Chapter 5

Symmetries of Variational Problems

The applications of symmetry groups to problems arising in the calculus of variations have their origins in the late papers of Lie, e.g., [**34**], which introduced the subject of "integral invariants". Lie showed how the symmetry group of a variational problem can be readily computed based on an adaptation of the infinitesimal method used to compute symmetry groups of differential equations. Moreover, for a given symmetry group, the associated invariant variational problems are completely characterized using the fundamental differential invariants and contact-invariant coframe, as presented in Chapter 3. This result lies at the foundation of modern physical theories, such as string theory and conformal field theory, which are constructed using a variational approach and postulating the existence of certain physical symmetries. The applications of Lie groups to the calculus of variations gained added importance with the discovery of Noether's fundamental theorem, [**41**], relating symmetry groups of variational problems to conservation laws of the associated Euler–Lagrange equations. We should also mention the applications of integral invariants to Hamiltonian mechanics, developed by Cartan, [**11**], which led to the modern symplectic approach to Hamiltonian systems, cf. [**35**]; unfortunately, space precludes us from pursuing this important theory here. More details on the applications of symmetry groups to variational problems and to Hamiltonian systems, along with many additional physical and mathematical examples, can be found in [**43**].

The Calculus of Variations

The starting point will be a discussion of some of the foundational results in the calculus of variations. As usual, we work over an open subset of the total space $E = X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$ coordinatized by independent variables $x = (x^1, \ldots, x^p)$ and dependent variables $u = (u^1, \ldots, u^q)$. The associated n^{th} jet space J^n is coordinatized by the derivatives $u^{(n)}$ of the dependent variables. Let $\Omega \subset X$ denote a connected open set with smooth boundary $\partial\Omega$. By an n^{th} order *variational problem*, we mean the problem of finding the extremals (maxima and/or minima) of a *functional*

$$\mathcal{L}[u] = \mathcal{L}_\Omega[u] = \int_\Omega L(x, u^{(n)})\, dx \tag{5.1}$$

over some space of functions $u = f(x)$, $x \in \Omega$. The integrand $L(x, u^{(n)})$, which is a smooth differential function on the jet space J^n, is referred to as the *Lagrangian* of the variational problem (5.1); the horizontal p-form $L\, dx = L\, dx^1 \wedge \cdots \wedge dx^p$ is the *Lagrangian form*. The precise space of functions upon which the functional (5.1) is to be extremized will depend on any boundary conditions which may be imposed — e.g., the Dirichlet conditions $u = 0$

on $\partial\Omega$ — as well as smoothness requirements. More generally, although this is beyond our scope, one may also impose additional constraints — holonomic, non-holonomic, integral, etc. In our applications, the precise nature of the boundary conditions will, by and large, be irrelevant. Moreover, as in our discussion of differential equations, we shall always restrict our attention to smooth extremals, leaving aside important, but technically more complicated problems for more general extremals.

The most basic result in the calculus of variations is the construction of the fundamental differential equations — the Euler–Lagrange equations — which must be satisfied by any smooth extremal.† The Euler–Lagrange equations constitute the infinite-dimensional version of the basic theorem from calculus that the maxima and minima of a smooth function $f(x)$ occur at the point where the gradient vanishes: $\nabla f = 0$. In the functional context, the gradient's role is played by the "variational derivative", whose components, in concrete form, are found by applying the fundamental Euler operators.

Definition 5.1. Let $1 \le \alpha \le q$. The differential operator $\mathrm{E} = (\mathrm{E}_1, \ldots, \mathrm{E}_q)$, whose components are

$$\mathrm{E}_\alpha = \sum_J \,(-D)_J\, \frac{\partial}{\partial u_J^\alpha}\,, \qquad \alpha = 1, \ldots, q, \tag{5.2}$$

is known as the *Euler operator*. In (5.2), the sum is over all symmetric multi-indices $J = (j_1, \ldots, j_k)$, $1 \le j_\nu \le p$, and $(-D)_J = (-1)^k D_J$ denotes the corresponding signed higher order total derivative.

