Symmetry and invariance in cognition — a mathematical perspective

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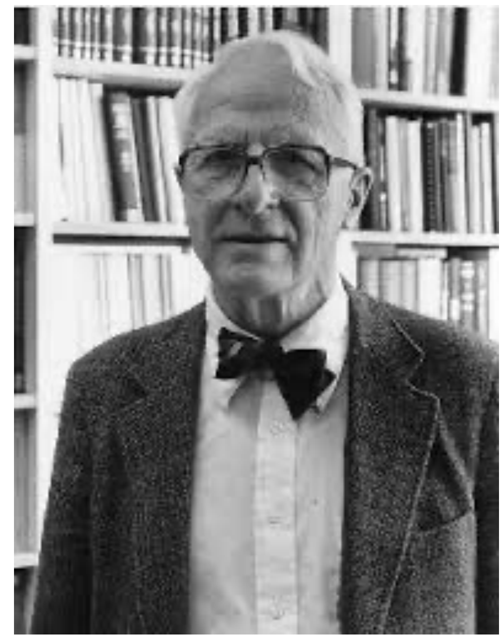
Harvard, April 2019

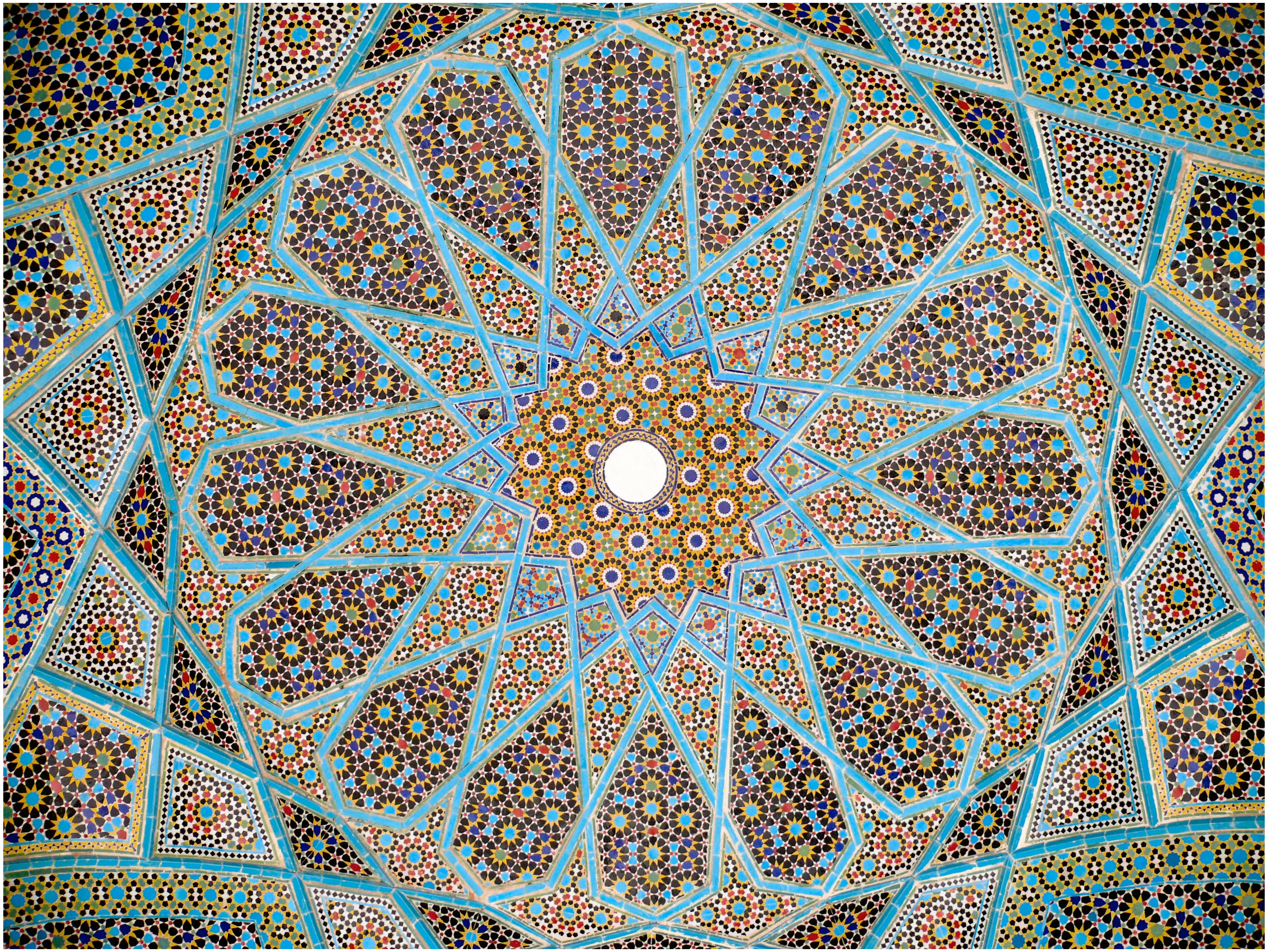
Sophus Lie
(1842–1899)

Elie Cartan
(1869–1951)



Garrett Birkhoff
(1911–1996)

Symmetry



Why are humans so attuned to symmetry?

Mathematically …

Mathematically …

Symmetry → *Group Theory*

Next to the concept of a function, which is the most important concept pervading the whole of mathematics, the concept of a group is of the greatest significance in the various branches of mathematics and its applications.

— P.S. Alexandroff

History of Group Theory

Solution of polynomial equations (exploiting symmetries of the roots)

- Quadratic formula — Diophantus, Brahmagupta, Al Khwarizmi, etc.
- Cubic — Ferro, Tartaglia, Cardano (1545)
- Quartic — Ferrara , Cardano (1545)
- Quintic and higher degree — Lagrange (1770), Ruffini (1799), Abel (1824), Galois (1832)

— finite and discrete groups

Solution of differential equations

- Sophus Lie (1876)
- Picard & Vessiot (1892)

— Lie groups and pseudo-groups

The Quadratic Formula

$$a x^2 + b x + c = 0$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

The Quadratic Formula

$$a\,x^2 + b\,x + c = 0$$

$$\begin{aligned} x &= -\,\frac{b}{2\,a} \pm \frac{\sqrt{b^2 - 4\,a\,c}}{2\,a} \\ &= \quad u \quad \pm \quad v \end{aligned}$$

The Quadratic Formula

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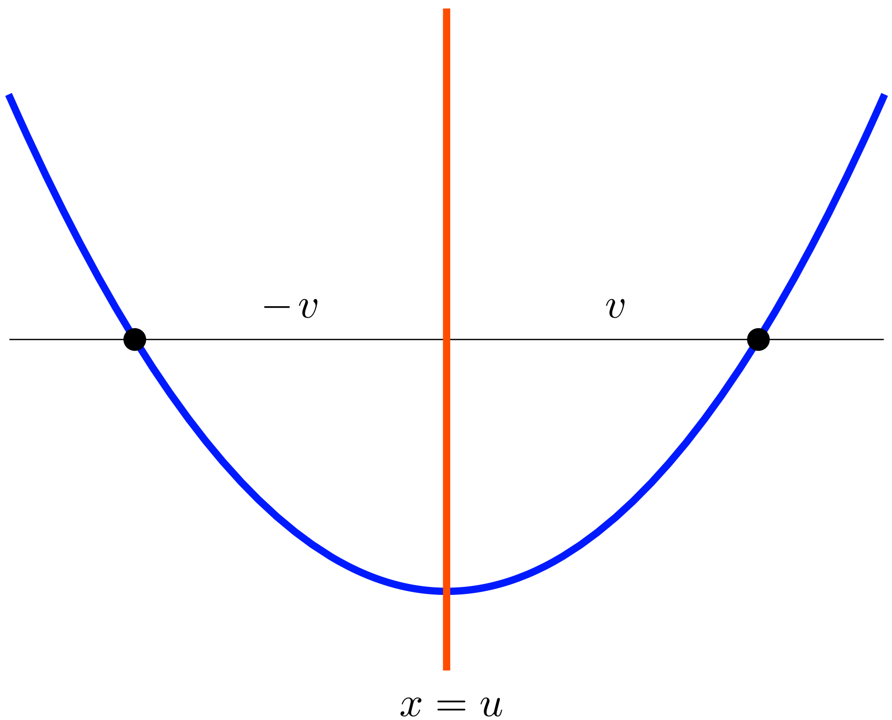
$$\begin{aligned} x &= -\frac{b}{2\,a} \pm \frac{\sqrt{b^2 - 4\,a\,c}}{2\,a} \\ &= \quad u \quad \pm \quad v \end{aligned}$$



Groups

Definition. A group G is a set with a binary operation $g \cdot h$ satisfying

- Associativity: $g \cdot (h \cdot k) = (g \cdot h) \cdot k$
- Identity: $g \cdot e = g = e \cdot g$
- Inverse: $g \cdot g^{-1} = e = g^{-1} \cdot g$

$\Longrightarrow$ not necessarily commutative: $g \cdot h \neq h \cdot g$

Examples of groups

The integers

…, -4, -3, -2, -1, 0, 1, 2, 3, 4, …

Group operation: addition $3 + 5 = 8$

Identity: zero $3 + 0 = 3 = 0 + 3$

Inverse: negative $7 + (-7) = 0 = (-7) + 7$

Examples of groups

The rational numbers (fractions)

Group operation: addition $1/4 + 5/3 = 23/12$

Identity: zero $5/3 + 0 = 5/3 = 0 + 5/3$

Inverse: negative $7/2 + (-7/2) = 0 = (-7/2) + 7/2$

Examples of groups

The positive rational numbers

Group operation: multiplication $1/4 \times 5/3 = 5/12$

Identity: one $5/3 \times 1 = 5/3 = 1 \times 5/3$

Inverse: reciprocal $7/2 \times 2/7 = 1 = 2/7 \times 7/2$

Examples of groups

The positive real numbers

Group operation: multiplication

$\sqrt{2}$ x π $= \sqrt{2}\,\pi$ = 4.44288293815836624701588099006....

Identity: one π x 1 = π = 1 x π

Inverse: reciprocal π x 1/ π = 1 = 1/ π x π

Examples of groups

Non-singular 2 x 2 matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad h = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \qquad ad - bc \neq 0 \neq xw - yz$$

Group operation:

$$g \cdot h = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} \neq \begin{pmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{pmatrix} = h \cdot g$$

Identity: $\quad e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e \cdot g = g = g \cdot e$

Inverse: $\quad g^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad g \cdot g^{-1} = e^{-1} = g \cdot g$

Symmetry Groups

A symmetry g of a geometric object S is
an invertible transformation that preserves it: $g \cdot S = S$

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The set of symmetries of a geometric object forms a group

The group operation is composition: $g \cdot h$ = first do h, then do g

The composition of two symmetries is a symmetry

The identity (do nothing) is always a symmetry

The inverse of a symmetry (undo it) is a symmetry

Symmetry

Definition. A symmetry of a set S is a transformation that preserves it:

$$g \cdot S = S$$

★★ The set of symmetries forms a group G_S, called the symmetry group of the set S.

Discrete Symmetry Group

Discrete Symmetry Group

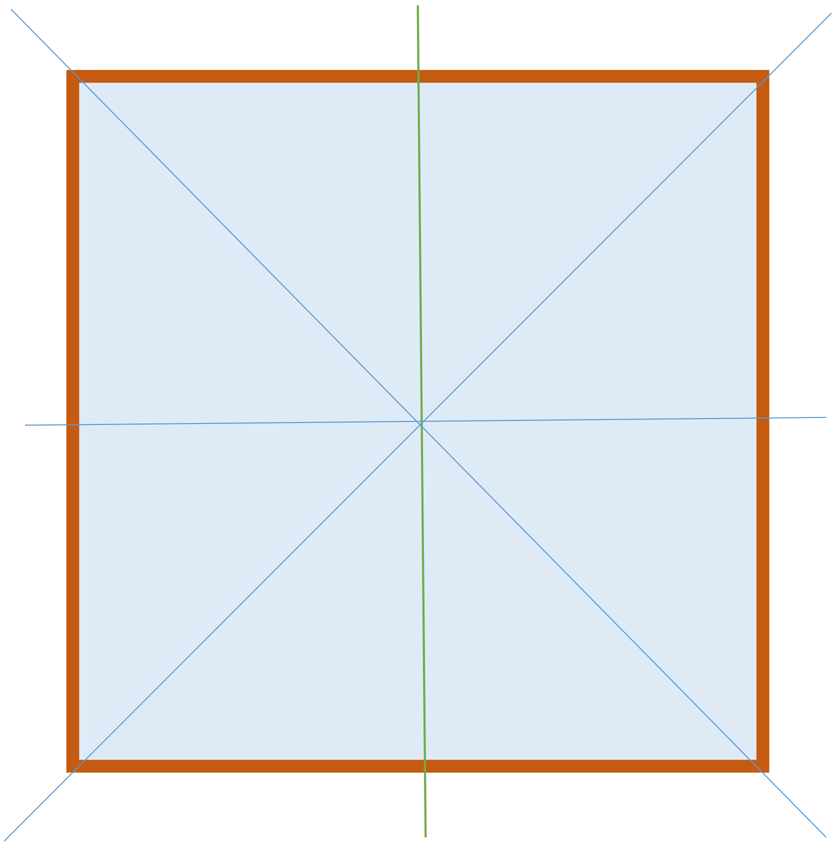
Rotations by 90º, 180º, 270º

Discrete Symmetry Group

Rotations by 90º, 180º, 270º

and 0º (identity)

Discrete Symmetry Group



Rotations by 90º, 180º, 270º

and 0º (identity)

