Chapter 4

Symmetries of Differential Equations

In this chapter we discuss the foundations and some applications of Lie's theory of symmetry groups of differential equations. The basic infinitesimal method for calculating symmetry groups is presented, and used to determine the general symmetry group of some particular differential equations of interest. The reader interested in pursuing these matters in greater depth should consult the texts [**9**, **43**, **44**, **46**], for a wide variety of additional examples, theoretical developments, and applications to both physical and mathematical problems. Extensive tables of symmetry groups of particular differential equations can be found in the recent handbook, [**26**].

Symmetry Groups and Differential Equations

Consider a general n^{th} order system of differential equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, m, \tag{4.1}$$

in p independent variables $x = (x^1, \ldots, x^p)$, and q dependent variables $u = (u^1, \ldots, u^q)$, with $u^{(n)}$ denoting the derivatives of the u's with respect to the x's up to order n. The system of differential equations (4.1), which we often abbreviate as $\Delta = 0$, is thus defined by the vanishing of a collection of differential functions $\Delta_\nu\colon \mathrm{J}^n \to \mathbb{R}$ defined on the n^{th} jet space J^n. (For simplicity, we shall restrict our attention to systems defined by smooth functions, although extensions of these methods to more general systems are possible.) The system (4.1) can therefore be viewed as defining (or defined by) a variety

$$\mathcal{S}_\Delta = \left\{ (x, u^{(n)}) \;\middle|\; \Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \ldots, m \right\}, \tag{4.2}$$

contained in the n^{th} order jet space, consisting of all points $(x, u^{(n)}) \in \mathrm{J}^n$ satisfying the system. The defining functions Δ_ν are assumed to be regular, as per Definition 1.8, in a neighborhood of $\mathcal{S}_\Delta$; in particular, this is the case if the Jacobian matrix of the functions Δ_ν with respect to the jet variables $(x, u^{(n)})$ has maximal rank m everywhere on S_Δ. Theorem 1.19 implies that $\mathcal{S}_\Delta$ is a submanifold of J^n of dimension $p + q\binom{p+n}{n} - r$, where r is the rank of the system. Essentially all systems of differential equations arising in applications satisfy this condition (at least away from singularities). We also assume that the projection $\pi^n_X\colon \mathrm{J}^n \to X$ maps $\mathcal{S}_\Delta$ onto an open subset of the space of independent variables, since, otherwise, the system would include constraints on the independent variables, and therefore fall outside the traditional realm of differential equations.

A (smooth) function $u = f(x)$ will define a *solution* to the system of differential equations (4.1) if and only if its n^{th} prolongation $f^{(n)}(x)$ satisfies the system, i.e.,

$\Delta_\nu(x, f^{(n)}(x)) = 0$, $\nu = 1, \ldots, m$. This is equivalent to the requirement that the graph $\Gamma_f^{(n)} = \{(x, f^{(n)}(x))\}$ of the n^{th} prolongation of f be entirely contained in the variety defined by the system: $\Gamma_f^{(n)} \subset \mathcal{S}_\Delta$. The reader should unravel the relevant definitions so as to be convinced that this is merely a geometric reformulation of the classical notion of solution to a system of differential equations.

As with systems of algebraic equations, a *symmetry* of the system of differential equations (4.1) means a transformation which maps (smooth) solutions to solutions. The most basic type of symmetry is a group of point transformations on the space of independent and dependent variables, as discussed in Chapter 3, although, for either physical or mathematical reasons, one may wish to restrict attention to fiber-preserving transformations, since they do not mix up independent and dependent variables, or other more restrictive classes, e.g., volume-preserving, symplectic, etc. (Indeed, many systems arising in applications only have fiber-preserving symmetries.)

Definition 4.1. A point transformation $g: E \to E$ acting on the space $E \simeq X \times U$ of independent and dependent variables is called a *symmetry* of the system of partial differential equations (4.1) if, whenever $u = f(x)$ is a solution to (4.1), and the transformed function $\bar{f} = g \cdot f$ is well defined, then $\bar{u} = \bar{f}(\bar{x})$ is also a solution to the system (4.1).

Definition 4.1 formulates the notion of symmetry of a system of differential equations in terms of its action on the space of solutions. A key geometrical reformulation of this property is obtained by considering the action of the prolonged group transformation on the associated variety (4.2).

Proposition 4.2. *Consider a system of n^{th} order differential equations* (4.1) *with associated variety $\mathcal{S}_\Delta \subset \mathrm{J}^n$ as in* (4.2). *If the n^{th} prolongation $g^{(n)}: \mathrm{J}^n \to \mathrm{J}^n$ of a point transformation $g: E \to E$ leaves $\mathcal{S}_\Delta$ invariant, so that $g^{(n)}(\mathcal{S}_\Delta) \subset \mathcal{S}_\Delta$, then g is a symmetry of the system.*

Proof: The proof follows directly from the basic definitions. As remarked above, a function $u = f(x)$ is a solution to the system if and only if the graph $\Gamma_f^{(n)}$ of $f^{(n)}$ is a subset of $\mathcal{S}_\Delta$. Now, invariance of $\mathcal{S}_\Delta$ implies that $g^{(n)} \cdot \Gamma_f^{(n)}$, which is the graph $\Gamma_{g \cdot f}^{(n)}$ of the prolongation of the transformed function $g \cdot f$, is also contained in $\mathcal{S}_\Delta$, and hence is also a solution. *Q.E.D.*

The converse to Proposition 4.2 is, perhaps surprisingly, not necessarily true. Whereas for a system of algebraic equations, every point in the corresponding variety is (tautologically) a solution, this is no longer necessarily the case for systems of differential equations. Only those points in $\mathcal{S}_\Delta$ which correspond to actual solutions are required to be mapped to each other by the prolonged symmetry transformation. We formalize the required concept with the following definition.

Definition 4.3. A system of differential equations is called *locally solvable* at a point $(x_0, u_0^{(n)}) \in \mathcal{S}_\Delta$ if there exists a smooth solution $u = f(x)$, defined in a neighborhood of x_0, which achieves the values of the indicated derivatives there: $u_0^{(n)} = f^{(n)}(x_0)$. A system of differential equations which satisfies both the regularity and local solvability conditions at each of its points is called *fully regular*.

Most of the systems arising in practice are fully regular. For instance, local solvability of a system of ordinary differential equations is the same as the property of existence of solutions to the standard initial value problem. For a system of partial differential equations, the local solvability problem is of a different character than the more usual Cauchy or boundary value problems, since the initial data are only specified at a single point. Nevertheless, the Cauchy–Kovalevskaya Existence Theorem proves that any normal, analytic system of partial differential equations is locally solvable. Frobenius' Theorem 1.31 and the Cartan–Kähler Theorem, [**44**], imply local solvability of more general involutive systems of partial differential equations. The principal types of systems which do not satisfy the local solvability criterion are systems of differential equations with nontrivial integrability conditions, and certain smooth, non-analytic systems of partial differential equations, first discovered by H. Lewy, [**32**], which have no solutions. See [**43**; Chapter 2] for a more detailed discussion of regularity and local solvability.