Theorem 5.2. *The smooth extremals $u = f(x)$ of a variational problem with Lagrangian $L(x, u^{(n)})$ must satisfy the system of Euler–Lagrange equations*

$$\mathrm{E}_\alpha(L) = \sum_J \,(-D)_J\, \frac{\partial L}{\partial u_J^\alpha} = 0, \qquad \alpha = 1, \ldots, q. \tag{5.3}$$

Note that, as with the total derivatives, even though the Euler operator (5.2) is defined using an infinite sum, for any given Lagrangian only finitely many summands are needed to compute the corresponding Euler–Lagrange expressions $\mathrm{E}(L)$.

Proof: The proof of Theorem 5.2 relies on the analysis of variations of the extremal u. In general, a one-parameter family of functions $u(x, \varepsilon)$ is called a *family of variations*‡ of a fixed function $u(x) = u(x, 0)$ provided that, outside a compact subset $K \subset \Omega$, the functions coincide: $u(x, \varepsilon) = u(x)$ for $x \in \Omega \setminus K$. In particular, all the functions in the family satisfy the same boundary conditions as u. Therefore, if u is, say, a minimum of the variational problem, then, for any family of variations $u(x, \varepsilon)$, the scalar function $h(\varepsilon) = \mathcal{L}[u(x, \varepsilon)]$, must have a minimum at $\varepsilon = 0$, and so, by elementary calculus, satisfies $h'(0) = 0$. In view

† See [**5**] for examples of variational problems with nonsmooth extremals which do *not* satisfy the Euler–Lagrange equations!

‡ In the usual approach, one employs a family of linear variations $u(x, \varepsilon) = u(x) + \varepsilon v(x)$, where $v(x)$ has compact support, since the inclusion of higher order terms in ε has no effect on the method.

of our smoothness assumptions, we can interchange the integration and differentiation to evaluate this derivative:

$$0 = \frac{d}{d\varepsilon}\,\mathcal{L}[u(x,\varepsilon)]\bigg|_{\varepsilon=0} = \int_\Omega \left[\sum_{\alpha=1}^{q} \sum_J \frac{\partial L}{\partial u_J^\alpha}\,(x,u^{(n)})\; D_J\, v^\alpha \right] dx, \tag{5.4}$$

where $v(x) = u_\varepsilon(x,0)$. The method now is to integrate (5.4) by parts. The Leibniz rule

$$P D_i Q = -Q D_i P + D_i[PQ], \qquad i = 1,\ldots,p, \tag{5.5}$$

for total derivatives implies, using the Divergence Theorem, the general integration by parts formula

$$\begin{aligned}\int_\Omega P D_i Q\, dx = & -\int_\Omega Q D_i P\, dx \,+ \\ & + \int_{\partial\Omega} (-1)^{i-1} P\, Q\, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^p,\end{aligned} \tag{5.6}$$

which holds for any smooth function $u = f(x)$. Applying (5.6) repeatedly to integral on the right hand side of (5.4), and using the fact that v and its derivatives vanish on $\partial\Omega$, we find

$$\begin{aligned} 0 &= \int_\Omega \left[\sum_{\alpha=1}^{q} \sum_J (-D)_J \left(\frac{\partial L}{\partial u_J^\alpha} \right) v^\alpha \right] dx \\ &= \int_\Omega \left[\sum_{\alpha=1}^{q} E_\alpha(L) v^\alpha \right] dx = \int_\Omega \big[\, \mathrm{E}(L) \cdot v \,\big]\, dx. \end{aligned}$$

Since the resulting integrand must vanish for *every* smooth function with compact support $v(x)$, the Euler–Lagrange expression $\mathrm{E}(L)$ must vanish everywhere in Ω, completing the proof. *Q.E.D.*

Let us specialize to the scalar case, when there is one independent and one dependent variable. Here, the Euler–Lagrange equation associated with an n^{th} order Lagrangian $L(x,u^{(n)})$ is the ordinary differential equation

$$\frac{\partial L}{\partial u} - D_x \left(\frac{\partial L}{\partial u_x} \right) + D_x^2 \left(\frac{\partial L}{\partial u_{xx}} \right) - \cdots + (-1)^n D_x^n \left(\frac{\partial L}{\partial u_n} \right) = 0. \tag{5.7}$$

For example, the Euler–Lagrange equation associated with the classical Newtonian variational problem $\mathcal{L}[u] = \int \big[\, \frac{1}{2} u_x^2 - V(u) \,\big]\, dx$ (which equals kinetic energy minus potential energy) is the second order differential equation $-u_{xx} - V'(u) = 0$ governing motion in a potential force field.