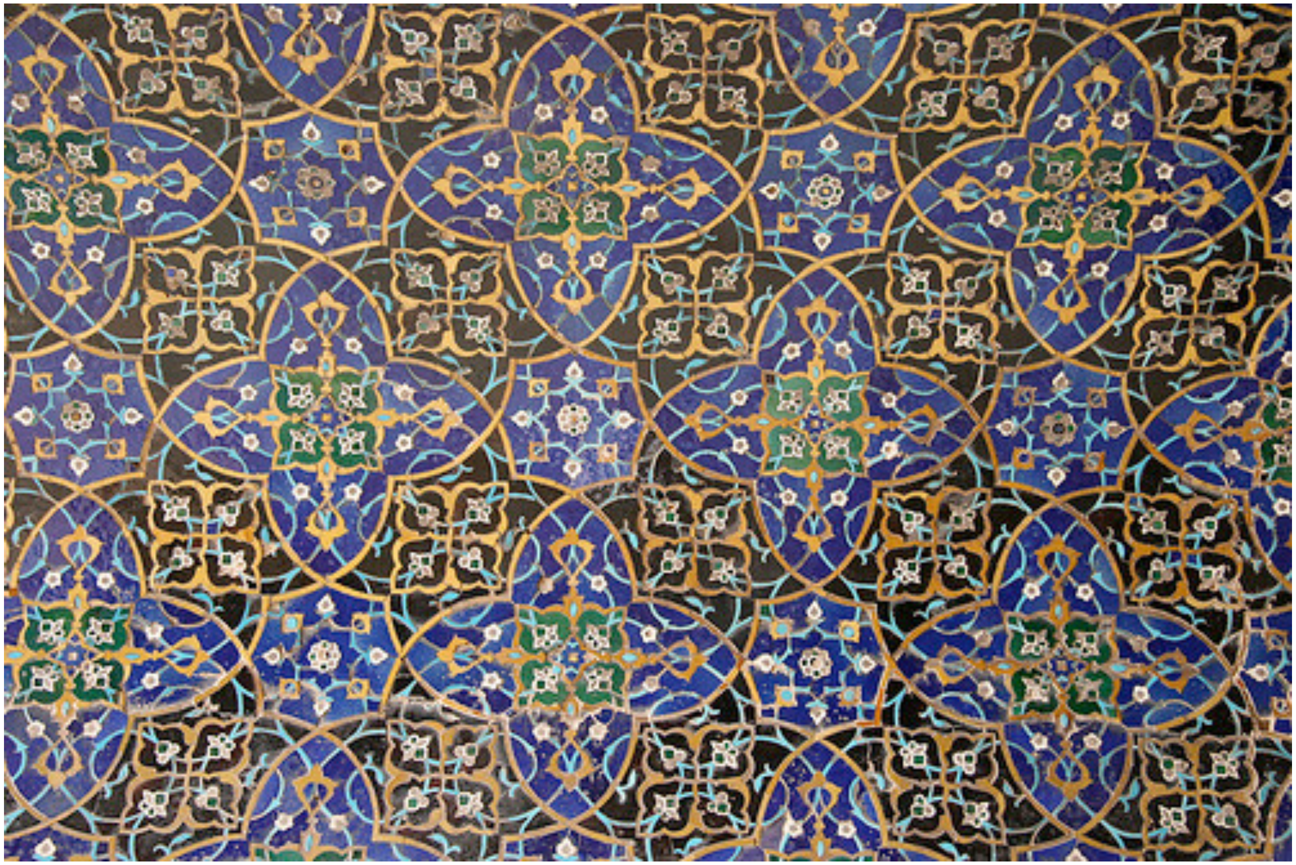
… and 4 reflections
(mirror image)

Wallpaper patterns

17 symmetry types

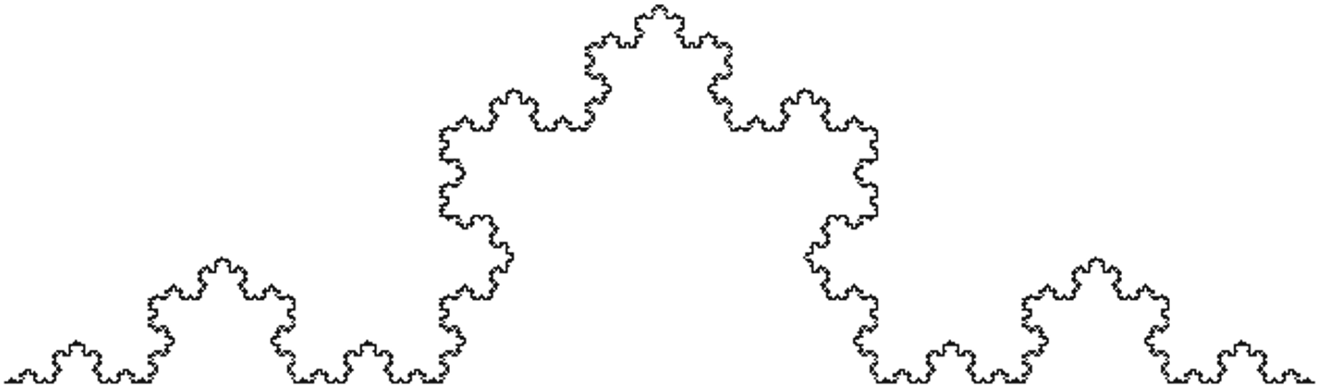
Tiling — The Alhambra, Spain

Tilings — *Jameh Mosque, Esfahan, Iran*

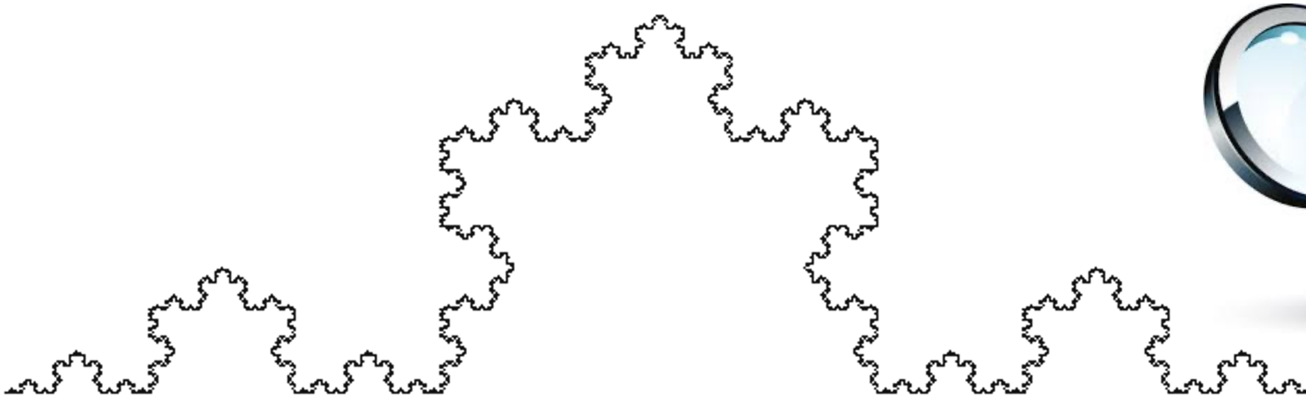



Crystallography
230 groups

The Koch snowflake — a fractal curve

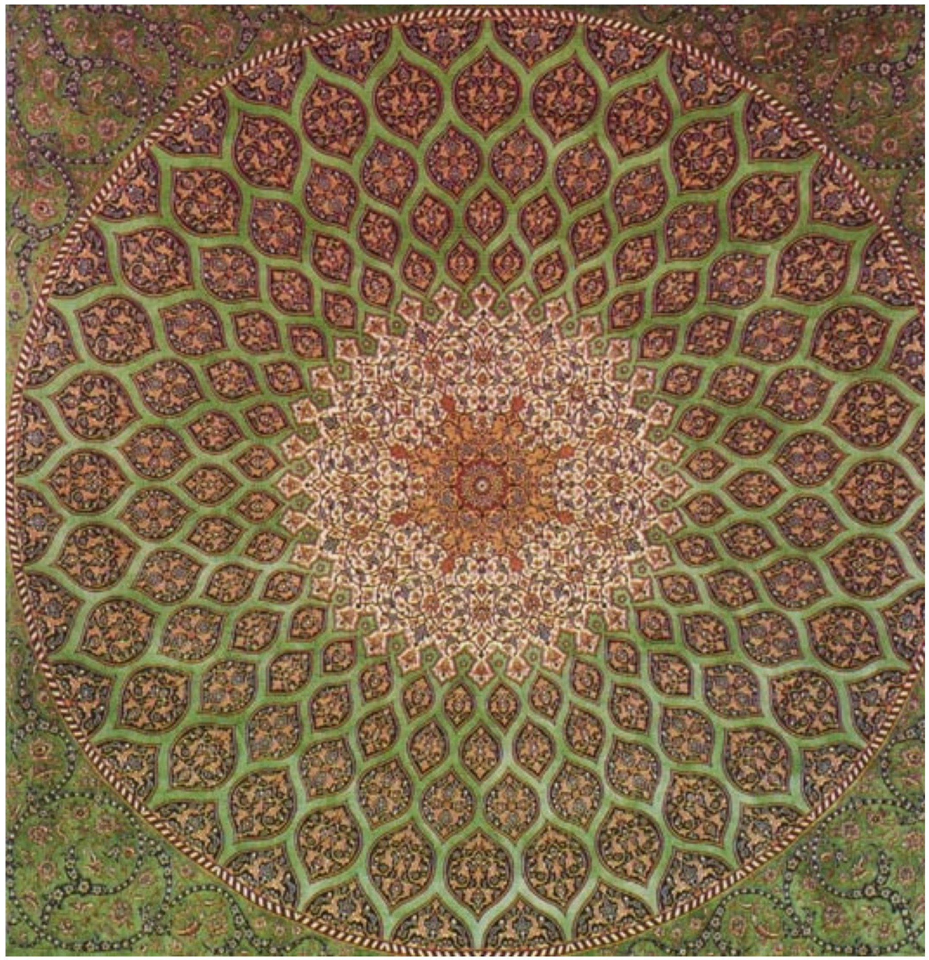


The Koch snowflake — a fractal curve

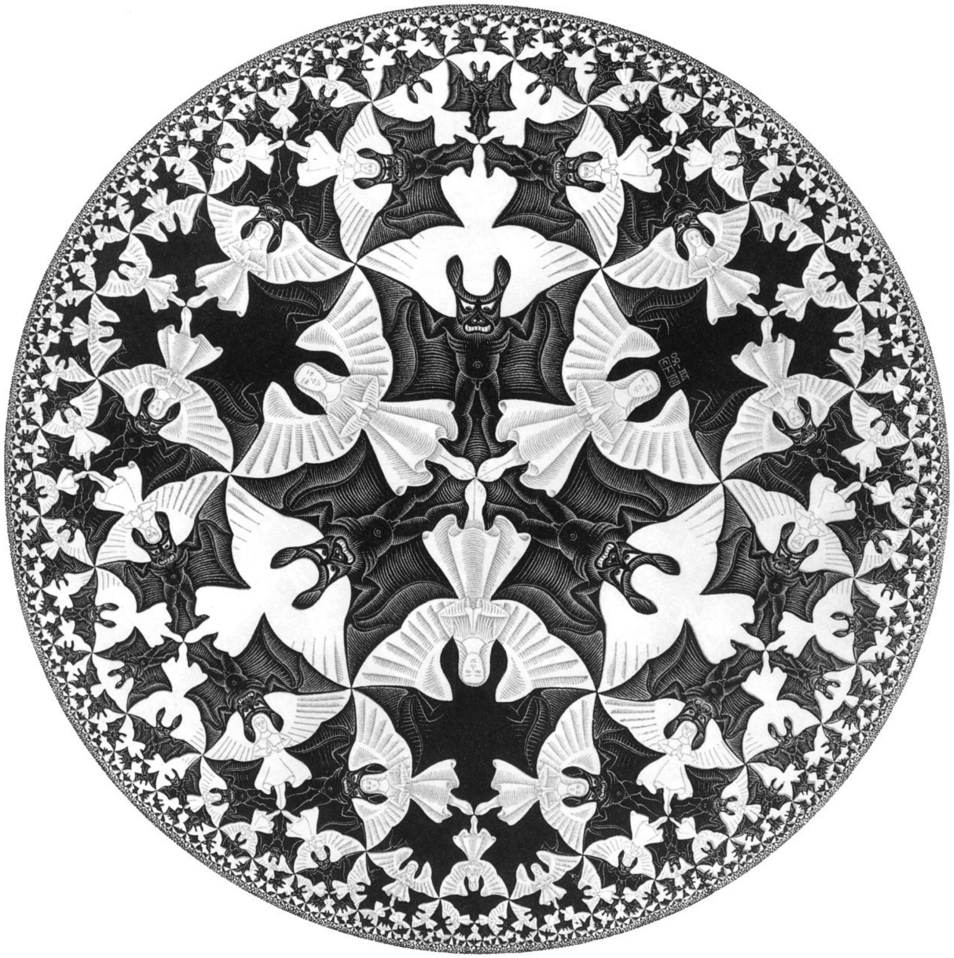


★ Scaling symmetry

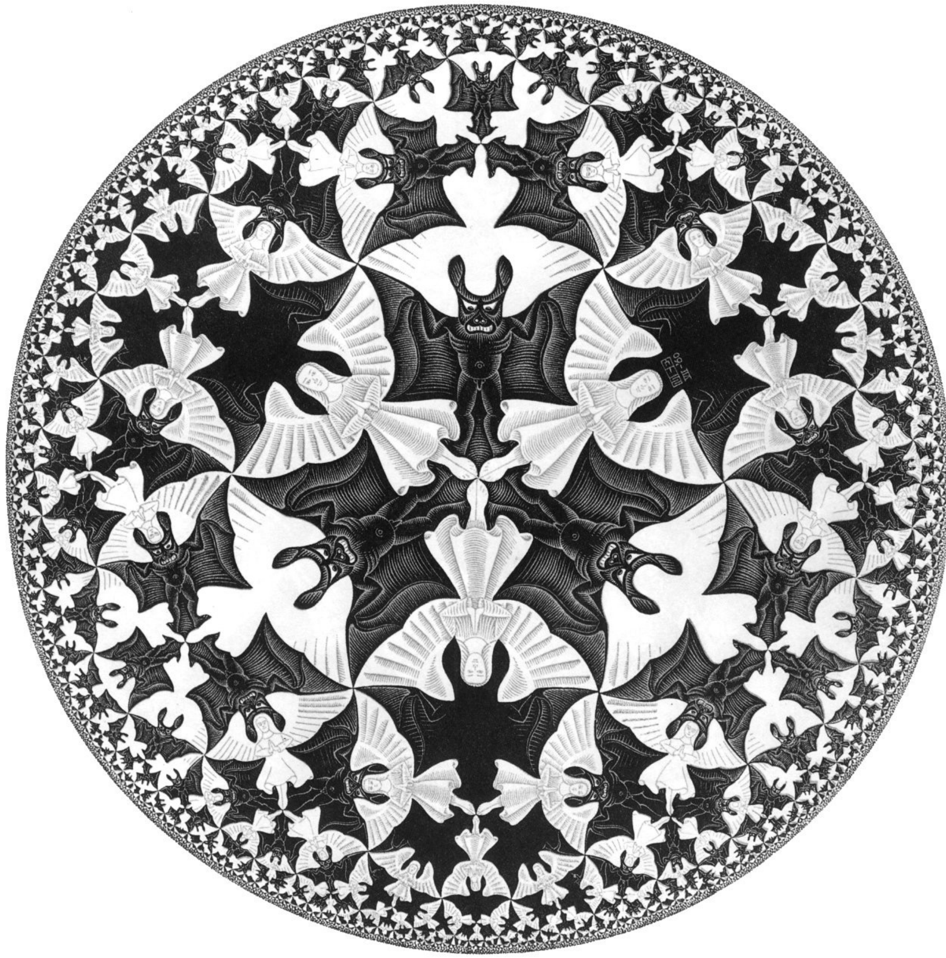
Dome of the Sheikh Lotfollah Mosque – Isfahan, Iran