Theorem 4.4. *A transformation g is a* symmetry *of a locally solvable system of differential equations* (4.1) *if and only if the corresponding variety $\mathcal{S}_\Delta$ is invariant under the prolonged transformation: $g^{(n)}(\mathcal{S}_\Delta) \subset \mathcal{S}_\Delta$.*

Proof: Given $z_0 = (x_0, u_0^{(n)}) \in \mathcal{S}_\Delta$, such that $\bar{z}_0 = g^{(n)} \cdot z_0$ is defined, we use the local solvability of $\mathcal{S}_\Delta$ to choose a solution $u = f(x)$ defined near x_0 with $u_0^{(n)} = f^{(n)}(x_0)$. Then $\bar{f} = g \cdot f$ is also a solution (at least near the image point $(\bar{x}_0, \bar{u}_0) = g \cdot (x_0, u_0)$), and hence $\bar{z}_0 = (\bar{x}_0, \bar{f}^{(n)}(\bar{x}_0)) \in \mathcal{S}_\Delta$, as desired. *Q.E.D.*

Infinitesimal Methods

We will henceforth assume that we are dealing with a connected group of point (or contact) transformations G. In the case of point transformations, the infinitesimal generators form a Lie algebra $\mathfrak{g}$ consisting of vector fields

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u) \frac{\partial}{\partial u^\alpha}, \tag{4.3}$$

on the space of independent and dependent variables. Let $\mathbf{v}^{(n)}$ denote the n^{th} prolongation of the vector field to the jet space J^n — the explicit formula appears in Theorem 3.14. Applying our basic infinitesimal symmetry criterion Theorem 2.71 and the geometric reformulation of symmetry in Theorem 4.4, we deduce the fundamental infinitesimal symmetry criterion for a system of differential equations.

Theorem 4.5. *A connected group of transformations G is a* symmetry group *of the fully regular system of differential equations $\Delta = 0$ if and only if the classical* infinitesimal symmetry conditions

$$\mathbf{v}^{(n)}(\Delta_\nu) = 0, \qquad \nu = 1, \ldots, r, \qquad \textit{whenever} \qquad \Delta = 0, \tag{4.4}$$

hold for every infinitesimal generator $\mathbf{v} \in \mathfrak{g}$ of G.

The conditions (4.4) are known as the *determining equations* of the symmetry group for the system; note in particular that they are only required to hold on points $(x, u^{(n)}) \in \mathcal{S}_\Delta$ satisfying the equations. If we substitute the explicit formulas (3.20, 21) for the coefficients of the prolongation $\mathbf{v}^{(n)}$, we find that the determining equations form a large, over-determined, linear system of partial differential equations for the coefficients ξ^i, φ^α of $\mathbf{v}$. In almost every example of interest, the determining equations are sufficiently elementary that they can be explicitly solved, and thereby one can determine the complete (connected) symmetry group of the system (4.1). However, the required calculations can become very lengthy and time consuming. Fortunately, there are now a wide variety of computer algebra packages available which will automate all the routine steps in the calculation of the symmetry group of a given system of partial differential equations; see [**22**] for surveys of the different packages available, including a discussion of their strengths and weaknesses.

One simplification, which is especially important in the case of contact symmetries, is to use the characteristic form (3.32) of the prolongation formula. Since any (smooth) solution to an equation $\Delta_\nu(x, u^{(n)}) = 0$ also satisfies all the equations $D_i \Delta_\nu = 0$, $i = 1, \ldots, p$, obtained by applying the total derivative operators (3.18), the infinitesimal symmetry conditions (4.4) have the alternative reformulation

$$\mathbf{v}_Q^{(n)}(\Delta_\nu) = 0, \qquad \nu = 1, \ldots, r, \qquad \text{on solutions to } \Delta = 0. \tag{4.5}$$

Note that in (4.5), the system *and* its derivatives must be taken into account.

Remark: The symmetry criterion (4.5) can be generalized to higher order "generalized symmetries", in which one allows the characteristic $Q = Q(x, u^{(k)})$ to depend on derivatives of a specified (finite) order. Although such higher order infinitesimal symmetries do *not* have a geometric, group-theoretic counterpart, they do play an essential role in the study of integrable soliton equations; we refer the interested reader to [**43**] for a complete development.

We now illustrate the practical use of the infinitesimal symmetry criterion (4.4) for determining the full (connected) symmetry group of several concrete differential equations of interest.

Example 4.6. We begin with an elementary example illustrating the basic techniques. Consider the second order ordinary differential equation

$$u_{xx} = 0. \tag{4.6}$$

(The reader may try to guess in advance the symmetry group of this equation based on the fact that its general solution is an affine function $u = ax + b$; thus every symmetry of (4.6) will map straight lines to straight lines.) An infinitesimal point symmetry of equation (4.6) will be a vector field of the general form $\mathbf{v} = \xi(x, u)\, \partial_x + \varphi(x, u)\, \partial_u$ on $E = \mathbb{R}^2$; our task is to determine which particular coefficient functions ξ, φ will produce infinitesimal symmetries. In order to apply Theorem 4.5, we must compute the second prolongation of $\mathbf{v}$, which is the vector field

$$\mathbf{v}^{(2)} = \xi(x, u)\, \frac{\partial}{\partial x} + \varphi(x, u)\, \frac{\partial}{\partial u} + \varphi^x(x, u^{(1)})\, \frac{\partial}{\partial u_x} + \varphi^{xx}(x, u^{(2)})\, \frac{\partial}{\partial u_{xx}}\,, \tag{4.7}$$

on J^2 whose coefficients φ^x, φ^{xx} can be found in (3.24). The infinitesimal symmetry criterion (4.4) in this case is simply

$$\varphi^{xx} = 0 \qquad \text{whenever} \qquad u_{xx} = 0. \tag{4.8}$$

Substituting the formula (3.24) for φ^{xx} into (4.8), and setting the second order derivative u_{xx} to zero, results in the explicit symmetry condition

$$\varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x + (\varphi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 = 0. \tag{4.9}$$

Since we can prescribe x, u, and u_x arbitrarily, and ξ and φ only depend on x, u, equation (4.9) will be satisfied if and only if the individual coefficients of the powers of u_x vanish. The resulting overdetermined system of partial differential equations

$$\varphi_{xx} = 0, \qquad 2\varphi_{xu} = \xi_{xx}, \qquad \varphi_{uu} = 2\xi_{xu}, \qquad \xi_{uu} = 0, \tag{4.10}$$

forms the complete set of *determining equations* for the symmetry group of the original ordinary differential equation (4.6). The general solution to the determining equations (4.10) is readily found:

$$\begin{aligned} \xi(x,u) &= c_1 x^2 + c_2 xu + c_3 x + c_4 u + c_5, \\ \varphi(x,u) &= c_1 xu + c_2 u^2 + c_6 x + c_7 u + c_8, \end{aligned}$$

where the c_i are arbitrary constants. Therefore, equation (4.6) admits an eight-dimensional Lie algebra of infinitesimal symmetries, spanned by the vector fields

$$\partial_x, \quad \partial_u, \quad x\partial_x, \quad x\partial_u, \quad u\partial_x, \quad u\partial_u, \quad x^2\partial_x + xu\partial_u, \quad xu\partial_x + u^2\partial_u. \tag{4.11}$$

Note that (4.11) appear as the Lie algebra in Case 2.3 of our classification tables. The corresponding Lie group is the projective group SL(3), acting via linear fractional transformations

$$(x,u) \longmapsto \left(\frac{ax + bu + c}{hx + ju + k}, \frac{dx + eu + f}{hx + ju + k} \right), \qquad \det \begin{vmatrix} a & b & c \\ d & e & f \\ h & j & k \end{vmatrix} \neq 0, \tag{4.12}$$

which forms the most general point symmetry of the elementary ordinary differential equation (4.6).