In general, the Euler–Lagrange equation (5.7) associated with an n^{th} order Lagrangian is an ordinary differential equation of order $2n$ provided the Lagrangian satisfies the classical *nondegeneracy condition*

$$\frac{\partial^2 L}{\partial u_n^2} \neq 0. \tag{5.8}$$

Isolated points at which the nondegeneracy condition (5.8) fails constitute singular points of the Euler–Lagrange equation. Note that, in this context, Lagrangians which are affine functions of the highest order derivative, $L(x, u^{(n)}) = A(x, u^{(n-1)})u_n + B(x, u^{(n-1)})$, are degenerate everywhere. However, a straightforward integration by parts will reduce such a Lagrangian to a nondegenerate one of lower order, and so the exclusion of such Lagrangians is not essential; see Exercise 5.6 below.

Of particular interest are the *null Lagrangians*, which, by definition, are Lagrangians whose Euler–Lagrange expression vanishes identically: $\mathrm{E}(L) \equiv 0$. The associated variational problem is completely trivial, since $\mathcal{L}[u]$ depends only on the boundary values of u, and hence every function provides an extremal.

Theorem 5.3. *A differential function $L(x, u^{(n)})$ defines a null Lagrangian, $\mathrm{E}(L) \equiv 0$, if and only if it is a total divergence, so $L = \operatorname{Div} P = D_1 P_1 + \cdots D_p P_p$, for some p-tuple $P = (P_1, \ldots, P_p)$ of differential functions.*

Proof: Clearly, if $L = \operatorname{Div} P$, the Divergence Theorem implies that the integral (5.1) only depends on the boundary values of u. Therefore, the functional is unaffected by any variations, and so $\mathrm{E}(L) \equiv 0$. Conversely, suppose $L(x, u^{(n)})$ is a null Lagrangian. Consider the expression

$$\frac{d}{dt} L(x, t\, u^{(n)}) = \sum_{\alpha, J} u_J^\alpha \, \frac{\partial L}{\partial u_J^\alpha} \, (x, t\, u^{(n)}).$$

Each term in this formula can be integrated by parts, using (5.5) repeatedly. The net result is, as in the proof of Theorem 5.2,

$$\frac{d}{dt} L(x, t\, u^{(n)}) = \sum_{\alpha=1}^{q} u^\alpha \mathrm{E}_\alpha(L)(x, t\, u^{(2n)}) + \operatorname{Div} R(t, x, u^{(2n)}), \tag{5.9}$$

for some well-defined p-tuple of differential functions $R = (R_1, \ldots, R_p)$ depending on L and its derivatives. Since $\mathrm{E}(L) = 0$ by assumption, we can integrate (5.9) with respect to t from $t = 0$ to $t = 1$, producing the desired divergence identity,

$$L(x, u^{(n)}) = L(x, 0) + \operatorname{Div} \widehat{P} = \operatorname{Div} P.$$

Here $\widehat{P}(x, u^{(2n)}) = \int_0^1 R(t, x, u^{(2n)})\, dt$, and $P = P_0 + \widehat{P}$, where $P_0(x)$ is any p-tuple such that $\operatorname{div} P_0 = L(x, 0)$. *Q.E.D.*

Remark: The proof of Theorem 5.3 assumes that $L(x, u^{(n)})$ is defined everywhere on the jet space J^n. A more general result will, as in the deRham theory, depend on the underlying topology of the domain of definition of L, cf. [**2**, **43**].

Corollary 5.4. *Two Lagrangians define the same Euler–Lagrange expressions if and only if they differ by a divergence: $\widetilde{L} = L + \operatorname{Div} P$.*

Remark: It is possible for two Lagrangians to give rise to the same Euler–Lagrange equations even though they do not differ by a divergence. For instance, both of the scalar variational problems $\int u_x^2\, dx$ and $\int \sqrt{1 + u_x^2}\, dx$ lead to the same Euler–Lagrange equation $u_{xx} = 0$, even though their Euler–Lagrange expressions are not identical. The characterization of such "inequivalent Lagrangians" is a problem of importance in the theory of integrable systems; see [**3**] and the references therein.