M.C. Escher — Circle Limit IV

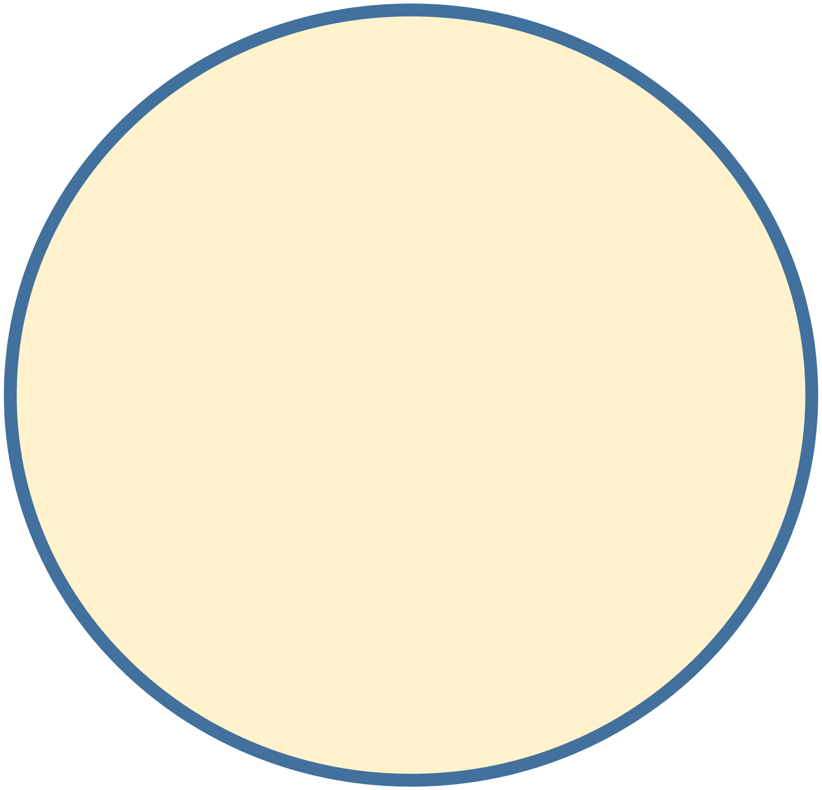


M.C. Escher — Circle Limit IV

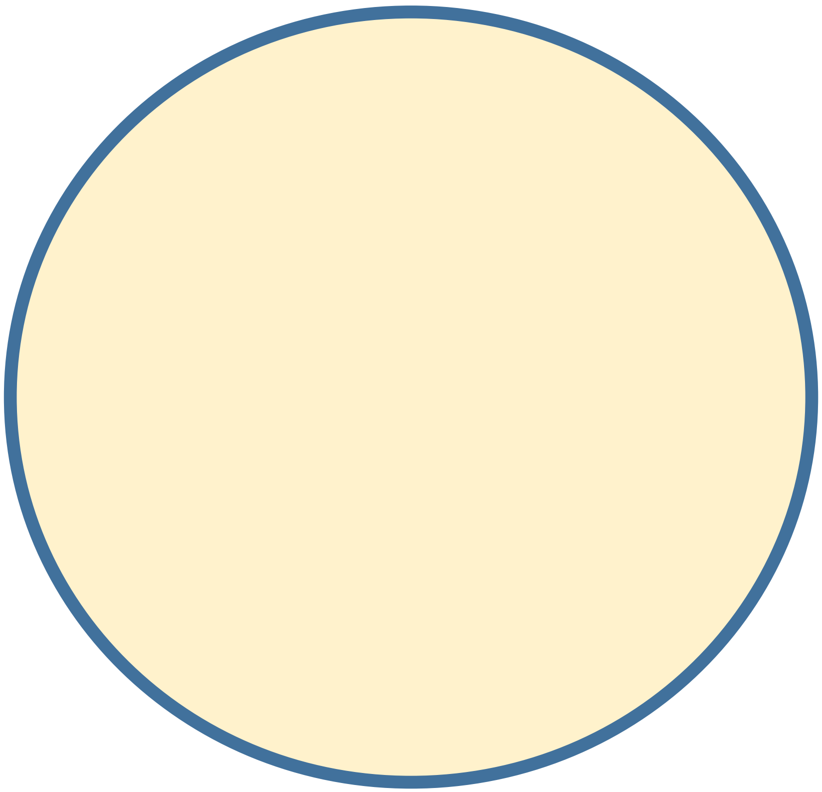


★ Conformal symmetry

Continuous Symmetry Group

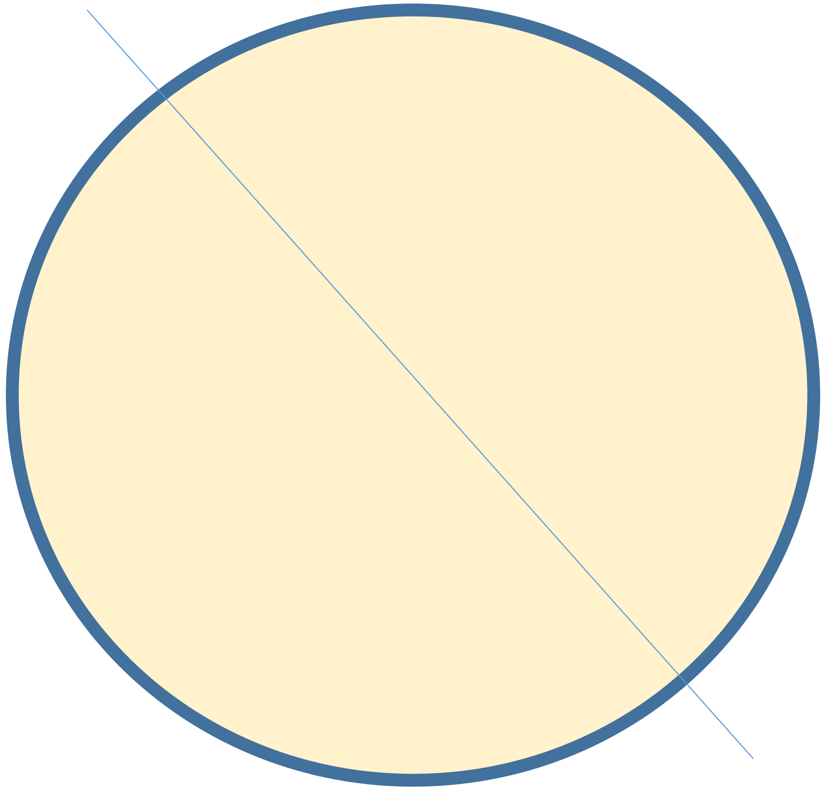


Continuous Symmetry Group



Rotations through any angle

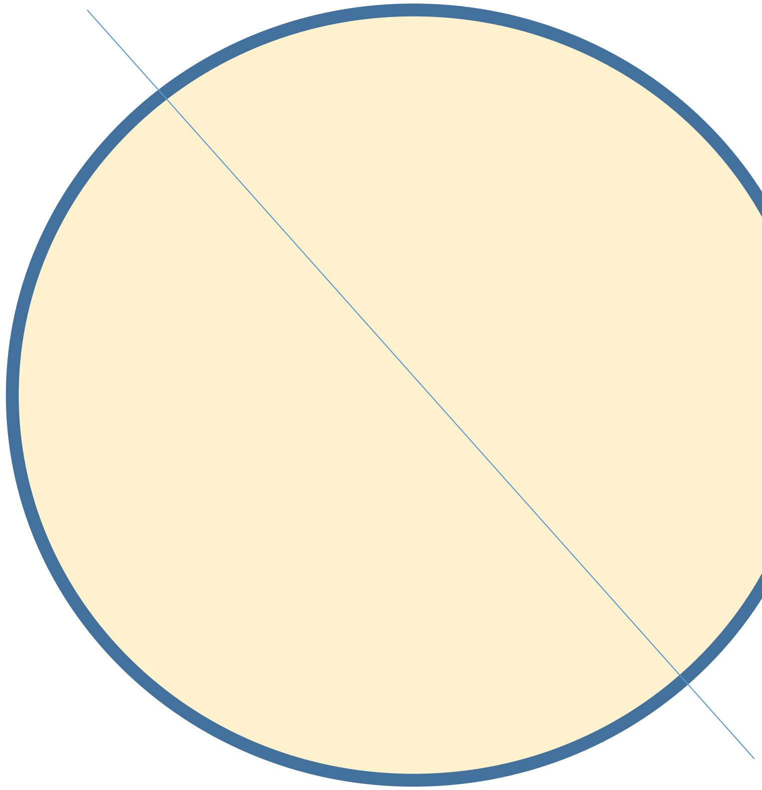
Continuous Symmetry Group



Rotations through any angle

and reflections

Continuous Symmetry Group



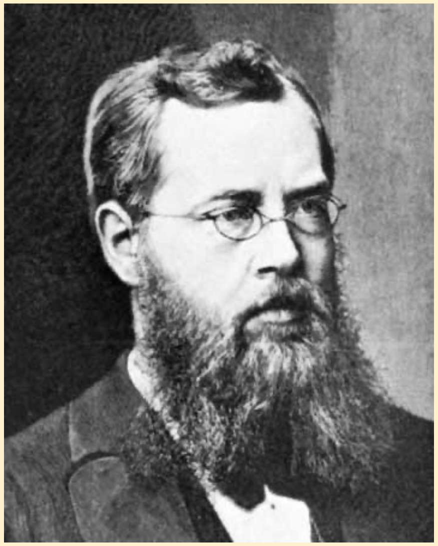
Rotations through any angle

and reflections

and conformal inversions

$$\overline{x} = \frac{x}{x^2 + y^2} \quad \overline{y} = \frac{y}{x^2 + y^2}$$

Continuous Symmetry Group = Lie Group



Rotations through any angle

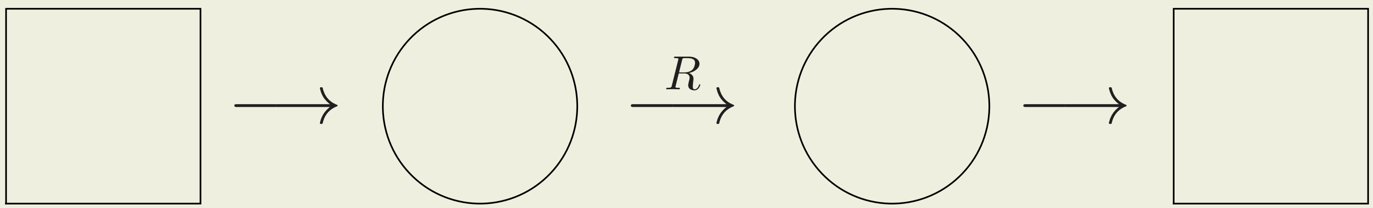
and reflections

and conformal inversions

$$\overline{x} = \frac{x}{x^2 + y^2} \qquad \overline{y} = \frac{y}{x^2 + y^2}$$

A continuous symmetry group is known as a Lie group in honor of the nineteenth century Norwegian mathematician Sophus Lie

Continuous Symmetries of a Square



Symmetry

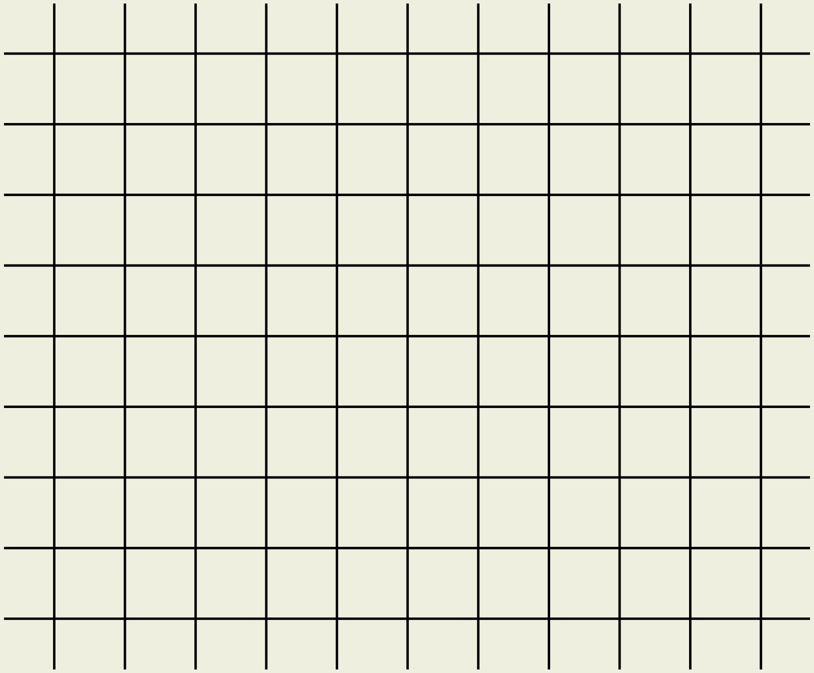
⋆ To define the set of symmetries requires a priori specification of the allowable transformations

G — transformation group containing all allowable transformations of the ambient space M

Definition. A symmetry of a subset $S \subset M$ is an allowable transformation $g \in G$ that preserves it:

$$g \cdot S = S$$

What is the Symmetry Group?