Exercise 4.7. Show that the second order ordinary differential equation $u_{xx} = [(x + x^2)e^u]_x$ has no (continuous) symmetries. Find the general solution to this equation.

Example 4.8. A classic example illustrating the basic techniques for computing symmetry groups of partial differential equations is the linear heat equation

$$u_t = u_{xx}. \tag{4.13}$$

An infinitesimal point symmetry of the heat equation will be given by a vector field of the form $\mathbf{v} = \xi\,\partial_x + \tau\,\partial_t + \varphi\,\partial_u$, whose coefficients ξ, τ, φ are functions of x, t, u. The infinitesimal symmetry criterion (4.4) is

$$\varphi^t = \varphi^{xx} \qquad \text{whenever} \qquad u_t = u_{xx}. \tag{4.14}$$

Here, φ^t and φ^{xx} are the coefficients of the terms $\partial_{u_t}, \partial_{u_{xx}}$ in the second prolongation of $\mathbf{v}$, cf. (3.26). Substituting the formulas (3.27) into (4.14), and replacing u_t by u_{xx} wherever it occurs, we are left with a polynomial equation involving the various derivatives of u whose coefficients are certain derivatives of ξ, τ, φ. Since ξ, τ, φ only depend on x, t, u we can equate the individual coefficients to zero, leading to the complete set of *determining equations*:

Coefficient	Monomial
$0 = -2\tau_u$	$u_x u_{xt}$
$0 = -2\tau_x$	u_{xt}
$0 = -\tau_{uu}$	$u_x^2 u_{xx}$
$-\xi_u = -2\tau_{xu} - 3\xi_u$	$u_x u_{xx}$
$\varphi_u - \tau_t = -\tau_{xx} + \varphi_u - 2\xi_x$	u_{xx}
$0 = -\xi_{uu}$	u_x^3
$0 = \varphi_{uu} - 2\xi_{xu}$	u_x^2
$-\xi_t = 2\varphi_{xu} - \xi_{xx}$	u_x
$\varphi_t = \varphi_{xx}$	1

The general solution to these elementary differential equations is

$$\xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt, \qquad \tau = c_2 + 2c_4 t + 4c_6 t^2,$$
$$\varphi = (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \alpha(x,t),$$

where c_i are arbitrary constants and $\alpha_t = \alpha_{xx}$ is an arbitrary solution to the heat equation. Therefore, the symmetry algebra of the heat equation is spanned by the vector fields

$$\begin{aligned} &\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = u\partial_u, \qquad \mathbf{v}_4 = x\partial_x + 2t\partial_t, \\ &\mathbf{v}_5 = 2t\partial_x - xu\partial_u, \qquad \mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u, \\ &\mathbf{v}_\alpha = \alpha(x,t)\partial_u, \qquad \text{where} \qquad \alpha_t = \alpha_{xx}. \end{aligned} \tag{4.15}$$

The corresponding one-parameter groups are, respectively, x and t translations, scaling in u, the combined scaling $(x,t) \longmapsto (\lambda x, \lambda^2 t)$, Galilean boosts, an “inversional symmetry”, and the addition of solutions stemming from the linearity of the equation. Each of these groups has the property of mapping solutions of the heat equation to other solutions. For example, the inversional group has the explicit form

$$(x,t,u) \longmapsto \left(\frac{x}{1-4\varepsilon t}, \frac{t}{1-4\varepsilon t}, u\sqrt{1-4\varepsilon t}\ \exp\left(\frac{-x\varepsilon x^2}{1-4\varepsilon t}\right)\right),$$

where ε is the group parameter. Note that this only defines a local transformation group. Computing the induced action on graphs of functions, we conclude that if $u = f(x,t)$ is any solution to the heat equation, so is

$$u = \frac{1}{1+4\varepsilon t}\ \exp\left(\frac{-\varepsilon x^2}{1+4\varepsilon t}\right) f\left(\frac{x}{1+4\varepsilon t}, \frac{t}{1+4\varepsilon t}\right),$$

a fact that can, of course, be checked directly. In particular, starting with the trivial constant solution $u = c$, we produce the non-trivial solution

$$u(t,x) = \frac{c}{\sqrt{1+4\varepsilon t}} \ \exp\left(-\varepsilon \frac{x^2}{1+4\varepsilon t} \right),$$

which is a multiple of the fundamental solution at the point $(0, -(4\varepsilon)^{-1})$. The reader should construct the actions of the other one-parameter groups and their combinations, and then verify that they all do take solutions to solutions. See [**43**] for more details.

Example 4.9. A similar computation can be used to determine the symmetry group of the nonlinear diffusion equation

$$u_t = u_{xx} + u_x^2, \tag{4.16}$$

known as the potential form of Burgers' equation. Infinitesimal symmetries have the same form as in the heat equation example, and the symmetry criterion (4.4) is

$$\varphi^t = \varphi^{xx} + 2\,u_x\,\varphi^x, \tag{4.17}$$

which must hold whenever (4.16) is satisfied. Again, we substitute the formulas (3.27) into (4.17), and replace u_t by $u_{xx} + u_x^2$. The general solution to the resulting determining equations is

$$\xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt, \qquad \tau = c_2 + 2c_4 t + 4c_6 t^2,$$
$$\varphi = c_3 - c_5 x - 2c_6 t - c_6 x^2 + \alpha(x,t)e^{-u},$$

where c_i are arbitrary constants and $\alpha_t = \alpha_{xx}$ is an arbitrary solution to the *heat equation*: $\alpha_t = \alpha_{xx}$. Therefore, the symmetry algebra of Burgers' equation is spanned by the vector fields

$$\begin{aligned} &\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = \partial_u, \qquad \mathbf{v}_4 = x\partial_x + 2t\partial_t, \\ &\mathbf{v}_5 = 2t\partial_x - x\partial_u, \qquad \mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2+2t)\partial_u, \\ &\mathbf{v}_\alpha = \alpha(x,t)e^{-u}\partial_u, \qquad \text{where} \qquad \alpha_t = \alpha_{xx}. \end{aligned} \tag{4.18}$$

The symmetry algebra (4.18) is isomorphic to that of the heat equation (4.15); indeed, the transformation $u \mapsto e^u$ maps the Burgers' symmetry generators to those of the heat equation, and, in fact, linearizes the potential Burgers' equation by mapping it to the heat equation. We have thus, based on a symmetry analysis, rediscovered the well-known Hopf–Cole transformation — see [**55**] for applications.

The preceding observation is, in fact, a particular case of a general linearization theorem for a system of partial differential equations — see Theorem 4.29 below. It also serves to illustrate the following simple general fact.

Proposition 4.10. *Two equivalent differential equations have isomorphic* symmetry groups.

