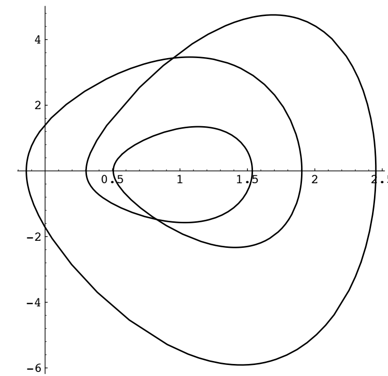
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

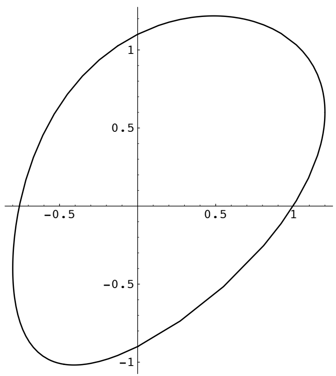


Euclidean Signature

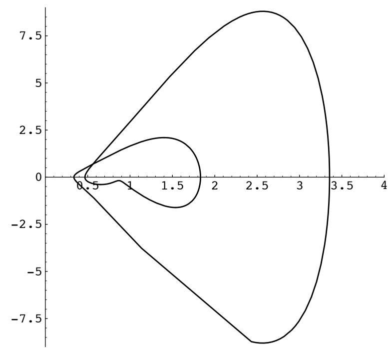


Affine Signature

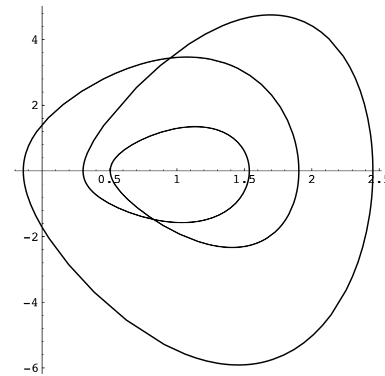
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