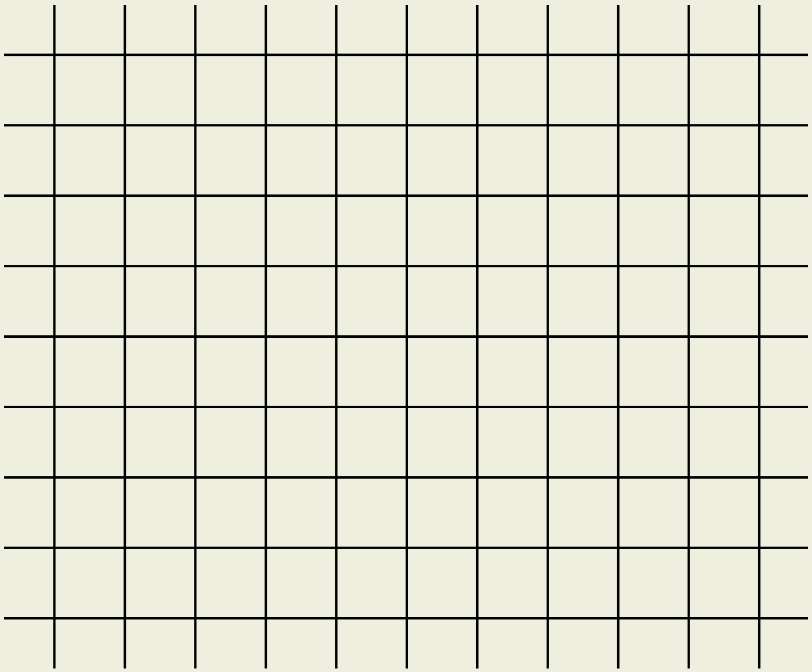


Allowable transformations:

Rigid motions

$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

What is the Symmetry Group?



Allowable transformations:

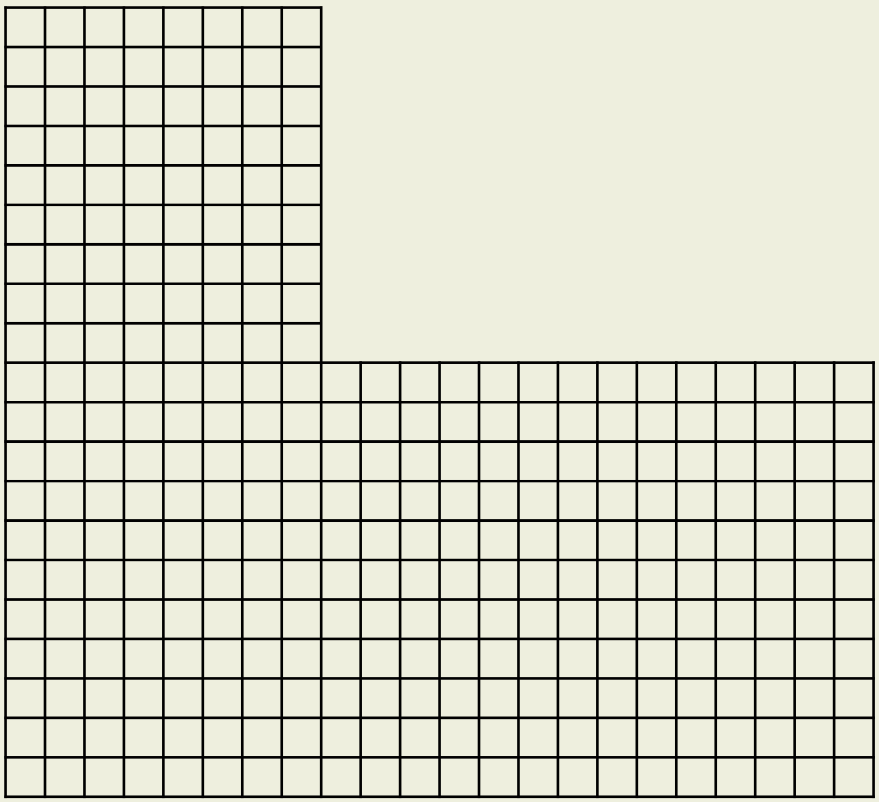
Rigid motions

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

Translations + rotations through 90 degrees:

$$G_S = \mathbb{Z}_4 \ltimes \mathbb{Z}^2$$

What is the Symmetry Group?



Allowable transformations:

Rigid motions

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

No symmetries!

$$G_S = \{e\}$$

Local Symmetries

Definition. $g \in G$ is a local symmetry of $S \subset M$ based at a point $z \in S$ if there is an open neighborhood $z \in U \subset M$ such that

$$g \cdot (S \cap U) = S \cap (g \cdot U)$$

Local Symmetries

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$$g \cdot (S \cap U) = S \cap (g \cdot U)$$

⋆⋆ The set of all local symmetries forms a groupoid!

Definition. A groupoid is a small category such that every morphism has an inverse.

- ⋆ Groupoids form the appropriate framework for studying objects with variable symmetry.
- ★ Symmetry groupoids are not necessarily Lie groupoids

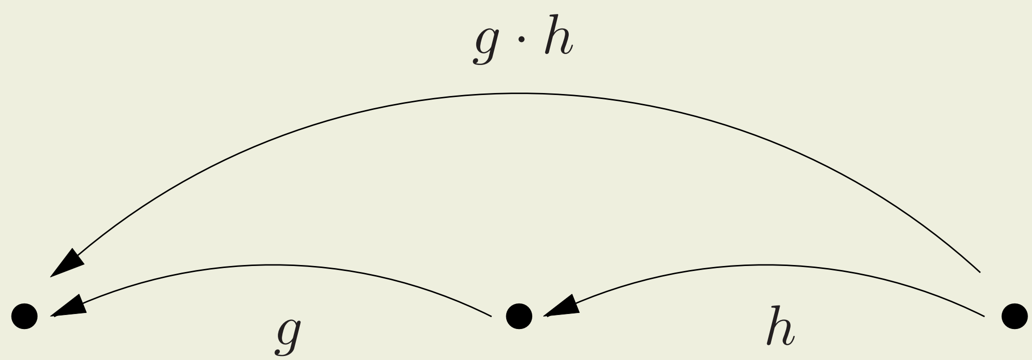
Groupoids

$\Longrightarrow$ In practice you are only allowed to multiply groupoid elements $g \cdot h$ when

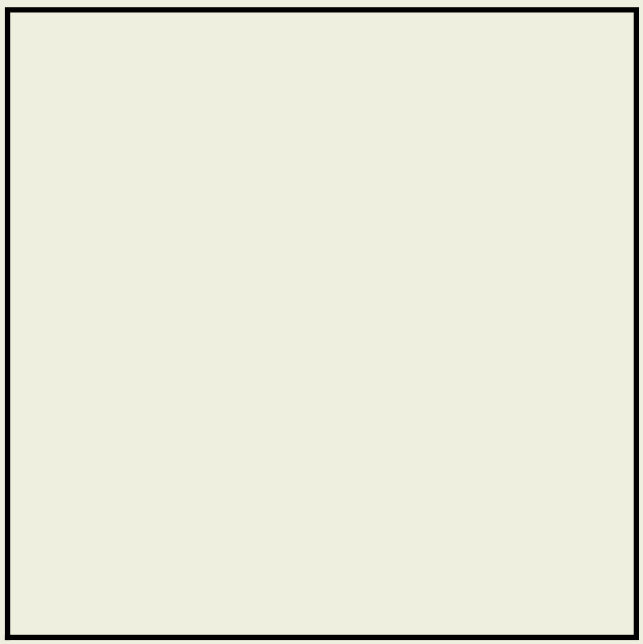
source (domain) of g = target (range) of h

Similarly for inverses g^{-1} and the identities e.

A groupoid is a "collection of arrows":



What is the Symmetry Groupoid?



$G = \mathrm{SE}(2)$

Corners:

$$G_z = G_S = \mathbb{Z}_4$$

Sides: G_z generated by

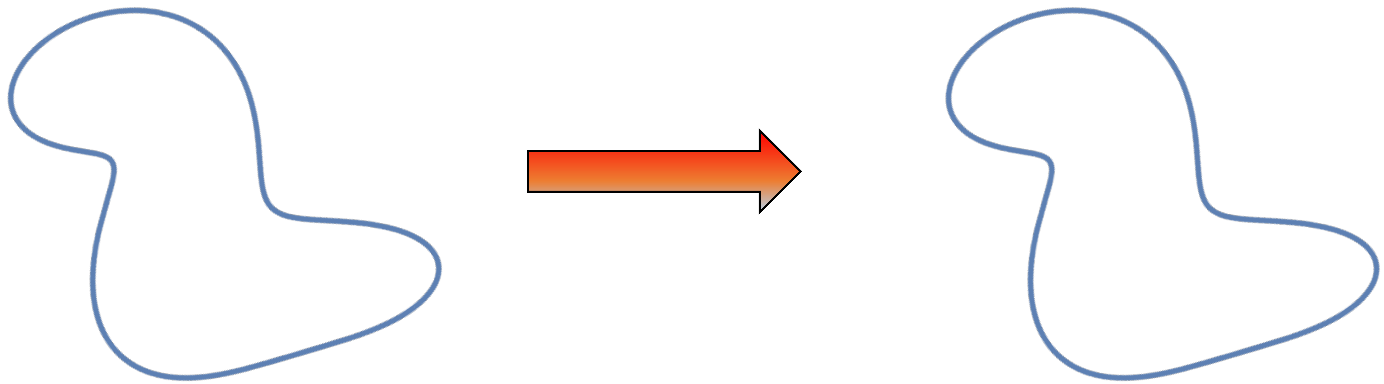
$G_S = \mathbb{Z}_4$

some translations

180° rotation around z

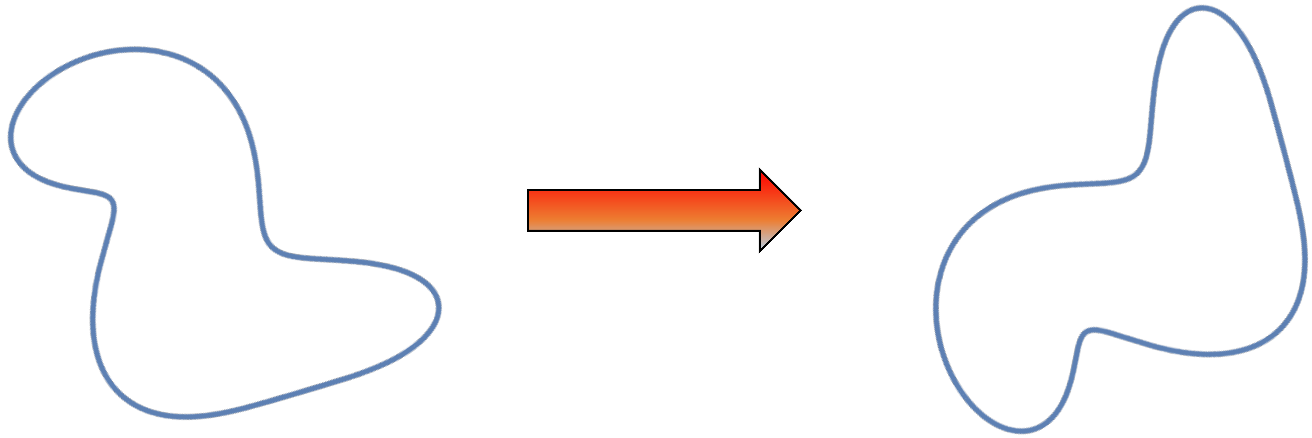
Geometric transformation groups

Translations

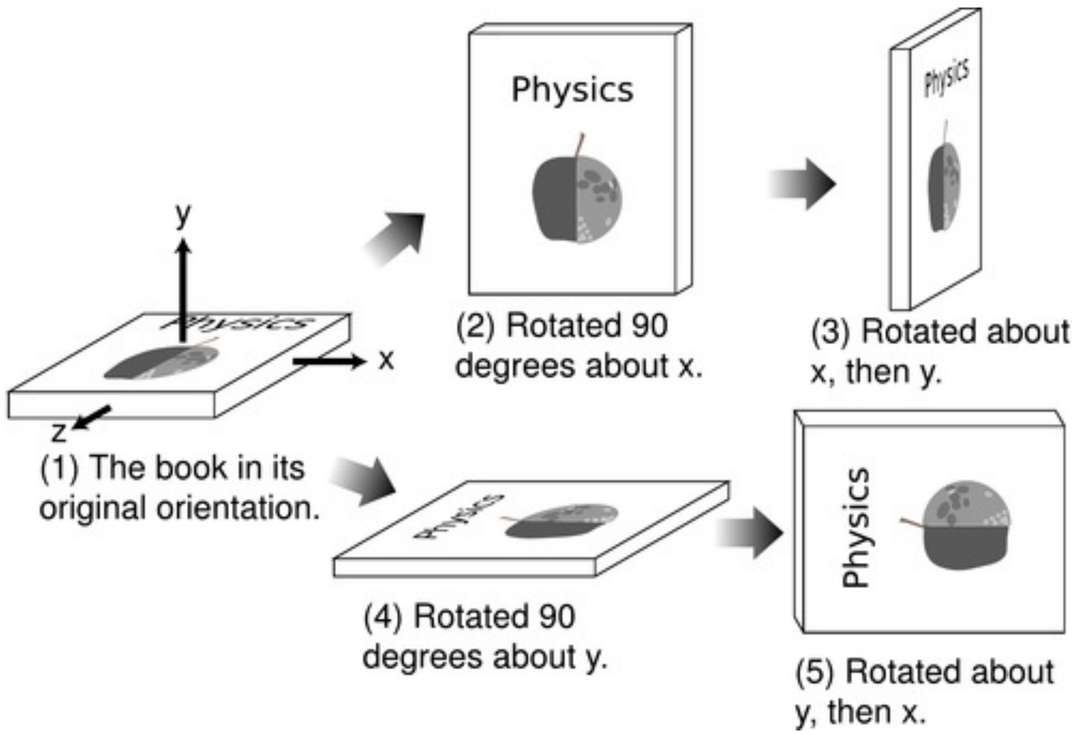


Geometric transformation groups

Rotations

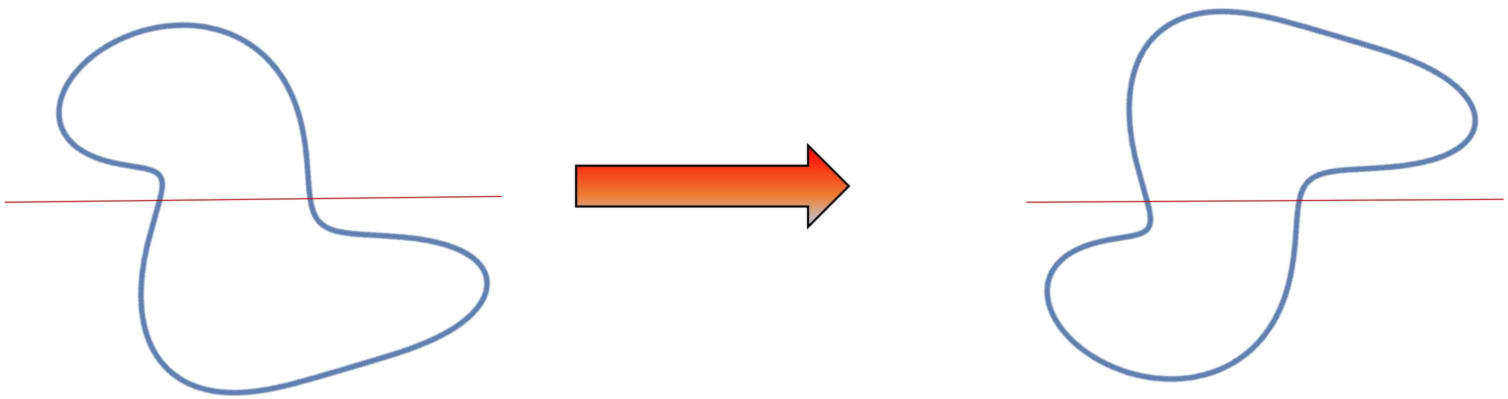


Noncommutativity of 3D rotations — order matters!



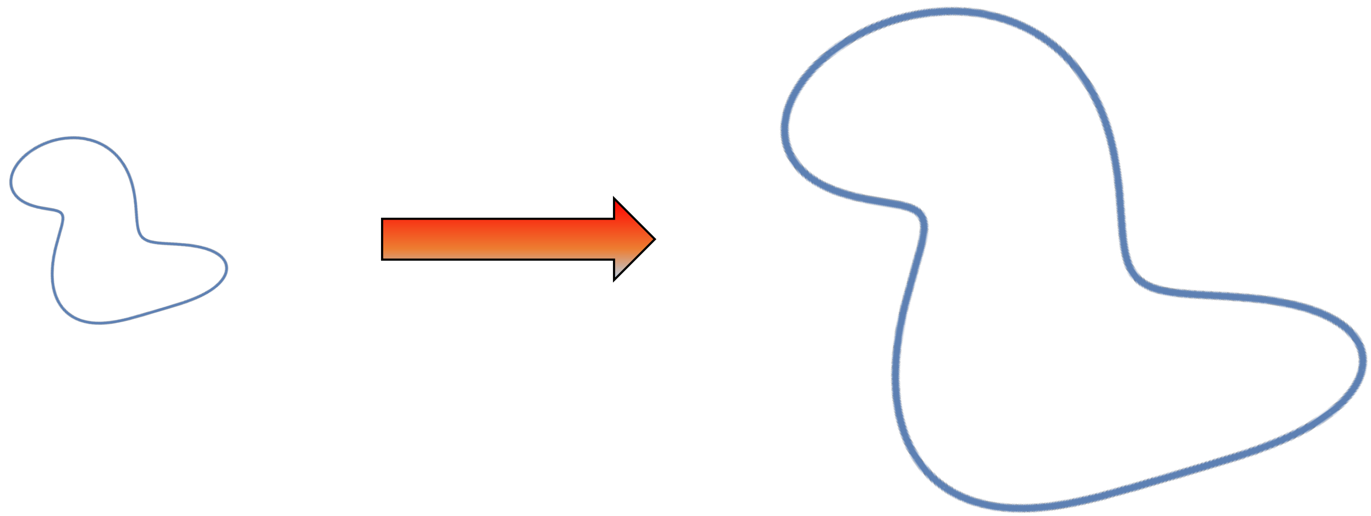
Geometric transformation groups

Reflections



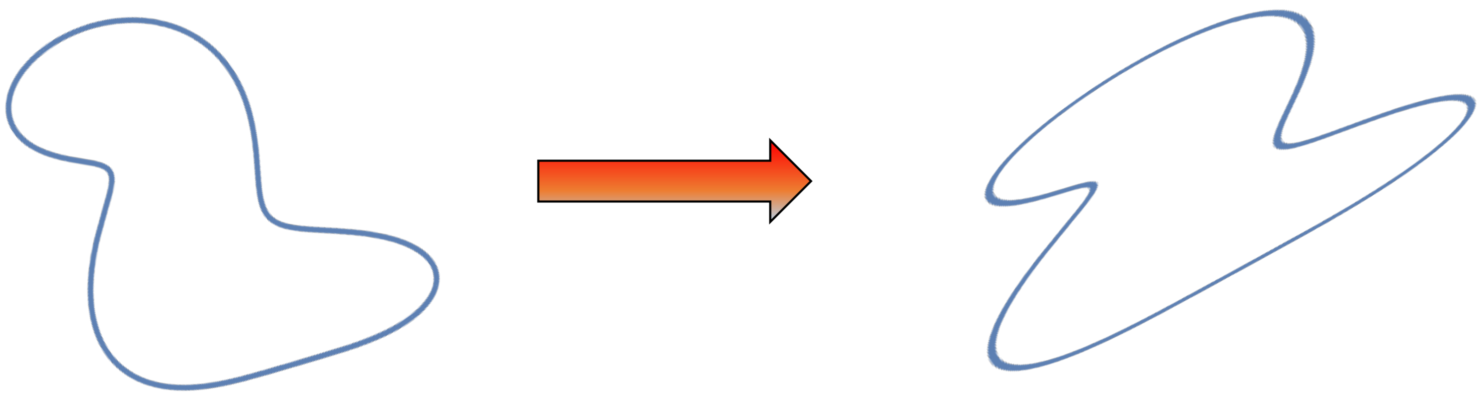
Geometric transformation groups

Scaling (similarity)



Geometric transformation groups

Projective and Equiaffine Transformations



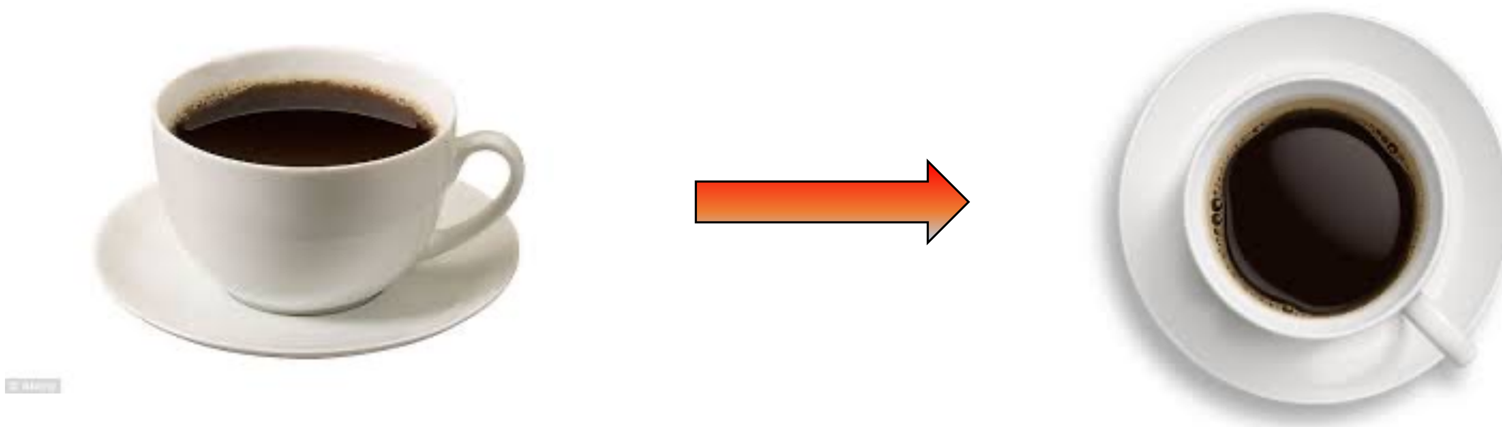
Geometric transformation groups

Projective Transformation

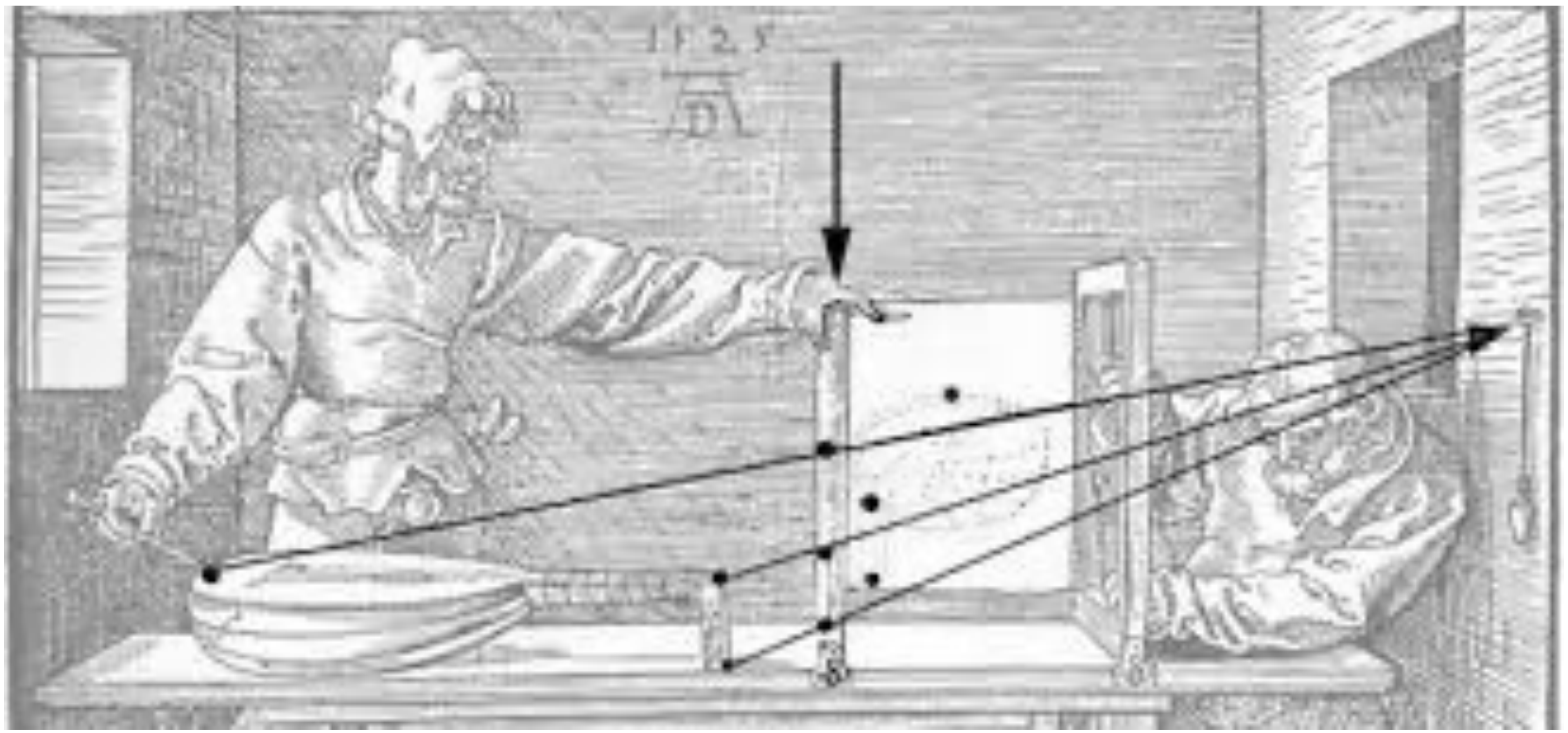
?