Indeed, if Φ is a transformation mapping one differential equation to another and g is a symmetry of the first equation, then $\bar{g} = \Phi \circ g \circ \Phi^{-1}$ is clearly a symmetry of the second equation. Of course, to interpret Proposition 4.10, one must be careful that the symmetries are included in the class of diffeomorphisms (i.e., fiber-preserving transformations, point transformations, contact transformations, etc.) which define equivalences. If we make a change of variables outside the class, then there is no guarantee that the symmetry groups will have anything to do with each other. For example, consider the more usual form

$$v_t = v_{xx} + 2 v v_x \tag{4.19}$$

of Burgers' equation, which is obtained by differentiating (4.16) and setting $v = u_x$, so that u plays the role of a potential function for v. The symmetry group of (4.19) is only five-dimensional; nevertheless we can map it to the linear heat equation $w_t = w_{xx}$, which has an infinite dimensional symmetry group, by the *nonlocal* Hopf–Cole map $v = (\log w)_x$. Thus, the symmetry group of a physical equation and that of the equation for the potential can be quite different, because local transformations in one case become nonlocal in the other (and vice versa). See [**9**] for the theory of *potential symmetries*, and [**36**, **43**] for an approach to such more general linearization questions based on higher order symmetries.

Example 4.11. The Boussinesq equation

$$u_{tt} + u u_{xx} + u_x^2 + u_{xxxx} = 0 \tag{4.20}$$

is a well-known soliton equation, and arises as a model equation for the unidirectional propagation of solitary waves in shallow water, [**40**]. The basic symmetry condition (4.4) now takes the form

$$\varphi^{tt} + u \varphi^{xx} + u_{xx} \varphi + 2 u_x \varphi^x + \varphi^{xxxx} = 0,$$

which must hold whenever (4.20) is satisfied. Here $\varphi^{tt}, \varphi^x, \varphi^{xx}, \varphi^{xxxx}$ are the coefficients of the vector fields $\partial_{u_{tt}}, \partial_{u_x}, \partial_{u_{xx}}, \partial_{u_{xxxx}}$ in the fourth prolongation of $\mathbf{v}$, with formulae similar to (3.27). A straightforward (but quite lengthy) calculation eventually yields the complete symmetry algebra, which is spanned by

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = x \partial_x + 2t \partial_t - 2u \partial_u. \tag{4.21}$$

In this example, the classical symmetry group is disappointingly trivial, consisting of easily guessed translations and scaling symmetries. Theorem 4.5 guarantees that these are the only continuous classical symmetries of the equation. There are, however, higher order generalized symmetries, [**43**], which account for the infinity of conservation laws of this equation, and, thus, its remarkable integrability properties.

Thus, the complicated calculation of the symmetry group of a system of differential equations sometimes yields only rather trivial symmetries; however, there are numerous examples where this is not the case and new and physically and/or mathematically important symmetries have arisen from a complete group analysis. See [**26**, **43**, **46**] for examples and references.

Exercise 4.12. Determine the form of the symmetry group of a nonlinear diffusion equation of the form

$$u_t = D_x\big(h(u)u_x\big). \tag{4.22}$$

The answer will depend on the form of $h(u)$.

Exercise 4.13. Prove that the Korteweg–deVries equation

$$u_t = u_{xxx} + u u_x, \tag{4.23}$$

has a four-parameter symmetry group. Find the symmetry transformations explicitly.

Integration of Ordinary Differential Equations

Lie made the remarkable observation that virtually all the classical methods for solving specific types of ordinary differential equations (separable, homogeneous, exact, etc.) are particular examples of a general method for integrating ordinary differential equations that admit a group of symmetries. In particular, knowledge of a one-parameter group of symmetries of an ordinary differential equation allows us to reduce its order by one. Before beginning, though, we must remark that the method cannot be used to find every solution to the equation.

Definition 4.14. Let $\mathbf{v}$ be a vector field on the space of independent and dependent variables. A function $u = f(x)$ is called *nontangential* provided $\mathbf{v}$ is nowhere tangent to the graph of f.

Theorem 4.15. *Let $\Delta(x, u^{(n)}) = 0$ be an n^{th} order scalar ordinary differential equation admitting a regular one-parameter symmetry group G. Then all nontangential solutions can be found by quadrature from the solutions to an ordinary differential equation $(\Delta/G)(x, u^{(n-1)}) = 0$ of order $n-1$, called the* symmetry reduced equation.

Remark: Note that a solution is everywhere tangential if and only if it is invariant under G, so the method will not, in particular, produce invariant solutions. However, in the scalar case, the graph of an invariant function must locally coincide with a one-dimensional orbit of the group, and hence the invariant solutions can be determined by inspection. See [**9**; §3.6] for applications to envelopes and separatrices.

Proof: Let us introduce rectifying coordinates $y = \chi(x,u)$, $v = \psi(x,u)$, in terms of which the infinitesimal generator of G is the vertical translation field ∂_v. If $\mathbf{v}$ is not tangent to the graph Γ of a solution $u = f(x)$, then, in terms of the new y, v coordinates, Γ remains transverse, and therefore will locally coincide with the graph of a smooth function $v = h(y)$. In the new coordinates, the group transformations, and their prolongations, are simply given by translation $v \mapsto v + \varepsilon$ in the v coordinate alone, the derivative coordinates remaining fixed. (The infinitesimal form is $\mathbf{v}^{(n)} = \partial_v$, $n \geq 0$.) Therefore the equation variety $\mathcal{S}_\Delta = \{\Delta(y, v^{(n)}) = 0\} \subset \mathrm{J}^n$ is invariant if and only if it does not depend on the variable v, and hence the equation is equivalent to one that does not depend explicitly on v itself (although it does depend on the derivatives of v with respect to y). Therefore, replacing v by $w = v_y$ reduces the equation to one of order $n-1$ for $w = w(y)$; moreover, we recover the solution to our original equation by quadrature: $v = \int w(y)\,dy$. *Q.E.D.*

In particular, a first order equation $u_x = P(x,u)$ admitting a one-parameter symmetry group can be solved by quadrature. However, the symmetry must be nontangential: The trivial symmetries $\mathbf{v} = \xi(x,u)\big(\partial_x + P(x,u)\partial_u\big)$ are everywhere tangential to solutions; moreover, the characteristic method for finding the rectifying coordinates of such a vector field is the essentially same problem as integrating the equation itself, so the reduction method is of no help. Indeed, the problem of determining the most general symmetry group of a first order equation is more complicated than solving the equation itself, so we can only successfully apply Lie's method if, by inspection (perhaps motivated by geometry or physics), we can detect a relatively simple symmetry group.

Example 4.16. A classical example is provided by the homogeneous equation

$$\frac{du}{dx} = x^{m-1}\, F\left(\frac{u}{x^m}\right), \tag{4.24}$$

which admits the scaling group $(x,u) \mapsto (\lambda x, \lambda^m u)$ with infinitesimal generator $x\partial_x + mu\partial_u$. For $x \neq 0$, rectifying coordinates are given by $y = u/x^m$, $v = \log x$, in terms of which the equation reduces to

$$\frac{dv}{dy} = \frac{1}{F(y) - my}.$$

This can clearly be integrated, $v = h(y) = \int dy/\{F(y) - my\}$, thereby defining u implicitly: $\log x = h(u/x^m)$. (The reader might enjoy comparing this method with the one taught in elementary ordinary differential equation texts for solving homogeneous equations.) In this case, the nontangentiality condition requires that $du/dx \neq mu/x$, meaning that $F(y) \neq my$, and this method (and the standard one) break down at such singularities. In particular, the scale-invariant function $u = cx^m$, which will be a solution provided $F(c) = mc$, cannot be recovered by this approach, and constitutes a singular solution to the equation.