The Original Curve



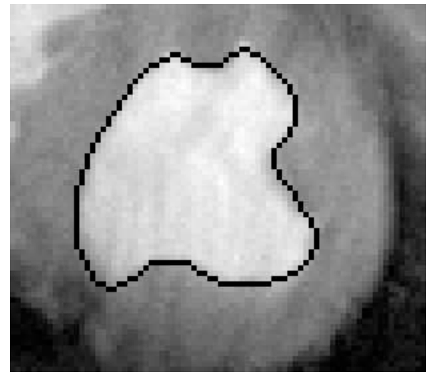
Euclidean Signature



Affine Signature

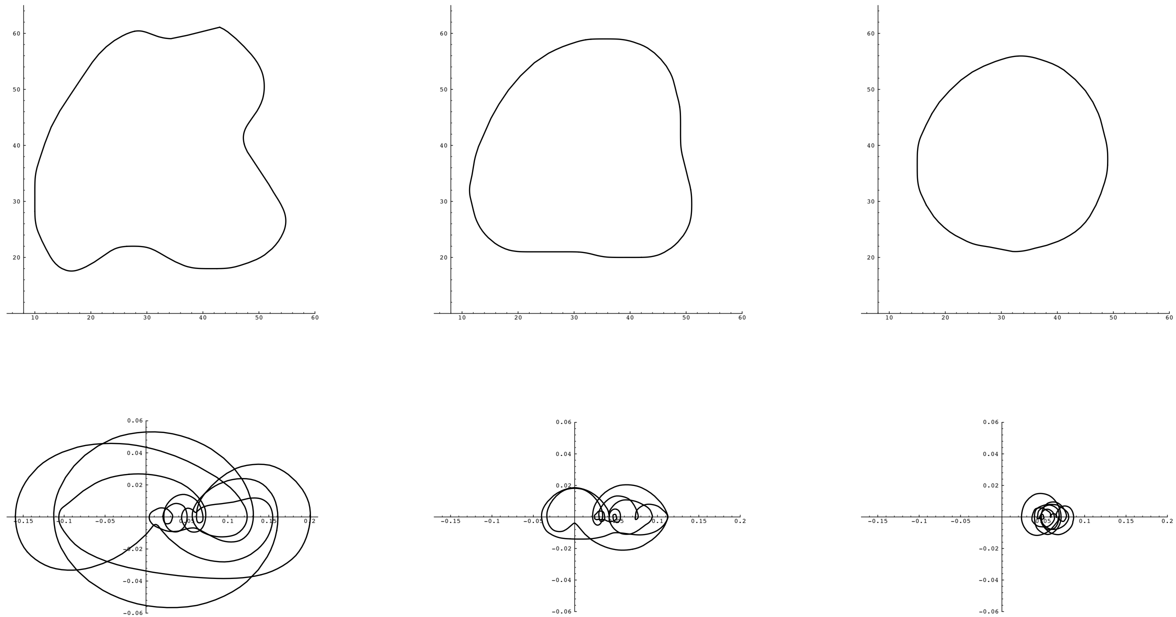
Canine Left Ventricle Signature

Original Canine Heart
MRI Image



Boundary of Left Ventricle

Smoothed Ventricle Signature



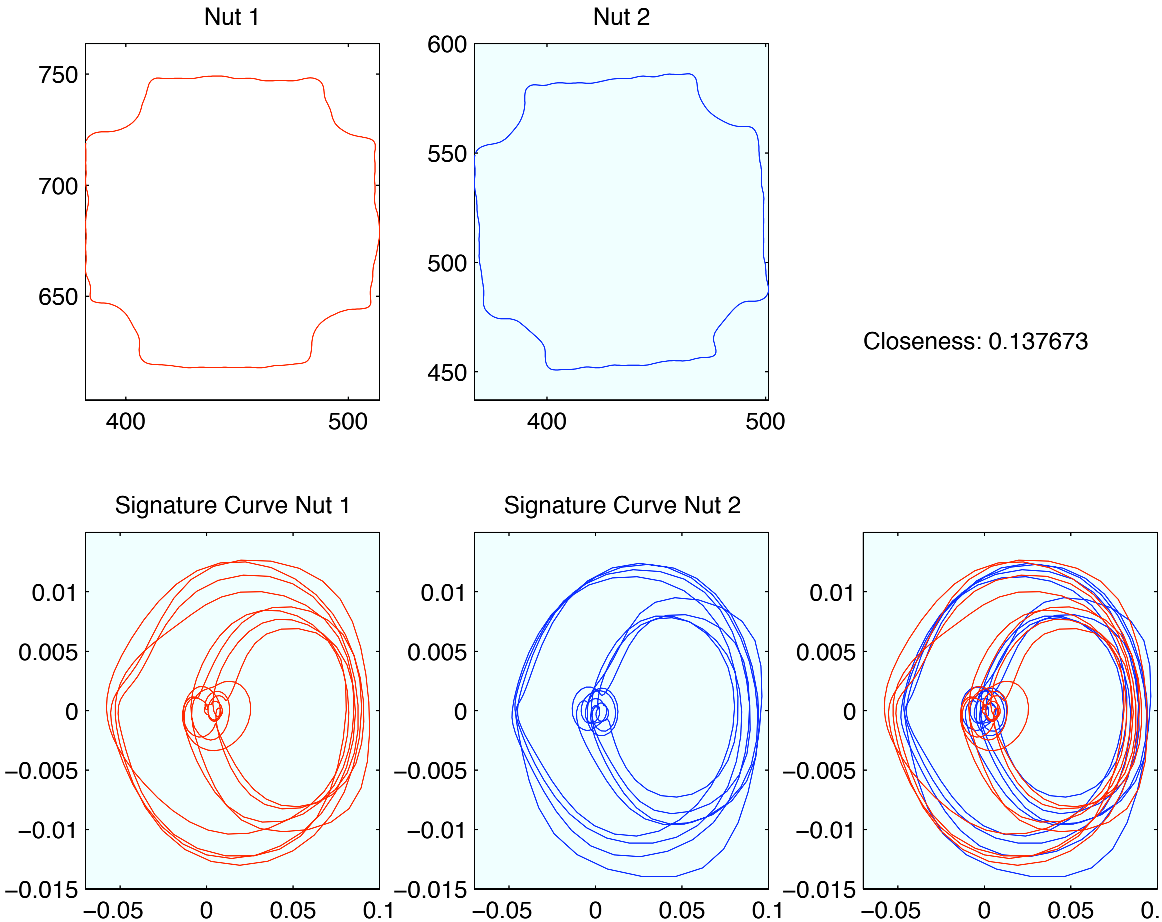
Evolution of Invariants and Signatures

Basic question: If the submanifold evolves according to an invariant evolution equation, how do its differential invariants & signatures evolve?

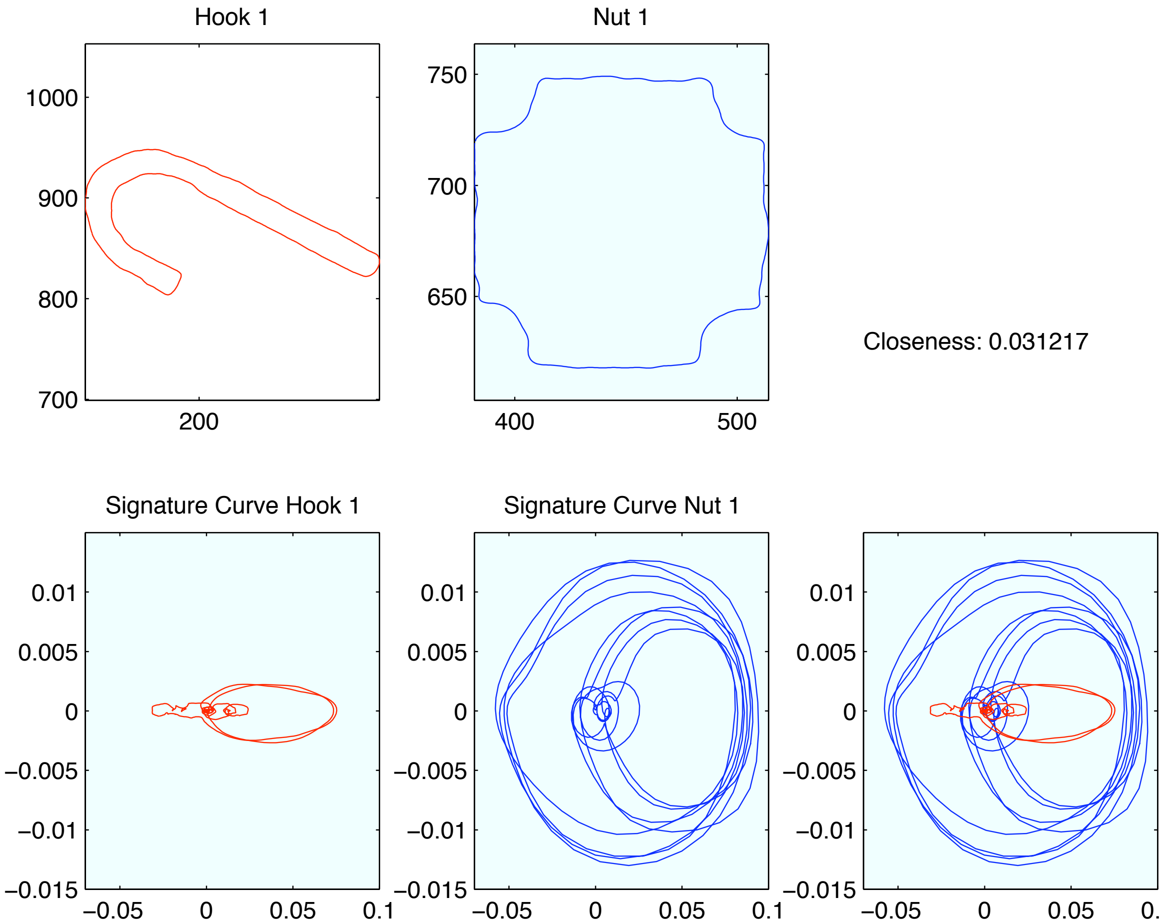
Theorem. Under the curve shortening flow $C_t = -\kappa\,\mathbf{n}$, the signature curve $\kappa_s = H(t,\kappa)$ evolves according to the parabolic equation

$$\frac{\partial H}{\partial t} = H^2\, H_{\kappa\kappa} - \kappa^3 H_\kappa + 4\,\kappa^2 H$$

$\Longrightarrow$ Steve Haker



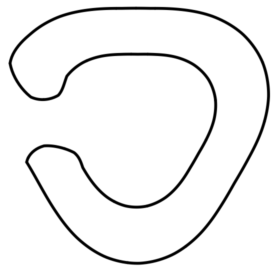
Nut 1
Nut 2
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2


Hook 1
Nut 1
1000
900
800
700
200
750
700
650
400
500
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

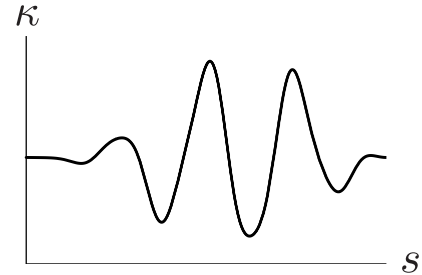
Signature Metrics

- Hausdorff
- Monge–Kantorovich transport
- Electrostatic repulsion
- Latent semantic analysis
- Histograms
- Gromov–Hausdorff & Gromov–Wasserstein