Geometric transformation groups

Projective Transformation

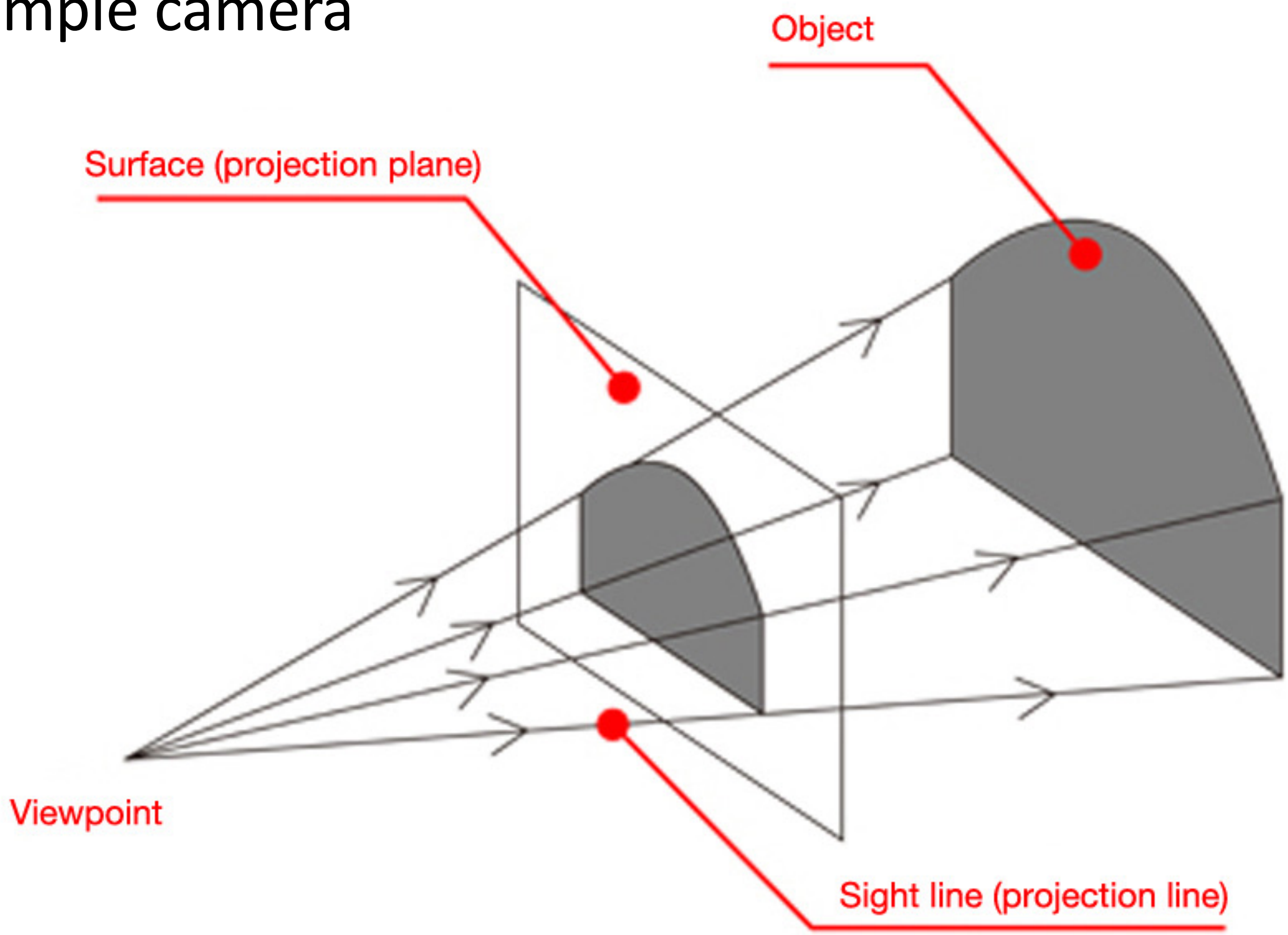


Projective transformations in art and photography



Albrecht Durer — 1500

A simple camera



Musashino Art University

Geometry = Group Theory

Felix Klein's Erlanger Programm (1872):

Each type of geometry is founded on a corresponding transformation group.

Geometry = *Group Theory*

Felix Klein's Erlanger Programm (1872):

Each type of geometry is founded on a corresponding transformation group.

Euclidean geometry:	rigid motions (translations and rotations)
"Mirror" geometry:	translations, rotations, and reflections
Similarity geometry:	translations, rotations, reflections, and scalings
Projective geometry:	all projective transformations

The Equivalence Problem

When are two shapes related by a group transformation?

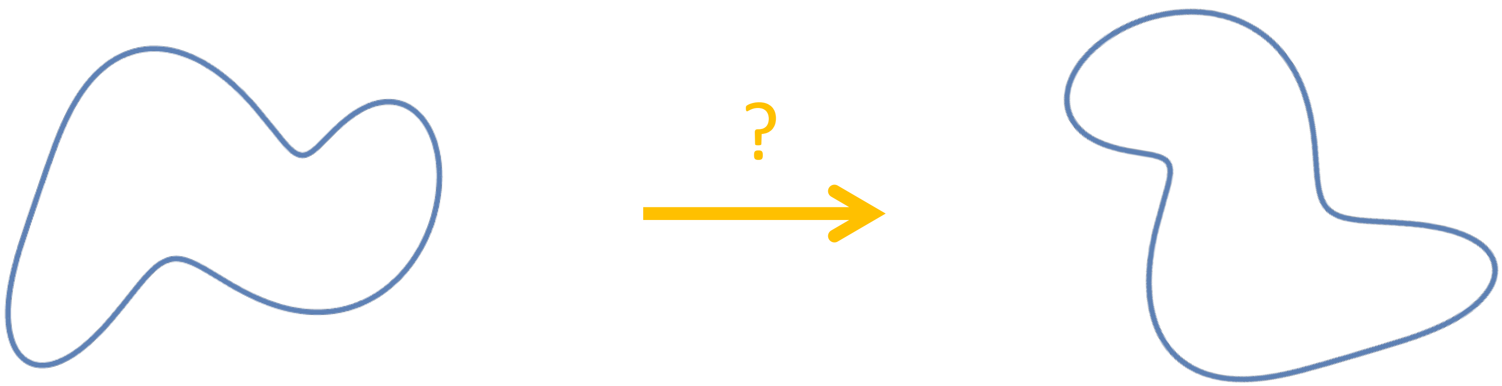
The Equivalence Problem

When are two shapes related by a group transformation?

- Rigid (Euclidean) equivalence (translations, rotations, reflections)
- Similarity equivalence
- Projective equivalence
- etc.

Rigid equivalence

When are two shapes related by a rigid motion?



Tennis, anyone?

?

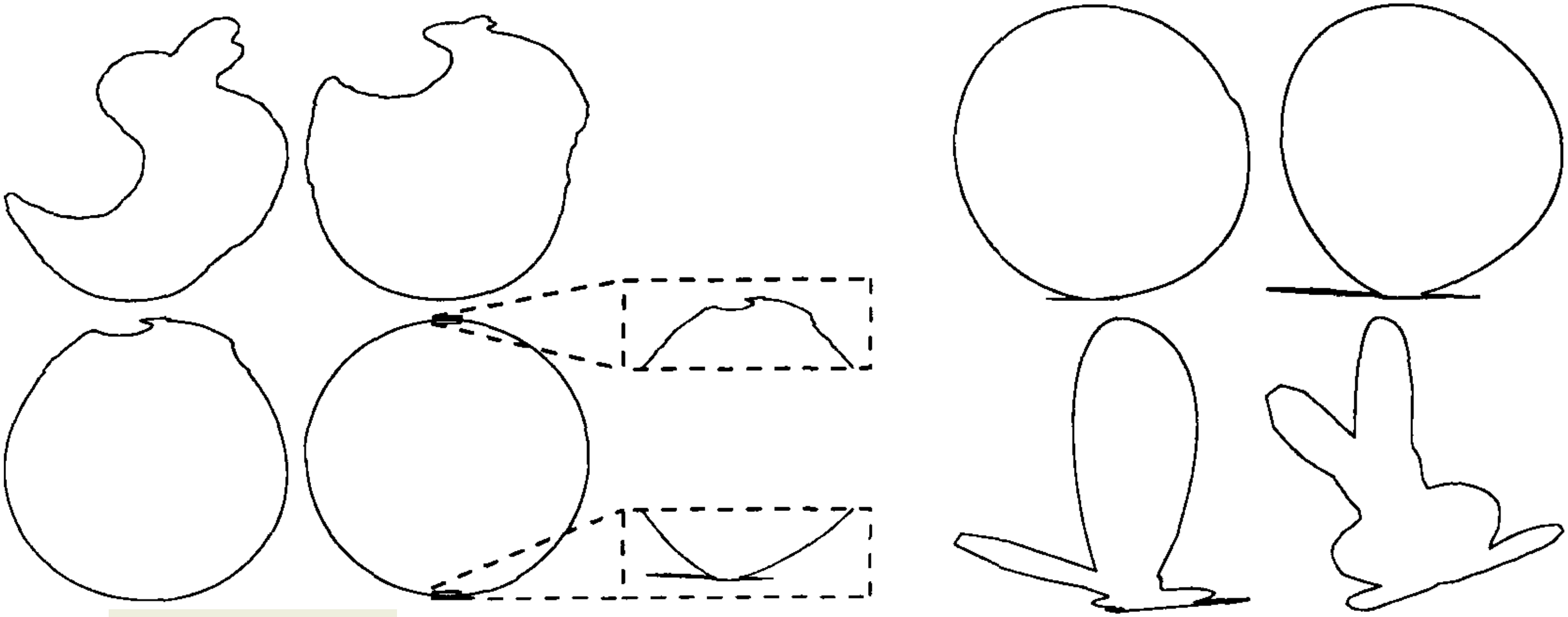
Tennis, anyone?

?

★ Projective (equiaffine) equivalence & symmetry

Duck = Rabbit?

Limitations of Projective Equivalence



$\Longrightarrow$ K. Åström (1995)

Limitations of Projective Equivalence

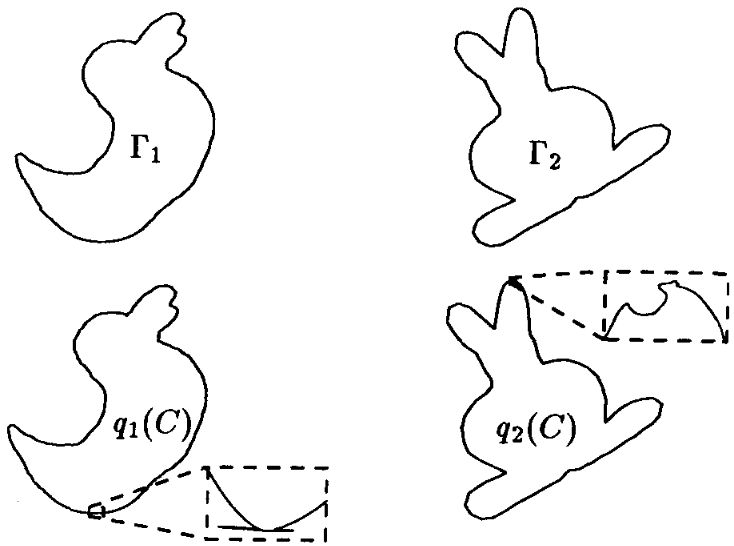
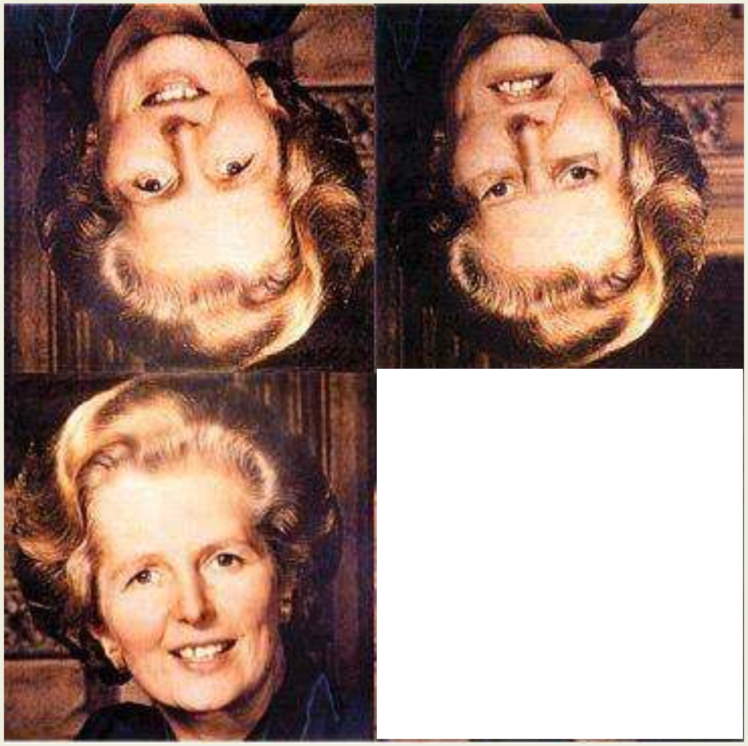


Fig. 3. The upper two curves are not projectively equivalent, but the lower two curves are. The lower curves are constructed by introducing small ripples along the convex hull, these are illustrated in the magnified pictures.

$\Longrightarrow$ K. Åström (1995)

Duck or Rabbit?