Example 5.5. In the case of one independent variable and one dependent variable, every null Lagrangian is a total derivative,

$$L(x,u^{(n)}) = D_x P(x,u^{(n-1)}) = u_n \,\frac{\partial P}{\partial u_{n-1}} + \widetilde{L}(x,u^{(n-1)}),$$

and hence an affine function of the top order derivative u_n. This is no longer the case for several dependent variables; for example the Hessian covariant

$$H = u_{xx}u_{yy} - u_{xy}^2 = D_x(u_x u_{yy}) - D_y(u_x u_{xy}),$$

is a divergence, and hence a null Lagrangian, even though it is quadratic in the top order derivatives. In fact, it can be shown that any null Lagrangian is a "total Jacobian polynomial" function of the top order derivatives of u, [**42**].

Remark: The problem of characterizing systems of differential equations which are the Euler–Lagrange equations for some variational problem is known as the "inverse problem" of the calculus of variations, and has been studied by many authors. See [**3**, **43**] for a discussion of results and the history of this problem.

Exercise 5.6. Suppose $p = 1$. Prove that any nontrivial Lagrangian is equivalent to a nondegenerate Lagrangian (not necessarily of the same order). Can this result be extended to Lagrangians involving several independent variables?

Symmetries of Variational Problems

Maps that preserve variational problems serve to define variational symmetries. The precise definition is as follows.

Definition 5.7. A point transformation g is called a *variational symmetry* of the functional (5.1) if and only if the transformed functional agrees with the original one, which means that for every smooth function f defined on a domain Ω, with transformed counterpart $\bar f = g\cdot f$ defined on $\overline{\Omega}$, we have

$$\int_\Omega L(x,f^{(n)}(x))\,dx = \int_{\overline{\Omega}} L(\bar x,\bar f^{(n)}(\bar x))\,d\bar x. \tag{5.10}$$

Thus, a transformation group G is a variational symmetry group if and only if the Lagrangian form $L(x,u^{(n)})\,dx$ is a contact-invariant p-form, so

$$(g^{(n)})^*\bigl[L(\bar x,\bar u^{(n)})\,d\bar x\bigr] = L(x,u^{(n)})\,dx + \Theta, \qquad g\in G, \tag{5.11}$$

for some contact form $\Theta = \Theta_g$, possibly depending on the group element g. In particular, if the group transformation g is fiber-preserving, then $\Theta = 0$, and the Lagrangian form is strictly invariant. In local coordinates, the contact invariance condition (5.11) takes the form

$$L(x,u^{(n)}) = L(\bar x,\bar u^{(n)})\det J, \qquad \text{when} \qquad (\bar x,\bar u^{(n)}) = g^{(n)}\cdot(x,u^{(n)}), \tag{5.12}$$

where $J = (D_i \chi^j)$ is the total Jacobian matrix. Since the Euler–Lagrange equations are correspondingly transformed under an equivalence map, we immediately deduce the following useful result.

Theorem 5.8. *Every* variational symmetry *group of a variational problem is a symmetry group of the associated* Euler–Lagrange equations.

Note, though, that the converse to Theorem 5.8 is not true. The most common examples of symmetries which fail to be variational are those generating groups of scaling transformations. For example, the variational problem $\int \frac{1}{2} u_x^2 \, dx$ has Euler–Lagrange equation $u_{xx} = 0$, admitting the two-parameter scaling symmetry group $(x, u) \mapsto (\lambda x, \mu u)$. However, this group does not leave the variational problem invariant, but, rather scales it by the factor μ^2/λ. Note, however, that the one-parameter subgroup $(x, u) \mapsto (\lambda^2 x, \lambda u)$ *is* a variational symmetry group. Therefore, to determine variational symmetries, one can proceed by first determining the complete symmetry group of the Euler–Lagrange equations using Lie's method, and then analyzing which of the usual symmetries satisfy the additional criterion of being variational. See also [**43**; Proposition 5.55] for an alternative approach based directly on the infinitesimal invariance of the Euler–Lagrange equations.