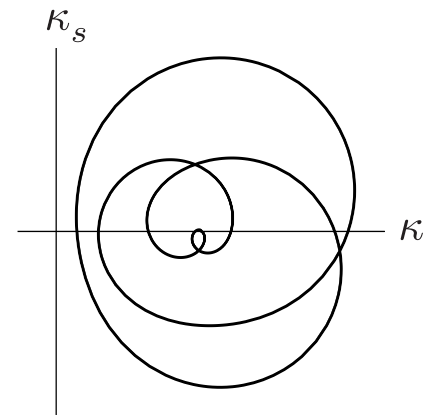
Signatures



Original curve

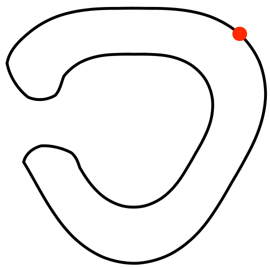


Classical Signature

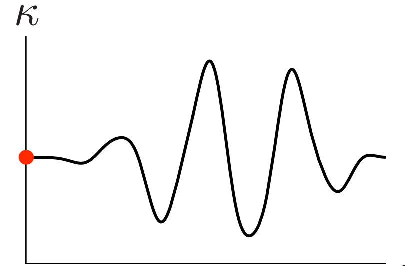


Differential invariant signature

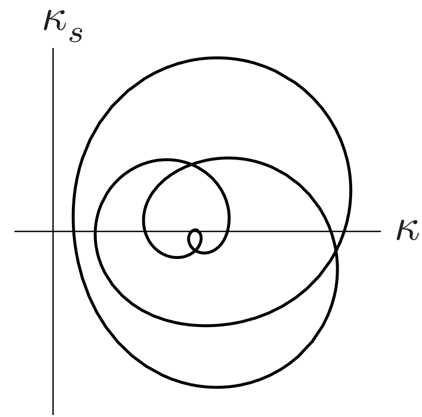
Signatures



Original curve

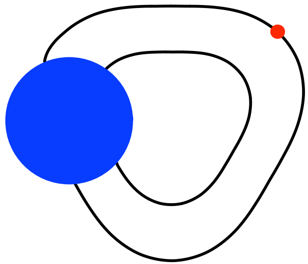


Classical Signature

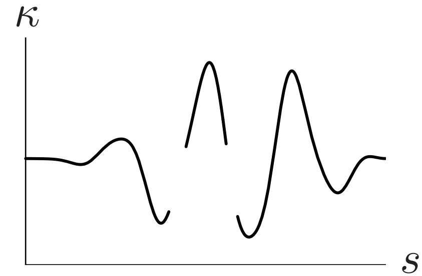


Differential invariant signature

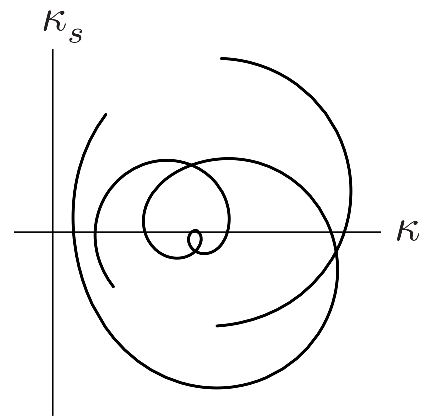
Occlusions



Original curve

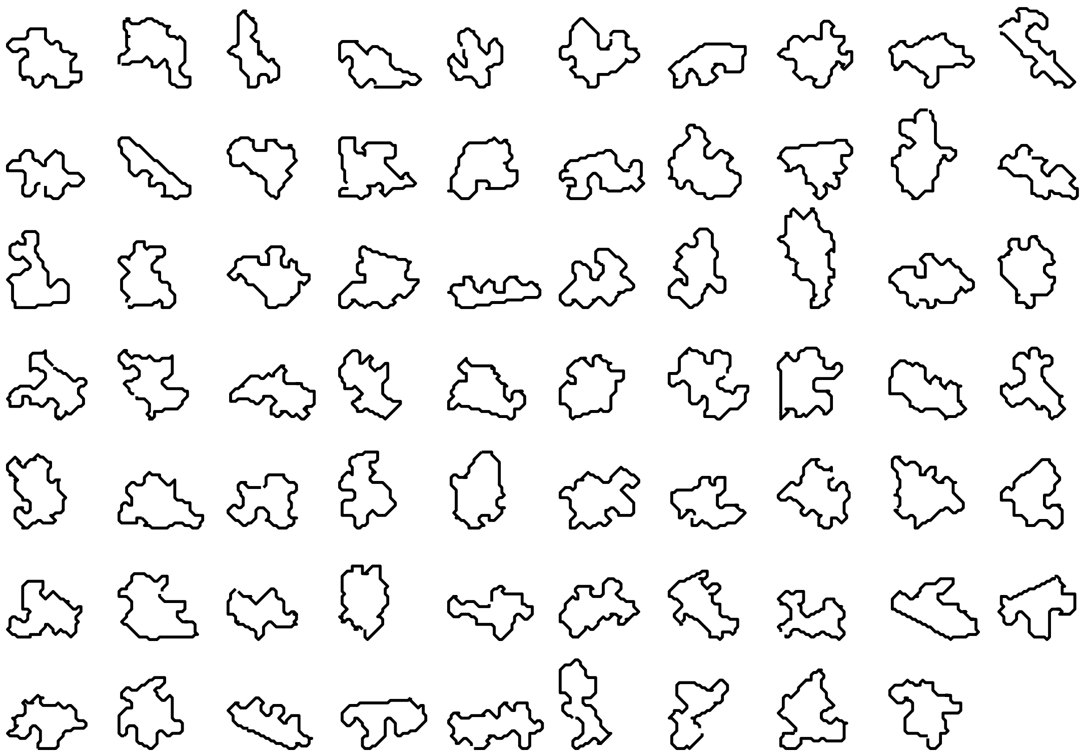


Classical Signature

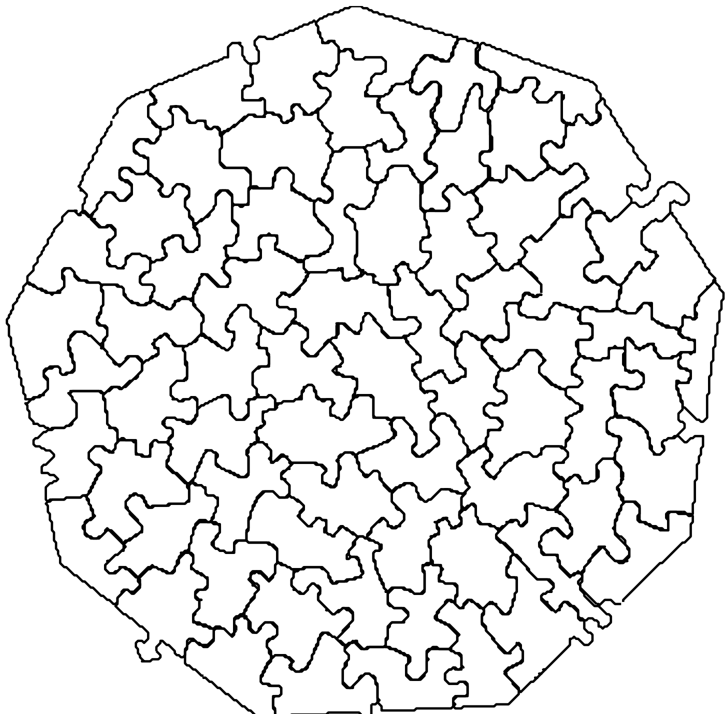


Differential invariant signature

The Baffler Jigsaw Puzzle



The Baffler Solved



$\Longrightarrow$ Dan Hoff

Classical Invariant Theory

$$M = \mathbb{R}^2 \setminus \{ u = 0 \}$$

$$G = \mathrm{GL}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \,\middle|\, \Delta = \alpha\,\delta - \beta\,\gamma \neq 0 \right\}$$

$$(x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \; \frac{u}{(\gamma x + \delta)^n} \right) \qquad n \neq 0, 1$$