Example 4.17. Consider the second order ordinary differential equation

$$u u_{xx} = \alpha u_x^2, \tag{4.25}$$

where α is a nonzero constant. The equation admits three obvious symmetries: a translation $x \mapsto x + \varepsilon$ in the independent variable, reflecting the fact that the equation is autonomous, and two independent scaling transformations $(x,u) \mapsto (\lambda x, \mu u)$. To reduce with respect to the translation group, we set $y = u$, $v = x$, so that $u_x = 1/v_y$, $u_{xx} = -v_{yy}/v_y^3$, and the equation reduces to a linear equation $y v_{yy} + \alpha v_y = 0$, with solution $v = c y^{1-\alpha} + d$ for $\alpha \neq 1$, or $v = c \log y + d$ for $\alpha = 1$. The corresponding solution of (4.25) is then $u = (ax+b)^{1/(1-\alpha)}$, or $u = \exp(ax+b)$. (Note that the translationally invariant solutions $u =$ constant are recovered as limiting cases of these solutions.) Alternatively, if we use the scaling symmetry in u, then the appropriate coordinates are x and $v = \log u$, in terms of which the equation becomes $v_{yy} = (\alpha - 1)v_y^2$, which reduces to a homogeneous (separable) equation for $w = v_y$. The reader may enjoy seeing what happens if we reduce with respect to the other scaling symmetry $x \mapsto \lambda x$ instead.

Example 4.18. Finally, consider a general homogeneous second order linear equation

$$u_{xx} + f(x)u_x + g(x)u = 0. \tag{4.26}$$

This clearly admits the scaling symmetry generated by $u\partial_u$. According to the general reduction procedure, as long as $u \neq 0$, we can introduce the new variable $v = \log u$, in terms of which the equation becomes $v_{xx} + v_x^2 + f(x)v_x + g(x) = 0$, which is a first order Riccati equation for $w = v_x = u_x/u$. We have thus recovered the well-known correspondence between second order linear equations and first order Riccati equations.

If a higher order equation admits several symmetries, then it may be reducible in order more than once. However, unless the symmetry group has additional structure, we may not be able to make a full reduction since the reduced equations may not inherit the full symmetry group of the original equation. (On the other hand, they may admit additional symmetries not shared by the original system!) See [**9**, **43**] for full details of the reduction techniques available.

Theorem 4.19. *Suppose $\Delta = 0$ is an n^{th} order ordinary differential equation admitting a symmetry group G. Let $H \subset G$ be a one-parameter subgroup. Then the H-reduced equation $\Delta/H = 0$ admits the quotient group G_H/H, where $G_H = \{\, g \mid g \cdot H \cdot g^{-1} \subset H \,\}$ is the* normalizer subgroup, *as a symmetry group.*

Proof: Let $y = \chi(x,u)$, $v = \psi(x,u)$ be the rectifying coordinates for the infinitesimal generator $\mathbf{v} = \partial_v$ of the subgroup H. According to Theorem 4.15, the original differential equation reduces to an $(n-1)^{\text{st}}$ order equation for $w = v_y = \omega(x,u,u_x)$. Consider an infinitesimal symmetry $\mathbf{w} \in \mathfrak{g}$ of the original equation, which we re-express in terms of the rectifying coordinates: $\mathbf{w} = \eta(y,v)\partial_y + \zeta(y,v)\partial_v$. Clearly, the vector field $\mathbf{w}$ will induce a point symmetry of the reduced equation if and only if its first prolongation $\mathbf{w}^{(1)} = \eta\partial_y + \zeta\partial_v + \zeta^y\partial_{v_y}$ can be reduced to a local vector field $\widehat{\mathbf{w}} = \eta(y,w)\partial_y + \theta(y,w)\partial_w$ depending on just y and w. This will happen if and only if η and ζ^y are independent of v. According to the prolongation formula (3.24), this occurs if and only if $\zeta = \alpha(y) + cv$ for c constant. Since $\mathbf{v} = \partial_v$, this condition is equivalent to the requirement that $[\mathbf{v},\mathbf{w}] = c\mathbf{v}$. Exercise 2.55 demonstrates that $\mathbf{w} \in \mathfrak{g}_H$, the subalgebra of $\mathfrak{g}$ corresponding to the normalizer subgroup G_H, thereby completing the proof. *Q.E.D.*

Example 4.20. Consider a second order equation of the form

$$x^2 u_{xx} = F(xu_x - u), \tag{4.27}$$

which admits the two-parameter symmetry group $(x,u) \mapsto (\lambda x, u + \mu x)$ with infinitesimal generators $\mathbf{v} = x\partial_x$, $\mathbf{w} = x\partial_u$. Since $[\mathbf{v},\mathbf{w}] = \mathbf{w}$, if we reduce with respect to $\mathbf{w}$, then the resulting first order equation will retain a symmetry corresponding to $\mathbf{v}$, and hence Theorem 4.19 guarantees that it can be integrated. In this case, we set $v = u/x$, $w = v_x = x^{-2}(xu_x - u)$, so that (4.27) reduces to $x^3 w_x = F(x^2 w) + 2x^2 w$. This equation admits a scaling symmetry generated by the reduced vector field $\widehat{\mathbf{v}} = x\partial_x - 2w\partial_w$, which means that it is of homogeneous form (4.24) and can be integrated as in Example 4.16. On the other hand, if we try to reduce the original equation with respect to $\mathbf{v}$ initially, using the variables $\tilde{y} = u$, $\tilde{v} = \log x$, equation (4.27) reduces to a first order equation

$w_y = -w\left[1 + F(w^{-1} - y)\right]$ having no obvious symmetry. It is worth remarking that the latter equation *can* be solved — just reverse the procedure so as to replace it by the original second order equation, and use the first reduction method.

An r-dimensional Lie group G is called *solvable* if there exists a sequence of subgroups $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_{r-1} \subset G_r = G$ such that each G_i is a normal subgroup of G_{i+1}. This is equivalent to the requirement that the corresponding subalgebras of $\mathfrak{g}$ satisfy $[\mathfrak{g}_i, \mathfrak{g}_{i+1}] \subset \mathfrak{g}_i$. A Theorem of Bianchi, [**6**], states that if an ordinary differential equation admits a (sufficiently regular) r-dimensional solvable symmetry group, then its solutions can be determined, by quadrature, from those to a reduced equation of order $n - r$; see also [**43**; Theorem 2.64].

Finally, it is worth reiterating the fact that not every integration method for ordinary differential equations is based on symmetry. Indeed, the equation appearing in Exercise 4.7 provides a simple example of an equation with no symmetries, but which can, nevertheless, be explicitly solved.