In the case of a connected transformation group, we let g in (5.11) belong to a one-parameter subgroup, and differentiate to retrieve the basic infinitesimal invariance criterion for variational symmetry groups. This requires that the Lie derivative of the Lagrangian p-form with respect to the prolonged vector field $\mathbf{v}^{(n)}$ be a contact form. In particular, as in Example 2.79, the horizontal component of the Lie derivative of the volume form with respect to $\mathbf{v}^{(n)}$ is

$$\mathbf{v}^{(n)}(dx^1 \wedge \cdots \wedge dx^p) = (\operatorname{Div} \xi)\, dx^1 \wedge \cdots \wedge dx^p, \tag{5.13}$$

where $\operatorname{Div} \xi = D_1 \xi^1 + \cdots D_p \xi^p$ denotes the total divergence of the base coefficients of $\mathbf{v}$. This suffices to prove the basic invariance criterion.

Theorem 5.9. *A connected transformation group G is a variational symmetry group of the Lagrangian $L(x, u^{(n)})$ if and only if the infinitesimal variational symmetry condition*

$$\mathbf{v}^{(n)}(L) + L \operatorname{Div} \xi = 0, \tag{5.14}$$

holds for every infinitesimal generator $\mathbf{v} \in \mathfrak{g}$.

Example 5.10. The Boussinesq equation (4.20) is not the Euler–Lagrange equation for any variational problem. However, replacing u by u_{xx}, we form the "potential Boussinesq equation"

$$u_{xxtt} + \tfrac{1}{2} D_x^2 (u_{xx}^2) + u_{xxxxxx} = 0, \tag{5.15}$$

which is the Euler–Lagrange equation for the variational problem

$$\mathcal{L}[u] = \int\!\!\int \left[\, \tfrac{1}{2} u_{xt}^2 + \tfrac{1}{6} u_{xx}^3 - \tfrac{1}{2} u_{xxx}^2 \,\right] dx \wedge dt. \tag{5.16}$$

The symmetry group of the potential form (5.15) is spanned by the translation and scaling vector fields

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = x\, \partial_x + 2t\, \partial_t, \tag{5.17}$$

and the two infinite families of vector fields

$$\mathbf{v}_f = f(t)\,\partial_u, \qquad \mathbf{v}_h = h(t)\,x\,\partial_u, \tag{5.18}$$

where $f(t)$ and $h(t)$ are arbitrary functions of t; the corresponding group action $u \mapsto u + f(t) + h(t)\,x$ indicates the ambiguity in our choice of potential. (Compare with the symmetry group of the usual form of the Boussinesq equation found in Example 4.11.) The most general variational symmetry is found by substituting a general symmetry vector field $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \mathbf{v}_f + \mathbf{v}_h$ into the infinitesimal criterion (5.14), which requires that

$$\tfrac{1}{4}c_3(-3u_{xt}^2 + u_{xx}^3 - 3u_{xxx}^2) + h'(t)u_{xt} = 0.$$

Therefore $c_3 = 0$ and h is constant, hence the two translations, and the group $u \mapsto u + c\,x + f(t)$, with c constant, are variational, whereas the scalings and the more general fields $\mathbf{v}_h$, h not constant, are not ordinary variational symmetries, but do define divergence symmetries, in the following sense.

Definition 5.11. A vector field $\mathbf{v}$ is a *divergence symmetry* of a variational problem with Lagrangian L if and only if it satisfies

$$\mathbf{v}^{(n)}(L) + L\ \mathrm{Div}\,\xi = \mathrm{Div}\,B, \tag{5.19}$$

for some p-tuple of functions $B = (B_1, \ldots, B_p)$.

A divergence symmetry is a divergence self-equivalence of the Lagrangian form, so that (5.11) holds modulo an exact p-form $d\,\Xi$. The divergence symmetry groups form the most general class of symmetries related to conservation laws. Indeed, Noether's Theorem provides a one-to-one correspondence between generalized divergence symmetries of a variational problem and conservation laws of the associated Euler-Lagrange equations; see [**43**] for a detailed discussion.