Prolongation:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} \qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u \qquad \Delta = \alpha\,\delta - \beta\,\gamma$$

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\, \sigma^{n-1}}$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2(n-1)\gamma\,\sigma\, u_x + n(n-1)\gamma^2\, u}{\Delta^2\, \sigma^{n-2}}$$

$$v_{yyy} = \cdots$$

Normalization:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} = 0 \qquad\qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u = 1 \qquad\qquad \Delta = \alpha\,\delta - \beta\,\gamma$$

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\ \sigma^{n-1}} = 0$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2(n-1)\,\gamma\,\sigma\, u_x + n(n-1)\gamma^2\, u}{\Delta^2\ \sigma^{n-2}} = \frac{1}{n(n-1)}$$

$$v_{yyy} = \cdots$$

Moving frame:

$$\alpha = u^{(1-n)/n}\sqrt{H} \qquad \beta = -x\,u^{(1-n)/n}\sqrt{H}$$

$$\gamma = \tfrac{1}{n}\,u^{(1-n)/n} \qquad \delta = u^{1/n} - \tfrac{1}{n}\,xu^{(1-n)/n}$$

Hessian:

$$H = n(n-1)u\,u_{xx} - (n-1)^2 u_x^2 \neq 0$$

Note: $H \equiv 0$ if and only if $Q(x) = (a\,x + b)^n$

$\Longrightarrow$ Totally singular forms

Differential invariants:

$$v_{yyy} \longmapsto \frac{J}{n^2(n-1)} \approx \kappa \qquad v_{yyyy} \longmapsto \frac{K + 3(n-2)}{n^3(n-1)} \approx \frac{d\kappa}{ds}$$

Absolute rational covariants:

$$J^2 = \frac{T^2}{H^3} \qquad K = \frac{U}{H^2}$$

$$H = \tfrac{1}{2}(Q,\, Q)^{(2)} \;= n(n-1)QQ'' - (n-1)^2 Q'^2 \;\sim Q_{xx}Q_{yy} - Q_{xy}^2$$

$$T = \;(Q,\, H)^{(1)} \;= (2n-4)Q'H - nQH' \;\sim Q_x H_y - Q_y H_x$$

$$U = \;(Q,\, T)^{(1)} \;= (3n-6)Q'T - nQT' \;\sim Q_x T_y - Q_y T_x$$

$$\deg Q = n \qquad \deg H = 2n-4 \qquad \deg T = 3n-6 \qquad \deg U = 4n-8$$

Signatures of Binary Forms

Signature curve of a nonsingular binary form $Q(x)$:

$$\mathcal{S}_Q = \left\{ (J(x)^2, K(x)) = \left(\frac{T(x)^2}{H(x)^3}, \frac{U(x)}{H(x)^2} \right) \right\}$$

Nonsingular: $\quad H(x) \neq 0$ and $(J'(x), K'(x)) \neq 0$.

Theorem. Two nonsingular binary forms are equivalent if and only if their signature curves are identical.

Maximally Symmetric Binary Forms

Theorem. If $u = Q(x)$ is a polynomial, then the following are equivalent:

- $Q(x)$ admits a one-parameter symmetry group
- T^2 is a constant multiple of H^3
- $Q(x) \simeq x^k$ is complex-equivalent to a monomial
- the signature curve degenerates to a single point
- all the (absolute) differential invariants of Q are constant
- the graph of Q coincides with the orbit of a one-parameter subgroup

Symmetries of Binary Forms

Theorem. The symmetry group of a nonzero binary form $Q(x) \not\equiv 0$ of degree n is:

- A two-parameter group if and only if $H \equiv 0$ if and only if Q is equivalent to a constant. $\Longrightarrow$ totally singular
- A one-parameter group if and only if $H \not\equiv 0$ and $T^2 = c\,H^3$ if and only if Q is complex-equivalent to a monomial x^k, with $k \neq 0, n$. $\Longrightarrow$ maximally symmetric
- In all other cases, a finite group whose cardinality equals the index of the signature curve, and is bounded by

$$\iota_Q \quad \leq \quad \begin{cases} 6n - 12 & \qquad U = cH^2 \\ 4n - 8 & \qquad \text{otherwise} \end{cases}$$

Noise Reduction

Strategy #1:

Use lower order invariants to construct a signature:

- joint invariants
- joint differential invariants
- integral invariants
- topological invariants
- . . .

Joint Invariants

A joint invariant is an invariant of the k-fold Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) \;=\; I(z_1, \ldots, z_k)}$$

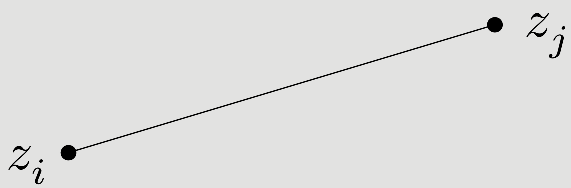
A joint differential invariant or semi-differential invariant is an invariant depending on the derivatives at several points $z_1, \ldots, z_k \in N$ on the submanifold:

$$\boxed{I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) \;=\; I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

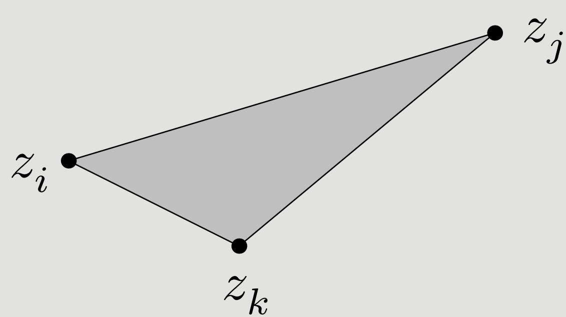
$$d(z_i, z_j) = \| \, z_i - z_j \, \|$$



Joint Equi–Affine Invariants

Theorem. Every planar joint equi–affine invariant is a function of the triangular areas

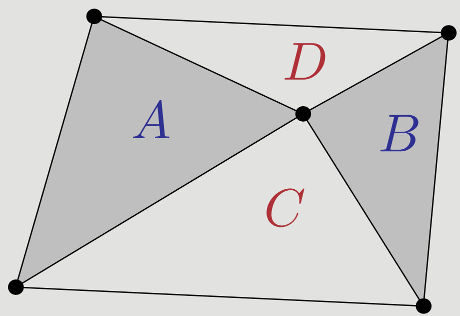
$$[\, i \ j \ k \,] = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$



Joint Projective Invariants

Theorem. Every joint projective invariant is a function of the planar cross-ratios