Characterization of Invariant Differential Equations

One of the most important uses of differential invariants is in the construction of general systems of differential equations (and variational problems) which admit a prescribed symmetry group. This is especially important in modern physical theories, where one begins by postulating the basic "symmetry group" of the theory, and then determines which field equations are admissible. The basic result that allows us to immediately write down the most general system of differential equations which is invariant under a prescribed transformation group is a direct corollary of Theorem 2.34 characterizing invariant systems of algebraic equations. Note that this result is valid for both ordinary and partial differential equations.

Theorem 4.21. *Let G be a Lie group acting on E. Assume that the n^{th} prolongation of G acts regularly and has a complete set of functionally independent n^{th} order differential invariants $I_1, \ldots, I_k$ on an open subset $W^{(n)} \subset \mathrm{J}^n$. A system of n^{th} order differential equations admits G as a symmetry group if and only if, on $W^{(n)}$, it can be rewritten in terms of the* differential invariants*:*

$$\Delta_\nu(x, u^{(n)}) = F_\nu(I_1(x, u^{(n)}), \ldots, I_k(x, u^{(n)})) = 0, \qquad \nu = 1, \ldots, l. \tag{4.28}$$

Example 4.22. Suppose we have just one independent variable and one dependent variable, and consider the usual rotation group SO(2) acting on $E = \mathbb{R} \times \mathbb{R}$. According to Definition 3.3, there are two first order differential invariants — the radius $r = \sqrt{x^2 + u^2}$ and the angular invariant $w = (xu_x - u)/(x + uu_x)$. Therefore, the most general first order ordinary differential equation admitting SO(2) as a symmetry group can be written in the form $F(r, w) = 0$. Solving for w, we deduce that the equation has the explicit form

$$\frac{xu_x - u}{x + uu_x} = H\left(\sqrt{x^2 + u^2}\,\right) \qquad \text{or} \qquad u_x = \frac{u + xH(\sqrt{x^2 + u^2})}{x - uH(\sqrt{x^2 + u^2})}. \tag{4.29}$$

In terms of the polar coordinates r, θ, equation (4.29) takes the separable form $\theta_r = r^{-1}H(r)$ and can thus be integrated.

The most general second order differential equation admitting a rotational symmetry group can be written in the form $F(r, w, \kappa) = 0$, where $\kappa = (1+u_x^2)^{-3/2} u_{xx}$ is the curvature. Solving for u_{xx}, we find

$$u_{xx} = (1 + u_x^2)^{3/2} H\left(\sqrt{x^2 + u^2},\ \frac{x u_x - u}{x + u u_x}\right). \tag{4.30}$$

This second order equation can also be rewritten in terms of polar coordinates:

$$r\theta_{rr} = (1 + r^2\theta_r^2) H(r, r\theta_r) - r^2\theta_r^3 - 2\theta_r.$$

Since the latter equation does not explicitly depend on θ, it can be reduced to a first order equation by introducing the variable $v = d\theta/dr$.

Example 4.23. Let $E = \mathbb{R}^2 \times \mathbb{R}$, and consider the action of the rotation group $\mathrm{SO}(2)$ acting on the independent variables x, y only. Every first order $\mathrm{SO}(2)$-invariant partial differential equation has the form $H(x u_x + y u_y, y u_x - x u_y, u, r) = 0$. Similarly, every second order $\mathrm{SO}(2)$-invariant partial differential equation can be written in terms of the second (and lower) order differential invariants

$$\begin{aligned} U &= x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}, \\ V &= -xy u_{xx} + (x^2 - y^2) u_{xy} + xy u_{yy}, \\ W &= y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy}. \end{aligned} \tag{4.31}$$

In particular, the rotational invariance of the Helmholtz equation follows from the identity

$$\Delta u + \lambda u = u_{xx} + u_{yy} + \lambda u = r^{-2}(U + W) + \lambda u.$$

The construction of Theorem 4.21 suggests an alternative reduction method for ordinary differential equations invariant under a symmetry group. Under the assumptions of Theorems 3.36 and 4.21, we can rewrite any n^{th} order ordinary differential equation admitting an r parameter symmetry group in the form

$$F\left(y, w, \frac{dw}{dy}, \ldots, \frac{d^{n-r} w}{dy^{n-r}}\right) = 0 \tag{4.32}$$

involving only the two fundamental differential invariants: $y = I(x, u^{(s)})$, $w = J(x, u^{(r)})$, which have orders $s < r = \dim G$, with $s = r - 1$ unless G is either intransitive, in which case $s = 0$, or pseudo-stabilizes, in which case $s = r - 2$. Therefore, we have reduced the original n^{th} order differential equation to an $(n - r)^{\text{th}}$ order differential equation in the differential invariants. However, once we have solved (4.32) for $w = h(y)$, we then must solve an auxiliary r^{th} order differential equation

$$J(x, u^{(r)}) = h\left[\, I(x, u^{(s)}) \,\right], \tag{4.33}$$

in order to recover u as a function of x. (If (4.32) only involves the first differential invariant y, then (4.33) has the form $I(x, u^{(s)}) = c$.) The Lie reduction method discussed above can be applied to equation (4.33) although, unless the Lie group is solvable, we will not in general be able to integrate it by quadrature alone.

A particularly interesting class of applications is provided by the three inequivalent planar actions of the special linear group SL(2); these appear as Cases 3.3, 1.1, and 1.2 in the tables at the end of the book. As noted in Example 3.18, all three actions are related via a process of prolongation and projection; this provides a ready means of simultaneously classifying and reducing the corresponding invariant ordinary differential equations.

For the first unimodular group action, any n^{th} order ordinary differential equation admitting the symmetry group generated by ∂_x, $x\partial_x$, $x^2\partial_x$ can be written in the form

$$\frac{d^{n-3}s}{du^{n-3}} = H\left(u, s, \frac{ds}{du}, \ldots, \frac{d^{n-4}s}{du^{n-4}}\right), \tag{4.34}$$

where

$$u \qquad \text{and} \qquad s = \frac{u_x u_{xxx} - \frac{3}{2}u_{xx}^2}{u_x^4} \tag{4.35}$$

are the two fundamental differential invariants. Once we know the solution $s = F(u)$ to the reduced ordinary differential equation (4.34), we recover the solution to the original equation by solving the invariant third order equation $s = F(u)$, or, explicitly,

$$u_x u_{xxx} - \tfrac{3}{2}u_{xx}^2 = u_x^4 F(u). \tag{4.36}$$

We can reduce (4.36) to a first order differential equation by applying the Lie reduction method associated with the two-dimensional solvable subgroup generated by ∂_x and $x\partial_x$. The two consequent reductions can be combined by setting

$$z = \frac{u_{xx}}{u_x^2}, \qquad \text{in terms of which} \qquad s = \frac{dz}{du} + \tfrac{1}{2}z^2. \tag{4.37}$$