Thatcher Illusion

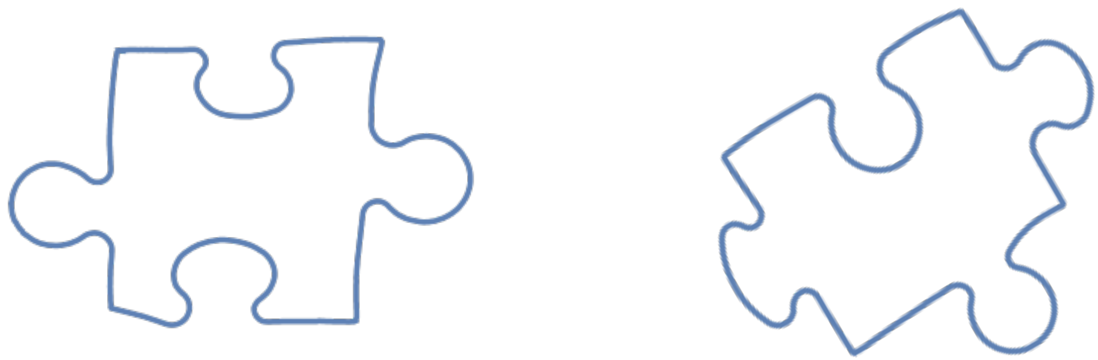


Thatcher Illusion

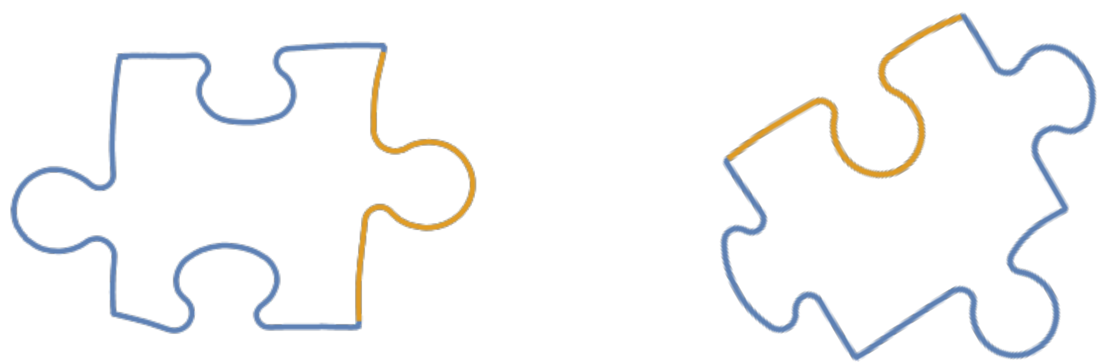
Thatcher Illusion

Local equivalence and symmetry — groupoids?
Probabilistic transformation group(oid)s?

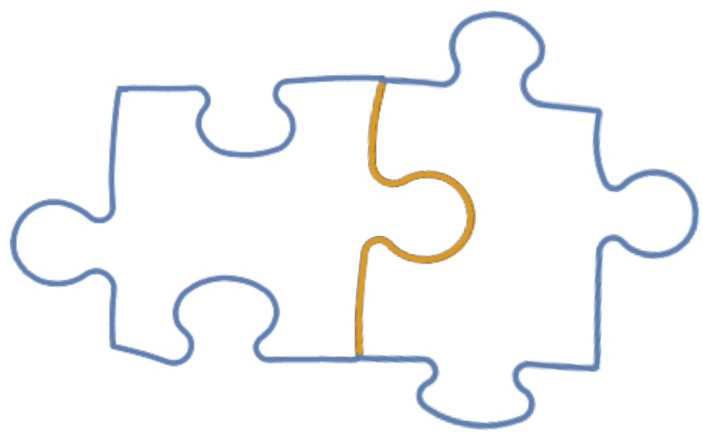
Equivalence of puzzle pieces



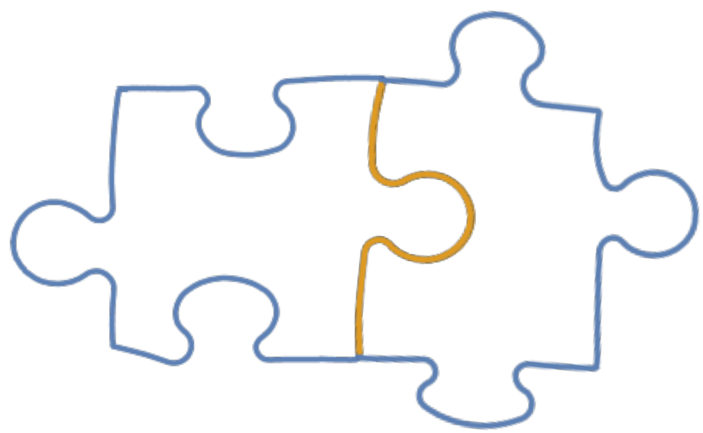
Local equivalence of puzzle pieces



Local equivalence of puzzle pieces



Local equivalence of puzzle pieces



- Occlusions and equivalence of parts

The Equivalence Problem

When are two shapes related by a group transformation?

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When are two shapes related by a group transformation?

Invariants

★ Solving the equivalence problem requires knowing the (appropriate) invariants

Invariants

Invariants are quantities that are unchanged by
the group transformations

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Invariants are quantities that are unchanged by the group transformations

- If two shapes are equivalent, they must have the same invariants.

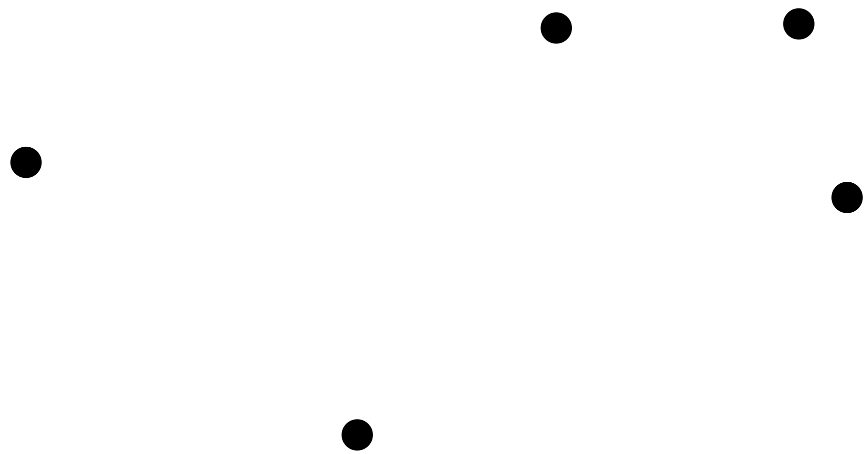
Invariants

The solution to an equivalence problem rests on understanding its invariants.

Definition. If G is a group acting on M, then an invariant is a real-valued function $I : M \to \mathbb{R}$ that does not change under the action of G:

$$I(g \cdot z) = I(z) \qquad \text{for all} \qquad g \in G, \quad z \in M$$

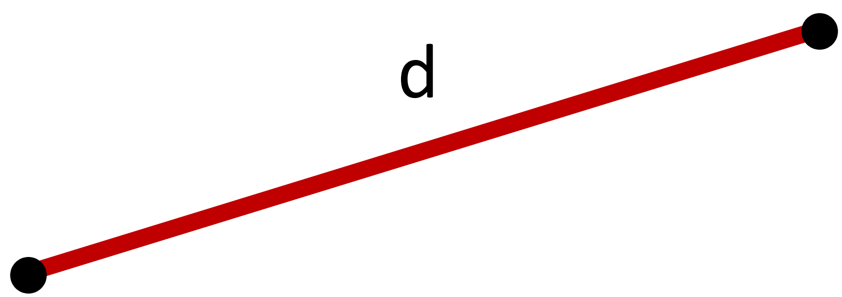
Joint invariants



An invariant that depends on several points is known as a joint invariant

Joint invariants

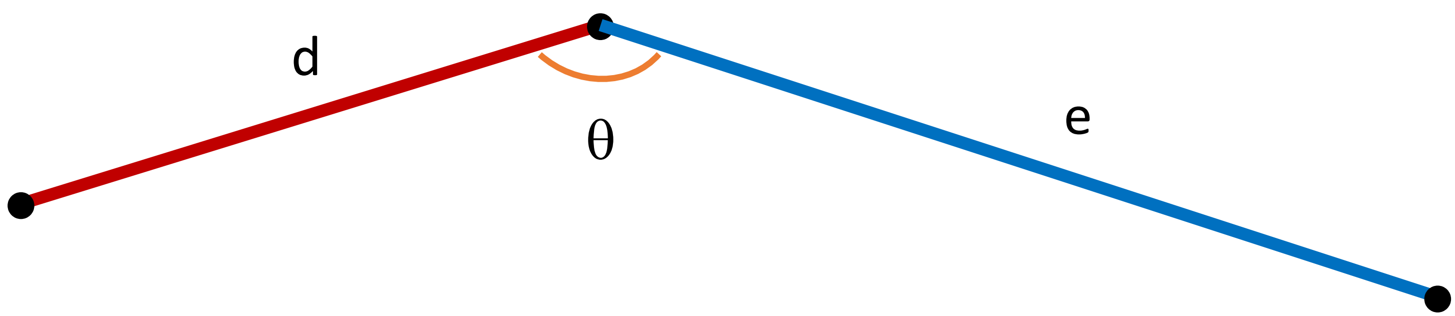
Rigid motions: distance between two points



Joint invariants

Similarity group:

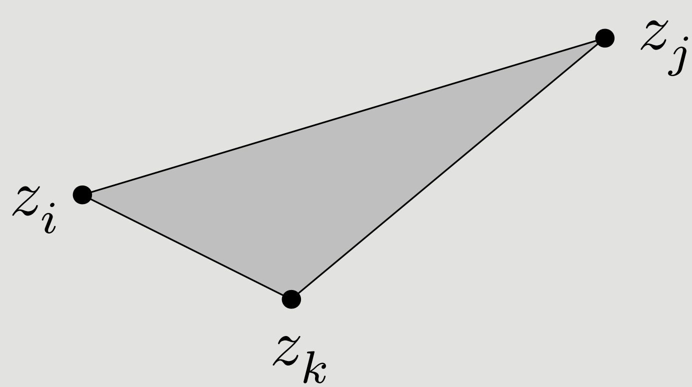
ratios of distances $R = d/e$ and angles θ



Joint Equi–Affine Invariants

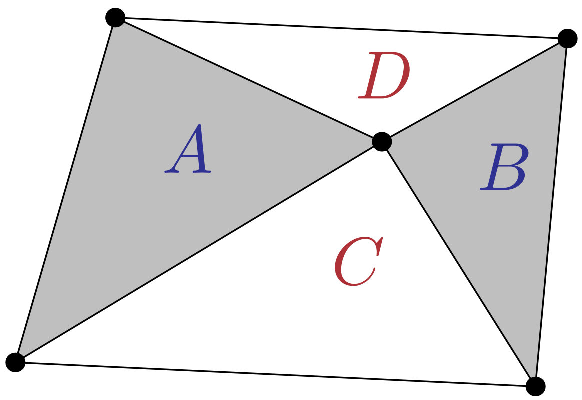
Theorem. Every planar joint equi–affine invariant is a function of the triangular areas

$$[\, i \; j \; k \,] = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$



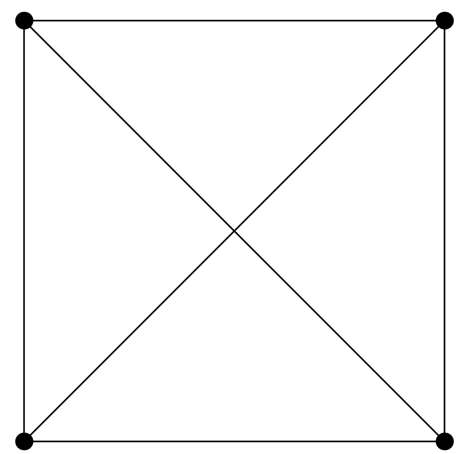
Joint invariants

Projective group: ratios of 4 areas



$$\frac{A\,B}{C\,D}$$

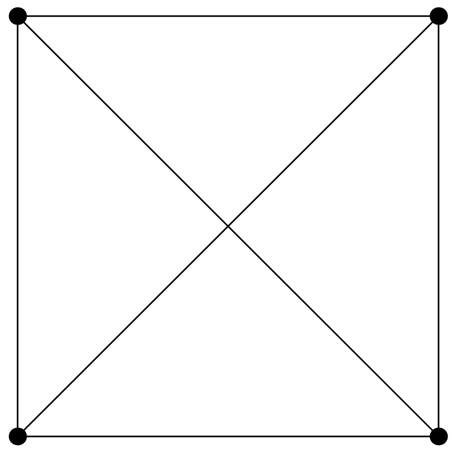
Distances between multiple points



$$1, \quad 1, \quad 1, \quad 1, \quad \sqrt{2}, \quad \sqrt{2}.$$

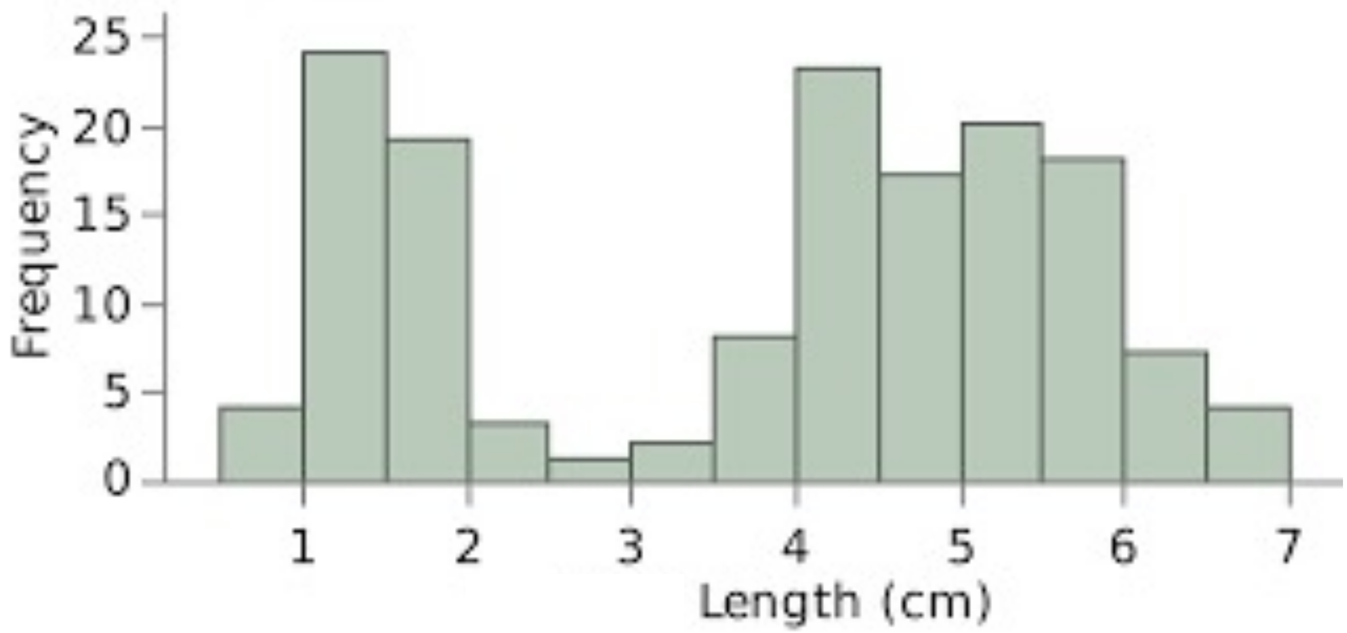
The Distance Histogram —

invariant under rigid motions



$$1, \quad 1, \quad 1, \quad 1, \quad \sqrt{2}, \quad \sqrt{2}.$$

Distance histograms



If two sets of points are equivalent up to rigid motion, they have the same distance histogram