Exercise 5.12. Prove that every divergence symmetry of a variational problem is an (ordinary) symmetry of its Euler–Lagrange equations.

Exercise 5.13. Show that the Korteweg–deVries equation $u_t = u_{xxx} + uu_x$ can be placed into variational form through the introduction of a potential $u = v_x$.

Invariant Variational Problems

As with differential equations, the most general variational problem admitting a given symmetry group can be readily characterized using the differential invariants of the prolonged group action. The key additional requirement is the existence of a suitable contact-invariant p-form, where p is the number of independent variables. The following theorem is a straightforward consequence of the infinitesimal variational symmetry criterion (5.14), and dates back to Lie, [**34**].

Theorem 5.14. *Let G be a transformation group, and assume that the n^{th} prolongation of G acts regularly on (an open subset of) J^n. Assume further that there exists a nonzero* contact-invariant *horizontal p-form $\Omega_0 = L_0(x, u^{(n)})\,dx$ on J^n. A variational problem admits G as a* variational symmetry *group if and only if it can be written in the form $\int I\,\Omega_0 = \int I L_0\,dx$, where I is a differential invariant of G.*

In particular, any contact-invariant coframe $\omega^1, \ldots, \omega^p$ provides a contact-invariant p-form $\Omega_0 = \omega^1 \wedge \cdots \wedge \omega^p$. Hence every G-invariant variational problem has the form

$$\mathcal{L}[u] = \int L(x, u^{(n)})\, dx = \int F\big(I_1(x, u^{(n)}), \ldots, I_k(x, u^{(n)})\big)\, \omega^1 \wedge \cdots \wedge \omega^p, \tag{5.20}$$

where $I_1, \ldots, I_k$ are a complete set of functionally independent n^{th} order differential invariants.

Example 5.15. Consider the rotation group SO(2) acting on $E \simeq \mathbb{R} \times \mathbb{R}$, cf. Example 3.2. Since the radius $r = \sqrt{x^2 + u^2}$ is an invariant, its total differential provides a contact-invariant one-form, which we slightly modify: $\omega = (x + u u_x)\, dx$. Therefore, according to Theorem 5.14, any first order variational problem admitting the rotation group as a variational symmetry group has the form

$$\int F(r, w)\, (x + u u_x)\, dx = \int (x + u u_x)\, F\left(\sqrt{x^2 + u^2}\,, \frac{x u_x - u}{x + u u_x}\right) dx.$$

According to Theorem 5.8, the Euler–Lagrange equation for such a variational problem is a second order ordinary differential equation admitting SO(2) as a symmetry group, and hence of the form (4.30). In polar coordinates, the variational problem becomes $\frac{1}{2} \int F(r, r\theta_r)\, r\, dr$, with Euler–Lagrange equation $D_r\left[\frac{1}{2} r^2 F(r, r\theta_r)\right] = 0$. The latter can be immediately integrated once, leading to $r^2 F(r, r\theta_r) = c$, an equation defining θ_r implicitly as a function of r, and thereby soluble by quadrature. This fact is, as we shall see, no accident.

Example 5.16. Consider the rotation group SO(2) acting on $E \simeq \mathbb{R}^2 \times \mathbb{R}$ by rotating the independent variables. The differential invariants were found in Example 4.23. The area form $dx \wedge dy$ is invariant, hence any first order variational problem admitting SO(2) as a variational symmetry group has the form

$$\mathcal{L}[u] = \int F\left(\sqrt{x^2 + y^2}\,, u, -y u_x + x u_y, x u_x + y u_y\right) dx \wedge dy.$$

Note that the one-forms $\omega^1 = x\, dx + y\, dy$, $\omega^2 = -y\, dx + x\, dy$ form a contact-invariant coframe, so $\omega^1 \wedge \omega^2 = (x^2 + y^2) dx \wedge dy$ provides an alternative contact-invariant two-form which, in view of Theorem 5.14, is an invariant multiple of the area form. Again, the Euler–Lagrange equation is rotationally invariant. For example, the Dirichlet variational problem has Lagrangian

$$u_x^2 + u_y^2 = \frac{(-y u_x + x u_y)^2 + (x u_x + y u_y)^2}{x^2 + y^2},$$

and is rotationally invariant; its Euler–Lagrange equation is the Laplace equation.