Equation (4.36) then reduces to the Riccati equation

$$\frac{dz}{du} + \tfrac{1}{2}z^2 = F(u). \tag{4.38}$$

Once we solve equation (4.38) for $z = z(u)$, we use (4.37) to recover the solution $u = f(x)$ to equation (4.36) by a pair of quadratures. The function $\psi(u) = \sqrt{u_x}$ is a solution to the second order, homogeneous, linear Schrödinger equation

$$\frac{d^2\psi}{du^2} - \tfrac{1}{2}F(u)\psi = 0. \tag{4.39}$$

We can recover $u(x)$ by a single quadrature:

$$\int^u \frac{d\hat{u}}{\psi(\hat{u})^2} = x + k. \tag{4.40}$$

Note that, according to the method of variation of parameters, cf. [**27**; p. 122], if $\psi(u)$ is one solution to the linear ordinary differential equation (4.39), then a second, linearly independent solution, is given by

$$\varphi(u) = \psi(u)\int^u \frac{d\hat{u}}{\psi(\hat{u})^2}. \tag{4.41}$$

Comparing with (4.40), and absorbing the integration constant, we conclude that the general solution to the invariant equation (4.36) is given, parametrically, as a ratio $x = \varphi(u)/\psi(u)$ of two arbitrary linearly independent solutions to the linear Schrödinger equation (4.39). The differential invariant s can be identified with the negative of the Schwarzian derivative of $x = x(u)$; therefore, our symmetry reduction of (4.36) provides a direct proof of a classical theorem due to Schwarz; see [**24**; Theorem 10.1.1].

Theorem 4.24. *The general solution to the* Schwarzian equation

$$\frac{x_u x_{uuu} - \frac{3}{2}x_{uu}^2}{x_u^2} = \widehat{F}(u) \tag{4.42}$$

has the form $x = \varphi(u)/\psi(u)$, where $\varphi(u)$ and $\psi(u)$ form two linearly independent, but otherwise arbitrary, solutions to the linear Schrödinger equation $\psi_{uu} + \frac{1}{2}\widehat{F}(u)\psi = 0$. Alternatively, $w = x_{uu}/x_u$ is an arbitrary solution to the Riccati equation $w_u = \frac{1}{2}w^2 + \widehat{F}(u)$.

Corollary 4.25. *The equation $x_u x_{uuu} = \frac{3}{2}x_{uu}^2$ has, as general solution, the* linear fractional functions $x = (\alpha u + \beta)/(\gamma u + \delta)$.

Consider the next unimodular group action, generated by the vector fields ∂_x, $x\partial_x - v\partial_v$, $x^2\partial_x - 2xv\partial_v$. According to Example 3.18, the group coincides with the first prolongation of the preceding action under the identification $v = u_x$. Any invariant n^{th} order ordinary differential equation has the form

$$\frac{d^{n-3}r}{ds^{n-3}} = H\left(s, r, \frac{dr}{ds}, \ldots, \frac{d^{n-4}r}{ds^{n-4}}\right), \tag{4.43}$$

where

$$s = \frac{vv_{xx} - \frac{3}{2}v_x^2}{v^4}, \qquad r = \frac{ds}{du} = \frac{v^2 v_{xxx} - 6vv_x v_{xx} + 6v_x^3}{v^6}, \tag{4.44}$$

are the two fundamental differential invariants. Once we know the solution $r = G(s)$ to the reduced ordinary differential equation (4.43), we recover the solution to the original ordinary differential equation by solving the invariant third order equation $r = G(s)$, or, explicitly,

$$v^2 v_{xxx} - 6vv_x v_{xx} + 6v_x^3 = v^6 G\left(\frac{vv_{xx} - \frac{3}{2}v_x^2}{v^4}\right). \tag{4.45}$$

Since $r = ds/du$, we can integrate equation (4.45) once to yield

$$H(s) = \int^s \frac{d\hat{s}}{G(\hat{s})} = u + k, \tag{4.46}$$

which, once solved for $s = F(u)$, reduces to the invariant third order equation (4.36), an equation we know how to solve in terms of a pair of quadratures and a Riccati equation. Alternatively, the solution based on the linear Schrödinger equation (4.39) can also be effectively employed. Note that $v = u_x = \psi(u)^2/\omega$, where $\omega = \varphi\psi_u - \varphi_u\psi$ denotes the Wronskian of the two solutions $\varphi(u)$ and $\psi(u)$ to (4.39), which, by Abel's formula, is constant. Therefore, the general solution to the invariant equation (4.45) can be written in parametric form as

$$x = \frac{\varphi(u)}{\psi(u)}, \qquad v = \frac{\psi(u)^2}{\omega}. \tag{4.47}$$

As for the third unimodular group action, the generators are ∂_x, $x\partial_x - w\partial_w$, $x^2\partial_x - (2xw+1)\partial_w$, where we identify $w = v_x/2v = u_{xx}/2u_x$. Any invariant n^{th} order ordinary differential equation has the form

$$\frac{d^{n-3}y}{dt^{n-3}} = H\left(t, y, \frac{dy}{dt}, \ldots, \frac{d^{n-4}y}{dt^{n-4}}\right), \tag{4.48}$$

where the fundamental differential invariants are

$$\begin{aligned} t &= \frac{r}{s^{3/2}} = \frac{w_{xx} - 6ww_x + 4w^3}{\sqrt{2}\,(w_x - w^2)^{3/2}}, \\ y &= \frac{2}{s^2}\,\frac{dr}{du} = 24 + \frac{w_{xxx} - 12ww_{xx} + 18w_x^2}{(w_x - w^2)^2}. \end{aligned} \tag{4.49}$$

Once we know the solution $y = 24 + K(t)$ to the reduced ordinary differential equation (4.48), we recover the solution to the original ordinary differential equation by solving the invariant third order equation

$$w_{xxx} - 12ww_{xx} + 18w_x^2 = (w_x - w^2)^2\, K\left(\frac{w_{xx} - 6ww_x + 4w^3}{\sqrt{2}(w_x - w^2)^{3/2}}\right). \tag{4.50}$$

Using (4.49), equation (4.50) takes the first order separable form

$$s\,\frac{dt}{ds} = \frac{K(t)}{t} - \frac{3t}{2}, \tag{4.51}$$

hence $t = M(s)$ can be found by quadrature. Then $r = s^{3/2}M(s) = G(s)$ has the form (4.45), and so can be solved as before. The general solution to the invariant equation (4.50) can thus be expressed in the parametric form

$$x = \frac{\varphi(u)}{\psi(u)}, \qquad w = \frac{\psi_x}{\psi} = \frac{\psi\psi_u}{\omega}, \tag{4.52}$$

where $\varphi(u)$ and $\psi(u)$ form two independent solutions to the second order linear equation (4.39), whose coefficient function $s = F(u)$ is found by inverting the integral (4.46).