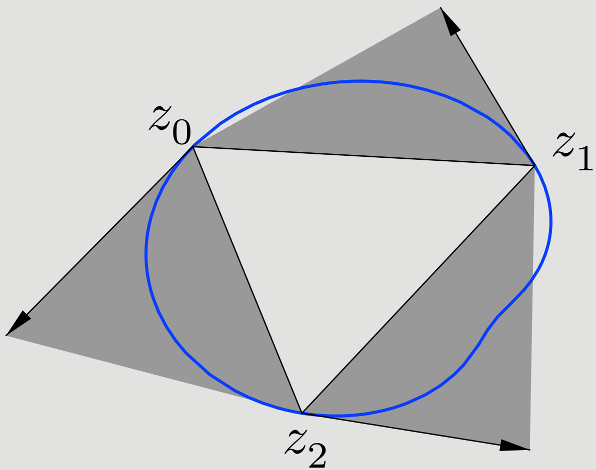
$$[\, z_i, z_j, z_k, z_l, z_m \,] \;=\; \frac{A\, B}{C\, D}$$

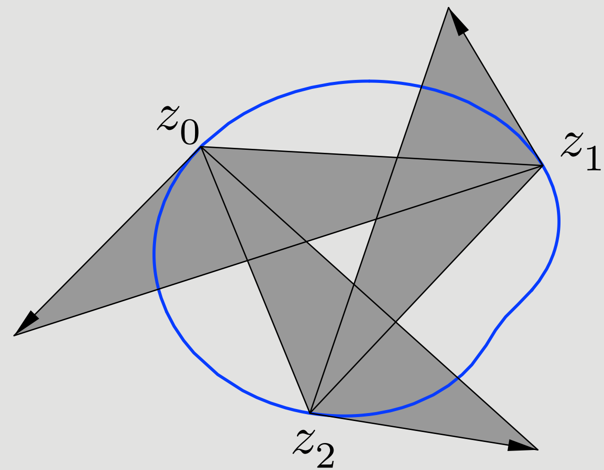


- Three–point projective joint differential invariant

  — tangent triangle ratio:

$$\frac{[\,0\;2\;\dot{0}\,]\,[\,0\;1\;\dot{1}\,]\,[\,1\;2\;\dot{2}\,]}{[\,0\;1\;\dot{0}\,]\,[\,1\;2\;\dot{1}\,]\,[\,0\;2\;\dot{2}\,]}$$





Joint Invariant Signatures

If the invariants depend on k points on a p-dimensional submanifold, then you need at least

$$\ell > k\,p$$

distinct invariants $I_1, \ldots, I_\ell$ in order to construct a syzygy. Typically, the number of joint invariants is

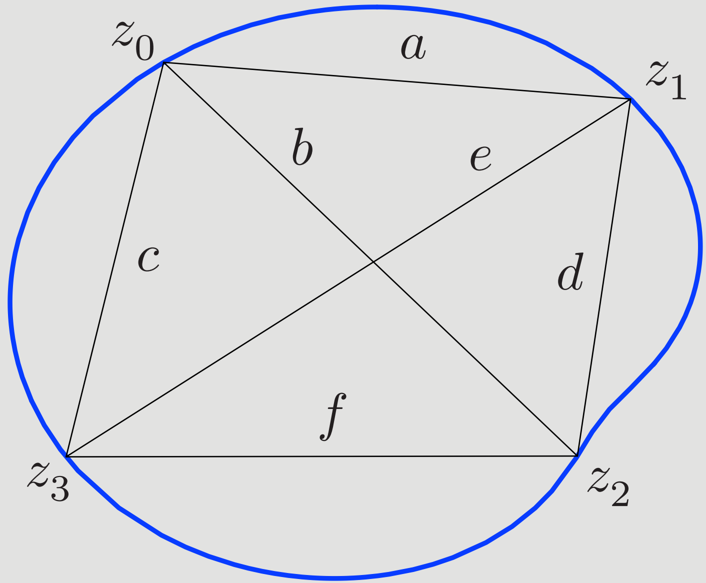
$$\ell = k\,m - r = (\#\text{points})\,(\dim M) - \dim G$$

Therefore, a purely joint invariant signature requires at least

$$k \;\geq\; \frac{r}{m-p} + 1$$

points on our p-dimensional submanifold $N \subset M$.

Joint Euclidean Signature



Joint signature map:

$$\Sigma : \mathcal{C}^{\times 4} \longrightarrow \mathcal{S} \subset \mathbb{R}^6$$

$$a = \| z_0 - z_1 \| \qquad b = \| z_0 - z_2 \| \qquad c = \| z_0 - z_3 \|$$

$$d = \| z_1 - z_2 \| \qquad e = \| z_1 - z_3 \| \qquad f = \| z_2 - z_3 \|$$

$$\Longrightarrow \quad \text{six functions of four variables}$$

Syzygies:

$$\Phi_1(a, b, c, d, e, f) = 0 \qquad \Phi_2(a, b, c, d, e, f) = 0$$

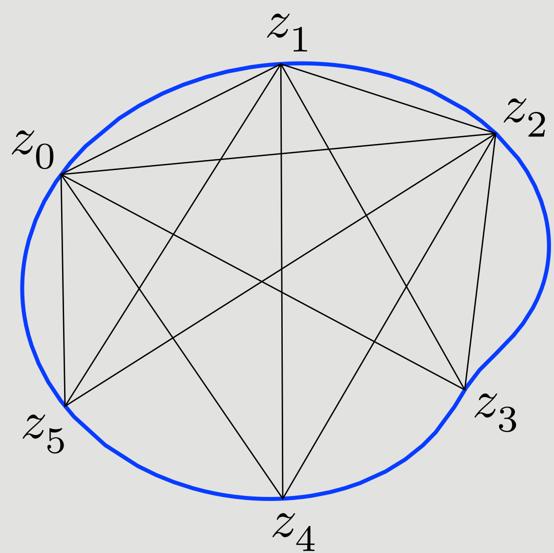
Universal Cayley–Menger syzygy $\quad \Longleftrightarrow \quad \mathcal{C} \subset \mathbb{R}^2$

$$\det \begin{vmatrix} 2\,a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\ a^2 + b^2 - d^2 & 2\,b^2 & b^2 + c^2 - f^2 \\ a^2 + c^2 - e^2 & b^2 + c^2 - f^2 & 2\,c^2 \end{vmatrix} = 0$$

Joint Equi–Affine Signature

Requires 7 triangular areas:

$$[\,0\ 1\ 2\,],\ [\,0\ 1\ 3\,],\ [\,0\ 1\ 4\,],\ [\,0\ 1\ 5\,],\ [\,0\ 2\ 3\,],\ [\,0\ 2\ 4\,],\ [\,0\ 2\ 5\,]$$



Joint Invariant Signatures

- The joint invariant signature subsumes other signatures, but resides in a higher dimensional space and contains a lot of redundant information.
- Identification of landmarks can significantly reduce the redundancies (Boutin)
- It includes the differential invariant signature and semi-differential invariant signatures as its "coalescent boundaries".
- Invariant numerical approximations to differential invariants and semi-differential invariants are constructed (using moving frames) near these coalescent boundaries.

Statistical Sampling

Idea: Replace high dimensional joint invariant signatures by increasingly dense point clouds obtained by multiply sampling the original submanifold.