Remark: If G is a given transformation group, then, as we have seen, any G-invariant variational problem, and its associated Euler–Lagrange equations, can both be written in terms of the differential invariants of G. The general formula for calculating the invariant formulation of the Euler–Lagrange equations directly from the invariant formula for the Lagrangian can be found in [**31**].

First Integrals

For systems of "conservative" ordinary differential equations — meaning systems of Euler–Lagrange equations — the power of the Lie's symmetry method for reducing the order is effectively doubled. This is due to a fundamental result of E. Noether, [**41**], that relates symmetries and first integrals of Euler–Lagrange equations.

Definition 5.17. For a system of ordinary differential equations $\Delta(x, u^{(n)}) = 0$, a *first integral* is a function $P(x, u^{(m)})$ which is constant on solutions.

Physical examples of first integrals include the usual conservation laws of linear and angular momentum and energy. The fact that P is constant is equivalent to the statement that its total derivative vanishes on solutions, so $D_x P = 0$ whenever $u = f(x)$ is a solution to $\Delta = 0$. In the regular case, this requires that P satisfy an identity of the form $D_x P = \sum_i Q_i D_x^i \Delta$ for some differential functions Q_i.

Theorem 5.18. *If* $\mathbf{v}$ *is an infinitesimal variational symmetry with* characteristic Q, *then the product* $Q\,\mathrm{E}(L) = D_x P$ *is a total derivative, and thus a first integral of the Euler–Lagrange equation* $\mathrm{E}(L) = 0$.

Proof: If $\mathbf{v}$ defines a variational symmetry, then according to (5.14) and (3.32),

$$0 = \mathbf{v}^{(n)}(L) + L\,D_x\xi = \mathbf{v}_Q^{(n)}(L) + D_x(L\xi) = \sum_{i=0}^{n}(D_x^i Q)\cdot\frac{\partial L}{\partial u_i} + D_x(L\xi).$$

Now, applying the basic integration by parts formula (5.5) repeatedly, we find

$$Q\cdot \mathrm{E}(L) - D_x P = Q\cdot\left[\sum_{i=0}^{n}(-D_x)^i\left(\frac{\partial L}{\partial u_i}\right)\right] - D_x P = 0$$

for some function P depending on Q, L, and their derivatives. Thus $D_x P$ is a multiple of the Euler–Lagrange equation, which suffices to prove the result. *Q.E.D.*

Corollary 5.19. *If a variational problem admits a one-parameter variational symmetry group, then its Euler–Lagrange equation can be reduced in order by 2.*

Proof: In terms of the rectifying coordinates y, v introduced in the proof of Theorem 4.15, the infinitesimal generator $\mathbf{v} = \partial_v$ is a variational symmetry if and only if the Lagrangian $L(y, v_y, v_{yy}, \ldots)$ is independent of v. Therefore, the Euler–Lagrange equations have the form $D_y P(y, v_y, v_{yy}, \ldots, v_{n-1}) = 0$, which can be integrated once. If $w = v_y$, then the reduced ordinary differential equation $P(y, w^{(n-2)}) = c$, for c constant, forms the Euler–Lagrange equation of order $n-2$ for the "reduced" Lagrangian $L(y, w, w_y, \ldots) - cw$. *Q.E.D.*

Thus, the fact that we could integrate the rotationally invariant Euler–Lagrange equation in Example 5.15 was no accident. For multi-dimensional groups, the reduced variational problem will not, in general, admit the original variational symmetries *unless* they commute with the reducing symmetry. Therefore, only abelian r-dimensional symmetry groups will yield complete reductions of the Euler–Lagrange equation by order $2r$. This fact is closely connected with the reduction theory for Hamiltonian systems, as described by Marsden and Weinstein, [**35**]; see also [**43**].

Noether's Theorem is considerably more general than the ordinary differential equation version of Theorem 5.18. In full generality, it prescribes a one-to-one correspondence between generalized variational symmetries and conservation laws of the Euler–Lagrange equations. See [**43**] for details.