The simplest class of invariant equations for the third unimodular group action is when the function K in (4.50) is constant. The resulting family of equations

$$w_{xxx} - 12ww_{xx} + 18w_x^2 = \alpha(w_x - w^2)^2, \tag{4.53}$$

are equivalent to the equations found by Chazy, [**14**], in his deep study of third order ordinary differential equations having the "Painlevé property", cf. [**27**; Chapter 14]. The Chazy equation appears as a similarity reduction of the Yang–Mills equations from particle physics, [**13**], and the Prandtl boundary layer equations from fluid mechanics, [**15**]. It also has deep connections to number theory and automorphic forms, [**52**]. Chazy showed that when

$$\alpha = 0, \qquad \text{or} \qquad \alpha = \frac{864}{36 - k^2}, \quad \text{where} \quad 6 < k \in \mathbb{N}, \tag{4.54}$$

then the nontrivial solutions $w = f(x)$ to (4.53) have a moveable natural boundary which forms a circle in the complex plane whose position depends on the initial data. Chazy's results, in fact, follow from the representation of the solution provided by the above analysis. First, in terms of the fundamental differential invariants r, s, the Chazy equation (4.53) takes the form

$$\frac{r}{s^2}\,\frac{dr}{ds} = \beta = \tfrac{1}{2}(\alpha - 24).$$

According to the general procedure, we solve for r, and then introduce a parameter u so that $r = ds/du$. The result is $r^2 = (ds/du)^2 = \frac{2}{3}\beta s^3 + c$, for c constant. If $\beta \neq 0$, i.e., $\alpha \neq 24$, then $s = F(u) = (6/\beta)\wp(u+k)$, where $\wp$ denotes the Weierstrass elliptic function with parameters $g_2 = 0$, $g_3 = -\frac{1}{36}\beta^2 c$. Therefore, we have the following result.

Theorem 4.26. *The general solution to the* Chazy equation (4.53) *for* $\alpha \neq 24$ *has the parametric form*

$$x = \frac{\varphi(u)}{\psi(u)}, \qquad y = \frac{1}{\psi}\,\frac{d\psi}{dx} = \frac{\psi\psi_u}{\omega}, \tag{4.55}$$

where $\varphi(u)$, $\psi(u)$, *are two linearly independent solutions of the* Lamé equation

$$\frac{d^2\psi}{du^2} - \frac{3\wp(u+k)}{\beta}\,\psi = 0, \tag{4.56}$$

and $\omega = \varphi\psi_u - \varphi_u\psi$ *is their* Wronskian.

Exercise 4.27. Show how, in the special case $\alpha = 24$ the solution to the Chazy equation can be recovered from that of the Airy equation $\psi_{uu} + cu\psi = 0$.

Exercise 4.28. Chazy, [**14**], expresses the general solution to (4.53) in the form

$$x = \frac{\varphi(y)}{\psi(y)}, \qquad w = \frac{1}{\psi(y)}\,\frac{d\psi}{dx}, \tag{4.57}$$

where $\varphi(y)$ and $\psi(y)$ are two arbitrary linearly independent solutions of the hypergeometric equation

$$y(1-y)\,\frac{d^2\psi}{dy^2} + \left(\frac{1}{2} - \frac{7}{6}y\right)\frac{d\psi}{dy} + \frac{1}{6(\alpha - 24)}\,\psi = 0. \tag{4.58}$$

Show how to transform (4.58) into the Lamé equation (4.56) and thereby connect the two solutions. It is interesting that equation (4.58) arises in Schwarz's theory of algebraic hypergeometric functions; in fact, for α given by (4.54), with $k = 2, 3, 4$, and 5, all of the solutions are algebraic functions; these parameter values correspond to hypergeometric equations admitting the discrete symmetry groups of dihedral triangle, tetrahedral, octahedral, and icosahedral class. See Hille, [**24**; §10.3], for details of Schwarz's theory.

Because of the preceding results, trivial linearizable ordinary differential equations are (as far as we know) uniquely characterized by the property that they admit a symmetry group of the maximal, finite dimension. Symmetry groups can also be used effectively to characterize linearizable systems of partial differential equations. The key remark is that a linear system of partial differential equations $\mathcal{D}[u] = 0$, where $\mathcal{D}$ is an n^{th} order linear differential operator, has (assuming the system is not overdetermined) an infinite-dimensional symmetry group since we can add any other solution to a given solution. The infinitesimal generators of the relevant infinite-dimensional symmetry group take the form

$$\mathbf{v}_\psi = \sum_{\alpha=1}^{q} \psi^\alpha(x) \frac{\partial}{\partial u^\alpha}\,, \tag{4.59}$$

where $\psi = (\psi^1(x), \ldots, \psi^q(x))$ is any solution to the linear system $\mathcal{D}[\psi] = 0$. Note that the vector fields (4.59) commute, and hence generate an infinite-dimensional abelian symmetry group. Assuming the system $\mathcal{D}[\psi] = 0$ is locally solvable, the operators (4.59) span a q-dimensional subspace, namely the space of vertical tangent directions in TE. According to Proposition 4.10, any equivalent system of partial differential equations must also admit such an infinite-dimensional symmetry group, and hence any system of partial differential equations which has only a finite-dimensional symmetry group is certainly not linearizable (at least by a local transformation). Versions of the following theorem appear in [**7**, **8**]; see also [**29**, **30**].

Theorem 4.29. *Let $\Delta(x, u^{(n)}) = 0$ be an n^{th} order system of q independent partial differential equations in $p \geq 2$ independent variables and q unknowns. If the system admits an infinite-dimensional abelian symmetry group, having q-dimensional orbits, and infinitesimal generators depending linearly on the general solution to an n^{th} order system of q independent linear partial differential equations $\mathcal{D}[\psi] = 0$, then it can, by a change of variables, be mapped to an* inhomogeneous *form of the linear system $\mathcal{D}[u] = f$.*

Proof: According to Frobenius' Theorem, any abelian transformation group with q-dimensional orbits can, through a change of variables, be mapped to a group generated by vector fields of the form (4.59). The additional hypothesis implies that the coefficient functions $\psi(x)$ form the general solution to a linear n^{th} order system of partial differential equations $\mathcal{D}[\psi] = 0$. Note that, in terms of the original coordinates, the generators take the form

$$\mathbf{v}_\psi = \sum_{\alpha=1}^{q} \psi^\alpha\big(\eta^1(x,u), \ldots, \eta^p(x,u)\big)\mathbf{w}_\alpha, \tag{4.60}$$

where the vector fields $\mathbf{w}_1, \ldots, \mathbf{w}_q$ are linearly independent, commute, and satisfy $\mathbf{w}_\alpha(\eta^i) = 0$ for all α, i. Now, in the new coordinates, the system must be equivalent to the inhomogeneous form of the linear system. To see this, it suffices to note that the only n^{th} order differential invariants of the infinite-dimensional group generated by the vector fields (4.59) are the components of $\mathcal{D}[u]$ and the variables x. Therefore, according to Theorem 4.21, any invariant system of differential equations must be isomorphic to one of the form $H(\mathcal{D}[u], x) = 0$. But, since the system consists of q independent equations, we can solve the system for the components of $\mathcal{D}[u]$, placing the system into the desired inhomogeneous form. *Q.E.D.*

Example 4.30. The nonlinear diffusion equation

$$u_t = u_x^{-2} u_{xx} \tag{4.61}$$

admits the following six symmetry generators:

$$\begin{aligned} &\partial_t, \qquad \partial_u, \qquad x\partial_x, \qquad 2t\partial_t + u\partial_u, \\ &xu\partial_x - 2t\partial_u, \qquad (2xt + xu^2)\partial_x - 4t^2\partial_t - 4tu\partial_u, \end{aligned}$$

as well as the infinite dimensional abelian subalgebra

$$\alpha(t,u)\partial_x, \qquad \text{where} \qquad \alpha_t = \alpha_{uu}.$$

The hodograph transformation $v = x$, $y = u$, linearizes (4.61) to the (homogeneous) heat equation $v_t = v_{yy}$.