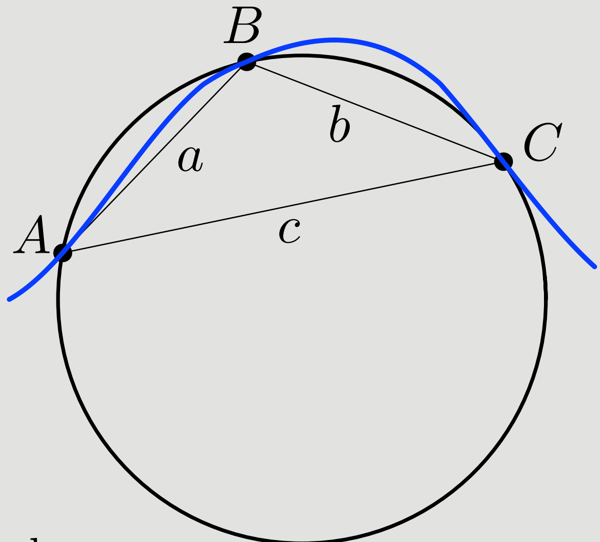
- The equivalence problem requires direct comparison of signature point clouds.
- Continuous symmetry detection relies on determining the underlying dimension of the signature point clouds.
- Discrete symmetry detection relies on determining densities of the signature point clouds.

Symmetry–Preserving Numerical Methods

- Invariant numerical approximations to differential invariants.
- Invariantization of numerical integration methods.

$\Longrightarrow$ Structure-preserving algorithms

Numerical approximation to curvature



Heron's formula

$$\widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

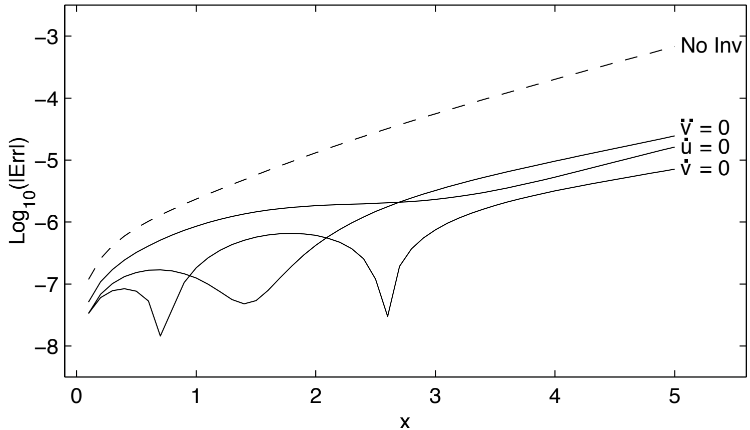
$$s = \frac{a+b+c}{2} \qquad \text{—} \qquad \text{semi-perimeter}$$

Invariantization of Numerical Schemes

$\Longrightarrow$ Pilwon Kim

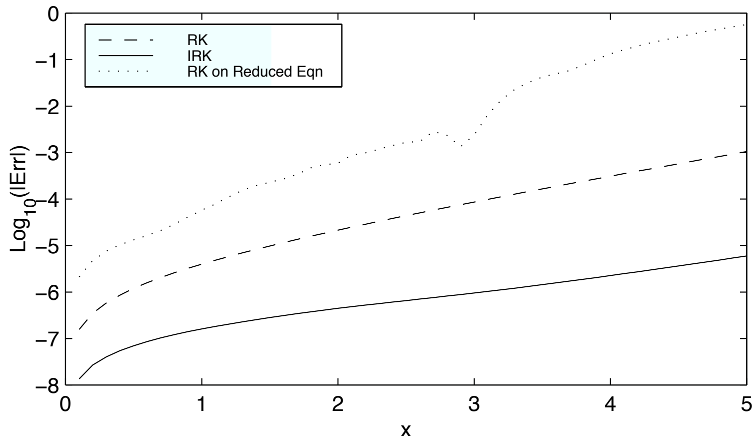
Suppose we are given a numerical scheme for integrating a differential equation, e.g., a Runge–Kutta Method for ordinary differential equations, or the Crank–Nicolson method for parabolic partial differential equations.

If G is a symmetry group of the differential equation, then one can use an appropriately chosen moving frame to invariantize the numerical scheme, leading to an invariant numerical scheme that preserves the symmetry group. In challenging regimes, the resulting invariantized numerical scheme can, with an inspired choice of moving frame, perform significantly better than its progenitor.



Invariant Runge–Kutta schemes

$$u_{xx} + x\,u_x - (x+1)u = \sin x, \qquad u(0) = u_x(0) = 1.$$

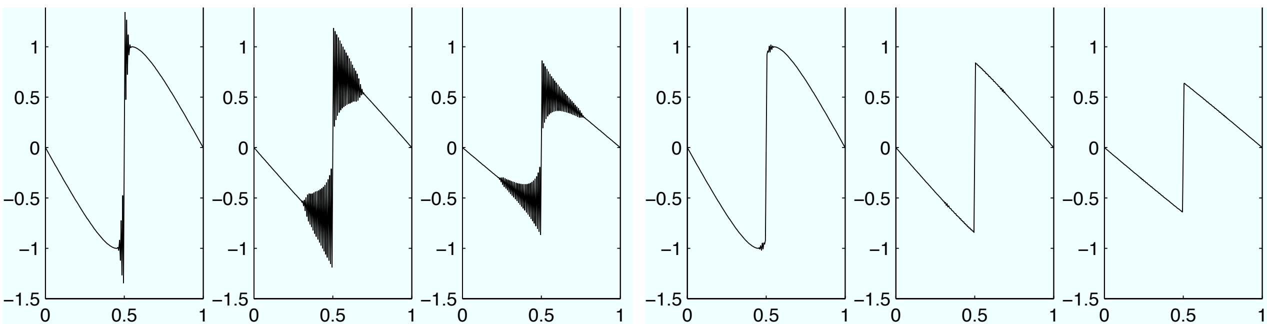


Comparison of symmetry reduction and invariantization for

$$u_{xx} + x\,u_x - (x+1)u = \sin x, \qquad u(0) = u_x(0) = 1.$$

Invariantization of Crank–Nicolson for Burgers' Equation

$$u_t = \varepsilon\, u_{xx} + u\, u_x$$



The Calculus of Variations

$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x}$ — variational problem

$L(x, u^{(n)})$ — Lagrangian

To construct the Euler-Lagrange equations: $\mathbf{E}(L) = 0$

- Take the first variation:

$$\delta(L\, d\mathbf{x}) = \sum_{\alpha, J} \frac{\partial L}{\partial u_J^\alpha}\, \delta u_J^\alpha\, d\mathbf{x}$$

- Integrate by parts:

$$\delta(L\, d\mathbf{x}) = \sum_{\alpha, J} \frac{\partial L}{\partial u_J^\alpha}\, D_J(\delta u^\alpha)\, d\mathbf{x}$$

$$\equiv \sum_{\alpha, J} (-D)^J \frac{\partial L}{\partial u_J^\alpha}\, \delta u^\alpha\, d\mathbf{x} = \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\, \delta u^\alpha\, d\mathbf{x}$$

Invariant Variational Problems

According to Lie, any G–invariant variational problem can be written in terms of the differential invariants:

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

If the variational problem is G-invariant, so

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}$$

then its Euler–Lagrange equations admit G as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$\mathbf{E}(L) \simeq F(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ ) = 0$$

Main Problem:

Construct F directly from P.

(*P. Griffiths, I. Anderson*)

Planar Euclidean group $\quad G = \mathrm{SE}(2)$

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \qquad \text{— curvature (differential invariant)}$$

$$ds = \sqrt{1 + u_x^2}\, dx \qquad \text{— arc length}$$

$$\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}} \frac{d}{dx} \qquad \text{— arc length derivative}$$

Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) \simeq F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

Euclidean Curve Examples

Minimal curves (geodesics):

$$\mathcal{I}[u] = \int ds = \int \sqrt{1 + u_x^2}\; dx$$

$$\mathbf{E}(L) = -\,\kappa = 0$$

$\Longrightarrow$ straight lines

The Elastica (Euler):

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\, ds = \int \frac{u_{xx}^2\, dx}{(1 + u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

To construct the invariant Euler-Lagrange equations:

Take the first variation:

$$\delta(P\, ds) = \sum_j \frac{\partial P}{\partial \kappa_j}\, \delta\kappa_j\, ds + P\, \delta(ds)$$

Invariant variation of curvature:

$$\delta\kappa = \mathcal{A}_\kappa(\delta u) \qquad\qquad \mathcal{A}_\kappa = \mathcal{D}^2 + \kappa^2$$

Invariant variation of arc length:

$$\delta(ds) = \mathcal{B}(\delta u)\, ds \qquad\qquad \mathcal{B} = -\,\kappa$$

$\Longrightarrow$ moving frame recurrence formulae

Integrate by parts:

$$\begin{aligned}\delta(P\,ds) &\equiv [\,\mathcal{E}(P)\,\mathcal{A}(\delta u) - \mathcal{H}(P)\,\mathcal{B}(\delta u)\,]\,ds \\ &\equiv [\,\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P)\,]\,\delta u\,ds = \mathbf{E}(L)\,\delta u\,ds\end{aligned}$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j}\,(-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

Euclidean–invariant Euler-Lagrange formula

$$\mathbf{E}(L) = \mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = (\mathcal{D}^2 + \kappa^2)\,\mathcal{E}(P) + \kappa\,\mathcal{H}(P) = 0.$$

The Elastica:

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\,ds \qquad P = \tfrac{1}{2}\,\kappa^2$$

$$\mathcal{E}(P) = \kappa \qquad\qquad \mathcal{H}(P) = -\,P = -\,\tfrac{1}{2}\,\kappa^2$$

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\,\kappa + \kappa\,(\,-\,\tfrac{1}{2}\,\kappa^2\,) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

The shape of a Möbius strip

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The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping[3] and paper crumpling[4,5]. This could give new insight into energy localization phenomena in unstretchable sheets[6], which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures[7–9].

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher[10]. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped $NbSe_3$ crystals under certain growth conditions involving a large temperature gradient[7,8].

Figure 1 Photo of a paper Möbius strip of aspect ratio 2π. The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.

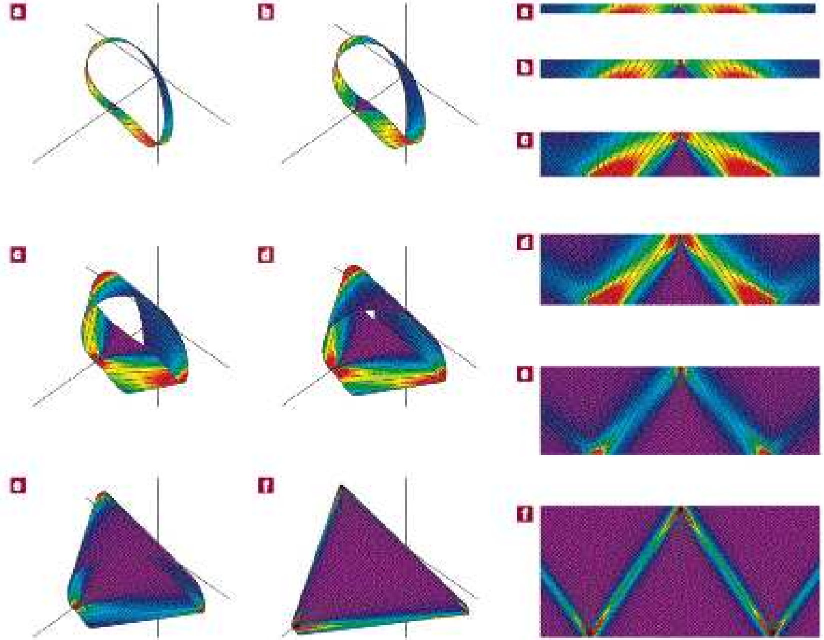


Figure 2 Computed Möbius strips. The left panel shows their three-dimensional shapes for $w = 0.1$ (**a**), 0.2 (**b**), 0.5 (**c**), 0.8 (**d**), 1.0 (**e**) and 1.5 (**f**), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from violet for regions of low bending to red for regions of high bending (scales are individually adjusted). Solution **c** may be compared with the paper model in Fig. 1 on which the generator field and density colouring have been printed.

Evolution of Invariants and Signatures

G — Lie group acting on $\mathbb{R}^2$

$C(t)$ — parametrized family of plane curves

G–invariant curve flow:

$$\frac{dC}{dt} = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- I, J — differential invariants
- $\mathbf{t}$ — "unit tangent"
- $\mathbf{n}$ — "unit normal"
- The tangential component $I\,\mathbf{t}$ only affects the underlying parametrization of the curve. Thus, we can set I to be anything we like without affecting the curve evolution.

Normal Curve Flows

$$C_t = J\,\mathbf{n}$$

Examples — Euclidean–invariant curve flows

- $C_t = \mathbf{n}$ — geometric optics or grassfire flow;
- $C_t = \kappa\,\mathbf{n}$ — curve shortening flow;
- $C_t = \kappa^{1/3}\,\mathbf{n}$ — equi-affine invariant curve shortening flow:
$$C_t = \mathbf{n}_{\text{equi-affine}}\,;$$
- $C_t = \kappa_s\,\mathbf{n}$ — modified Korteweg–deVries flow;
- $C_t = \kappa_{ss}\,\mathbf{n}$ — thermal grooving of metals.

Intrinsic Curve Flows

Theorem. The curve flow generated by

$$\mathbf{v} = I\,\mathbf{t} + J\,\mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D}\,I = 0.$$

$\mathcal{D}$ — invariant arc length derivative

$\mathcal{B}$ — invariant arc length variation

$$\delta(ds) = \mathcal{B}(\delta u)\,ds$$

Normal Evolution of Differential Invariants

Theorem. Under a normal flow $C_t = J\,\mathbf{n}$,

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J), \qquad \frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J).$$

Invariant variations:

$$\delta\kappa = \mathcal{A}_\kappa(\delta u), \qquad \delta\kappa_s = \mathcal{A}_{\kappa_s}(\delta u).$$

$\mathcal{A}_\kappa = \mathcal{A}$ — invariant variation of curvature;

$\mathcal{A}_{\kappa_s} = \mathcal{D}\,\mathcal{A} + \kappa\,\kappa_s$ — invariant variation of κ_s.

Euclidean–invariant Curve Evolution

Normal flow: $\quad C_t = J\,\mathbf{n}$

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J) = (\mathcal{D}^2 + \kappa^2)\,J,$$

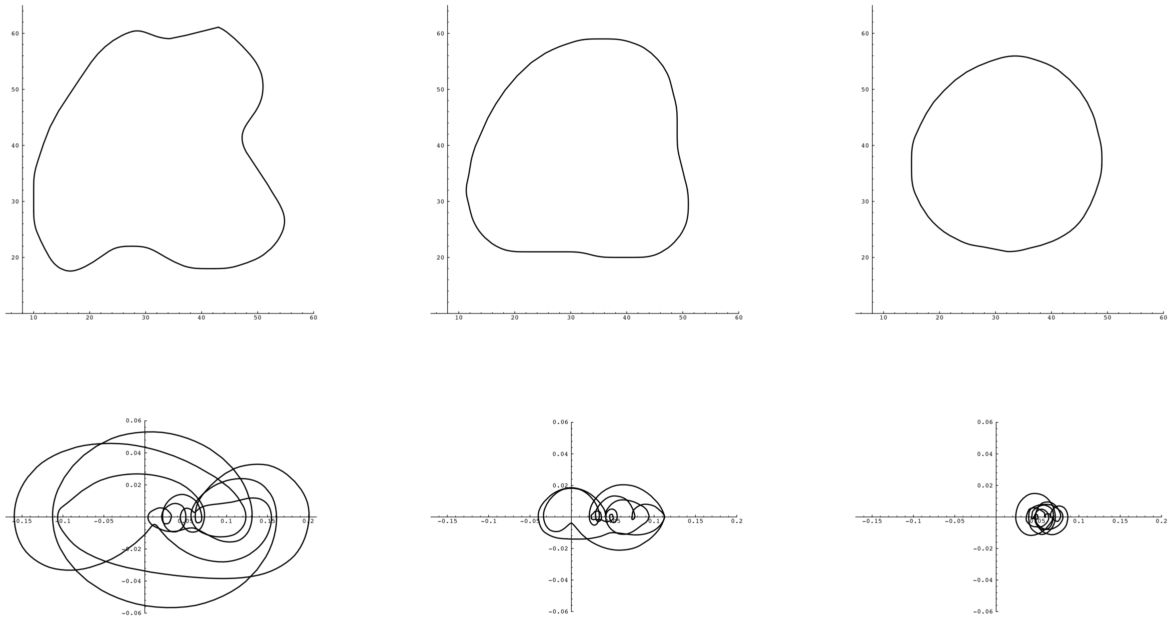
$$\frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J) = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\,J.$$

Warning: For non-intrinsic flows, ∂_t and ∂_s *do not commute*!

Theorem. Under the curve shortening flow $C_t = -\kappa\,\mathbf{n}$, the signature curve $\kappa_s = H(t,\kappa)$ evolves according to the parabolic equation

$$\frac{\partial H}{\partial t} = H^2\,H_{\kappa\kappa} - \kappa^3 H_\kappa + 4\,\kappa^2 H$$

Smoothed Ventricle Signature



Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \qquad \text{where} \qquad \mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} \qquad (*)$$

In surprisingly many situations, (*) is a well-known integrable evolution equation, and $\mathcal{R}$ is its recursion operator!

$\Longrightarrow$ Hasimoto

$\Longrightarrow$ Langer, Singer, Perline

$\Longrightarrow$ Marí–Beffa, Sanders, Wang

$\Longrightarrow$ Qu, Chou, Anco, and many more ...

Euclidean plane curves

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$\mathcal{A} = \mathcal{D}^2 + \kappa^2 \qquad\qquad \mathcal{B} = -\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\boxed{\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\kappa^2 \kappa_s}$$

$\Longrightarrow$ modified Korteweg-deVries equation

Equi-affine plane curves

$$G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$$

$$\mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2$$

$$\mathcal{B} = \tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s\mathcal{D}^{-1}\mathcal{B}$$

$$= \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{4}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2 + \tfrac{2}{9}\,\kappa_s\mathcal{D}^{-1}\cdot\kappa$$

$$\boxed{\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + \tfrac{5}{3}\,\kappa\,\kappa_{sss} + \tfrac{5}{3}\,\kappa_s\kappa_{ss} + \tfrac{5}{9}\,\kappa^2\kappa_s}$$

$\Longrightarrow$ Sawada–Kotera equation

Recursion operator: $\widehat{\mathcal{R}} = \mathcal{R}\cdot(\mathcal{D}^2 + \tfrac{1}{3}\,\kappa + \tfrac{1}{3}\,\kappa_s\mathcal{D}^{-1})$

Euclidean space curves

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \frac{2\tau}{\kappa} D_s^2 + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \frac{1}{\kappa} D_s^3 - \frac{\kappa_s}{\kappa^2} D_s^2 + \frac{\kappa^2 - \tau^2}{\kappa} D_s + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} \kappa & 0 \end{pmatrix}$$

$$\mathcal{R} = \mathcal{A} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B} \qquad \boxed{\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix}}$$

$\Longrightarrow$ vortex filament flow (Hasimoto)

The Recurrence Formula

For *any* function or differential form Ω:

$$\boxed{d\,\iota(\Omega) = \iota(d\,\Omega) + \sum_{k=1}^{r} \nu^k \wedge \iota\,[\mathbf{v}_k(\Omega)]}$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for $\mathfrak{g}$ — infinitesimal generators

$\nu^1, \ldots, \nu^r$ — dual invariantized Maurer–Cartan forms

★ ★ The ν^k are uniquely determined by the recurrence formulae for the phantom differential invariants

$$\boxed{\boxed{d\,\iota(\Omega) = \iota(d\Omega) + \sum_{k=1}^{r} \nu^k \wedge \iota\left[\mathbf{v}_k(\Omega)\right]}}$$

★ ★ ★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal recurrence formula by letting Ω range over the basic functions and differential forms!

$$\boxed{d\,\iota(\Omega) = \iota(d\Omega) + \sum_{k=1}^{r} \nu^k \wedge \iota\,[\mathbf{v}_k(\Omega)]}$$

★ ★ ★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal recurrence formula by letting Ω range over the basic functions and differential forms!

★ ★ ★ Therefore, the entire structure of the differential invariant algebra and invariant variational bicomplex can be completely determined using only linear differential algebra; this does not require explicit formulas for the moving frame, the differential invariants, the invariant differential forms, or the group transformations!

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}$ is generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim N$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ *Lie, Tresse, Ovsiannikov, Kumpera*

$\star$ Moving frames provides a constructive proof.

Minimal Generating Invariants

A set of differential invariants is a generating system if all other differential invariants can be written in terms of them and their invariant derivatives.

Euclidean curves $C \subset \mathbb{R}^3$:

- curvature κ and torsion τ

Equi–affine curves $C \subset \mathbb{R}^3$:

- affine curvature κ and torsion τ

Euclidean surfaces $S \subset \mathbb{R}^3$:

- mean curvature H

$\star$ Gauss curvature $K = \Phi(\mathcal{D}^{(4)} H)$.

Equi–affine surfaces $S \subset \mathbb{R}^3$:

- Pick invariant P.