If two sets of points are equivalent up to rigid motion, they have the same distance histogram

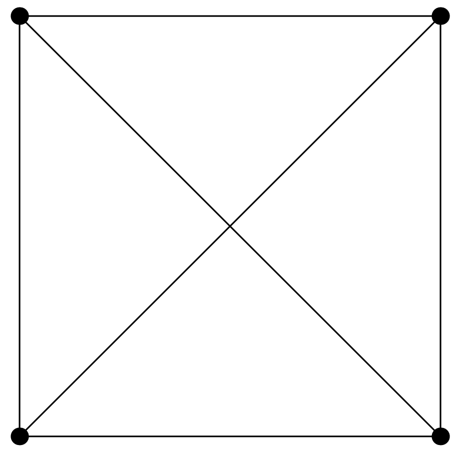
Does the distance histogram uniquely determine a set of points up to rigid motion?

Does the distance histogram uniquely determine a set of points up to rigid motion?

Answer: Yes for most sets of points, but there are some exceptions!

★ Mireille (Mimi) Boutin and Gregor Kemper (2004)

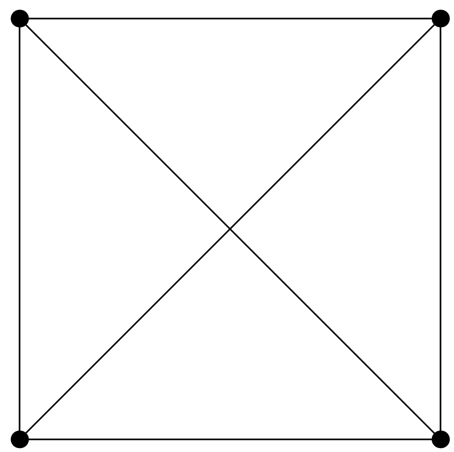
Does the distance histogram uniquely determine a set of points up to rigid motion?



$$1, \quad 1, \quad 1, \quad 1, \quad \sqrt{2}, \quad \sqrt{2}.$$

Does the distance histogram uniquely determine a set of points up to rigid motion?

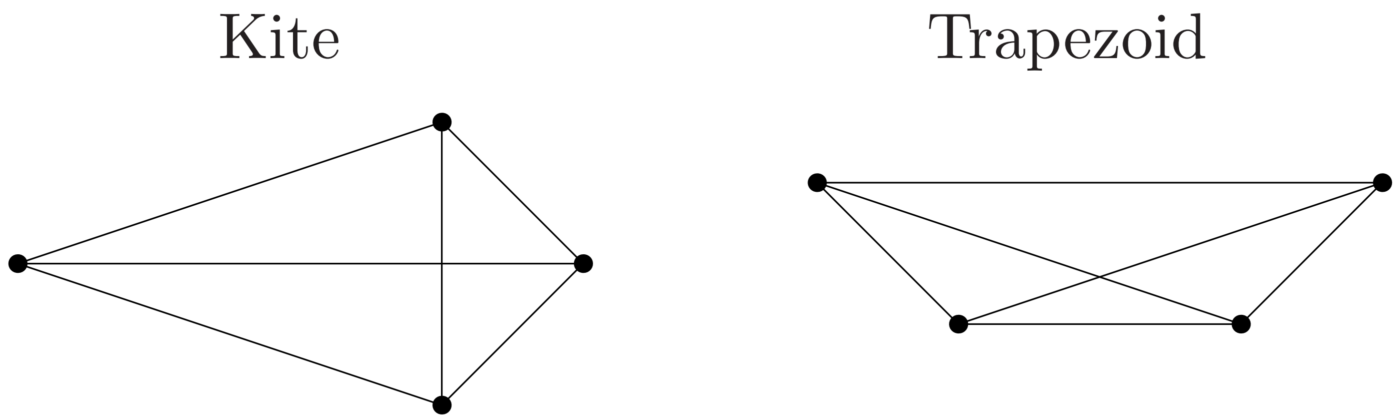
Yes:



$$1, \quad 1, \quad 1, \quad 1, \quad \sqrt{2}, \quad \sqrt{2}.$$

Does the distance histogram uniquely determine a set of points up to rigid motion?

No:



$$\sqrt{2}, \quad \sqrt{2}, \quad 2, \quad \sqrt{10}, \quad \sqrt{10}, \quad 4.$$

Distance histogram for points on a line

Does the distance histogram uniquely determine a set of points on a line up to translation?

Distance histogram for points on a line

No:

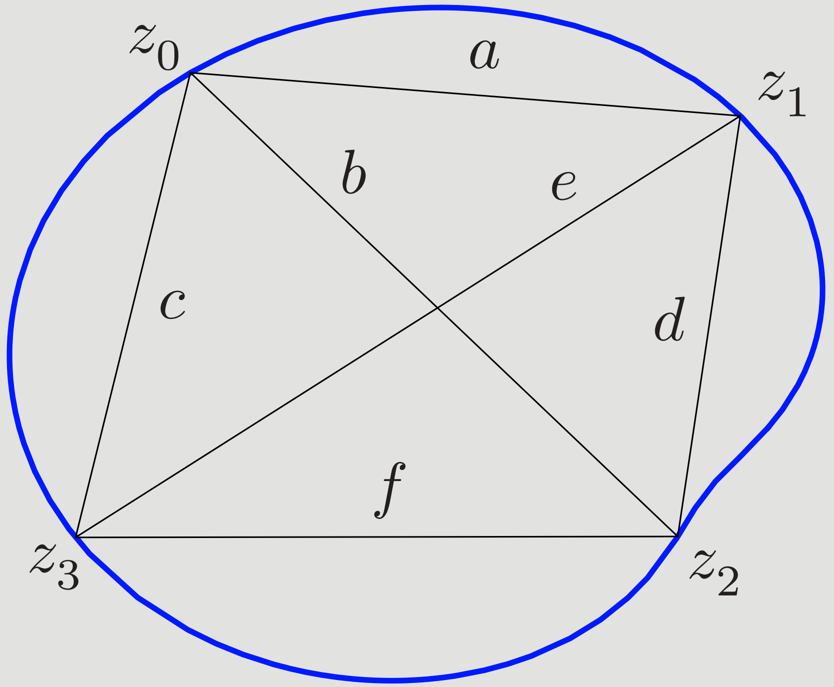
$$P = \{0, 1, 4, 10, 12, 17\}$$

$$Q = \{0, 1, 8, 11, 13, 17\}$$

$$\eta = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17$$

$\Longrightarrow$ G. Bloom, *J. Comb. Theory, Ser. A* **22** (1977) 378–379

Joint Euclidean Signature



Joint signature map:

$$\Sigma : \mathcal{C}^{\times 4} \longrightarrow \mathcal{S} \subset \mathbb{R}^6$$

$$a = \| z_0 - z_1 \| \qquad b = \| z_0 - z_2 \| \qquad c = \| z_0 - z_3 \|$$

$$d = \| z_1 - z_2 \| \qquad e = \| z_1 - z_3 \| \qquad f = \| z_2 - z_3 \|$$

$\Longrightarrow$ six functions of four variables

Syzygies:

$$\Phi_1(a, b, c, d, e, f) = 0 \qquad \Phi_2(a, b, c, d, e, f) = 0$$

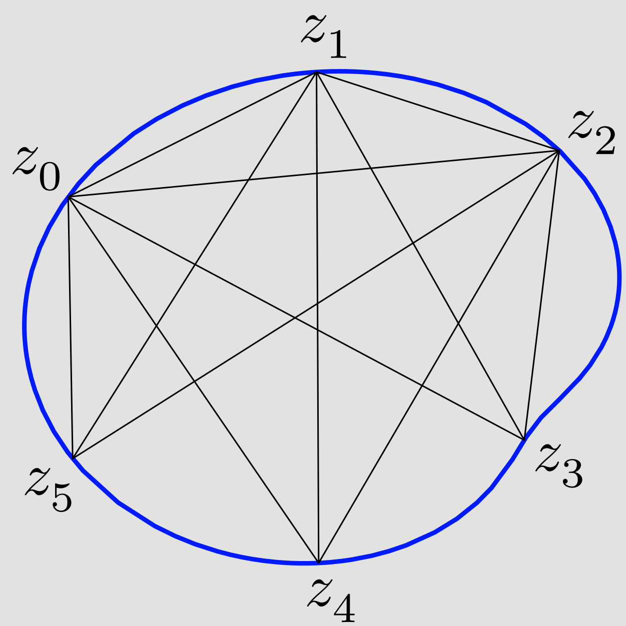
Universal Cayley–Menger syzygy $\quad \Longleftrightarrow \quad \mathcal{C} \subset \mathbb{R}^2$

$$\det \begin{vmatrix} 2\,a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\ a^2 + b^2 - d^2 & 2\,b^2 & b^2 + c^2 - f^2 \\ a^2 + c^2 - e^2 & b^2 + c^2 - f^2 & 2\,c^2 \end{vmatrix} = 0$$

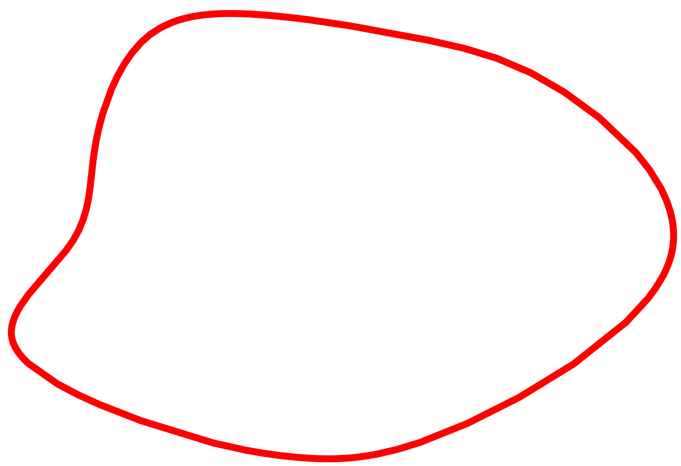
Joint Equi–Affine Signature

Requires 7 triangular areas:

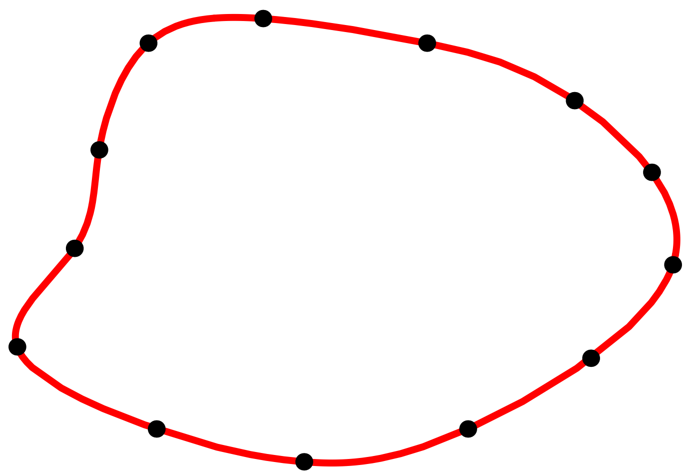
$[\,0\ 1\ 2\,],\ [\,0\ 1\ 3\,],\ [\,0\ 1\ 4\,],\ [\,0\ 1\ 5\,],\ [\,0\ 2\ 3\,],\ [\,0\ 2\ 4\,],\ [\,0\ 2\ 5\,]$



Limiting Curve Histogram

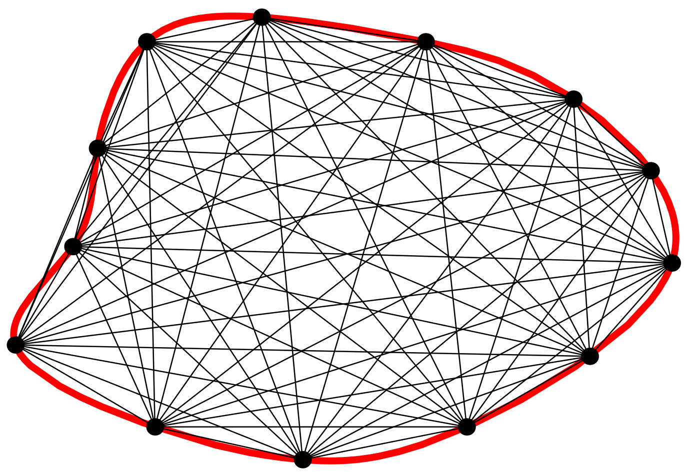


Limiting Curve Histogram



Integral Invariants

Limiting Curve Histogram

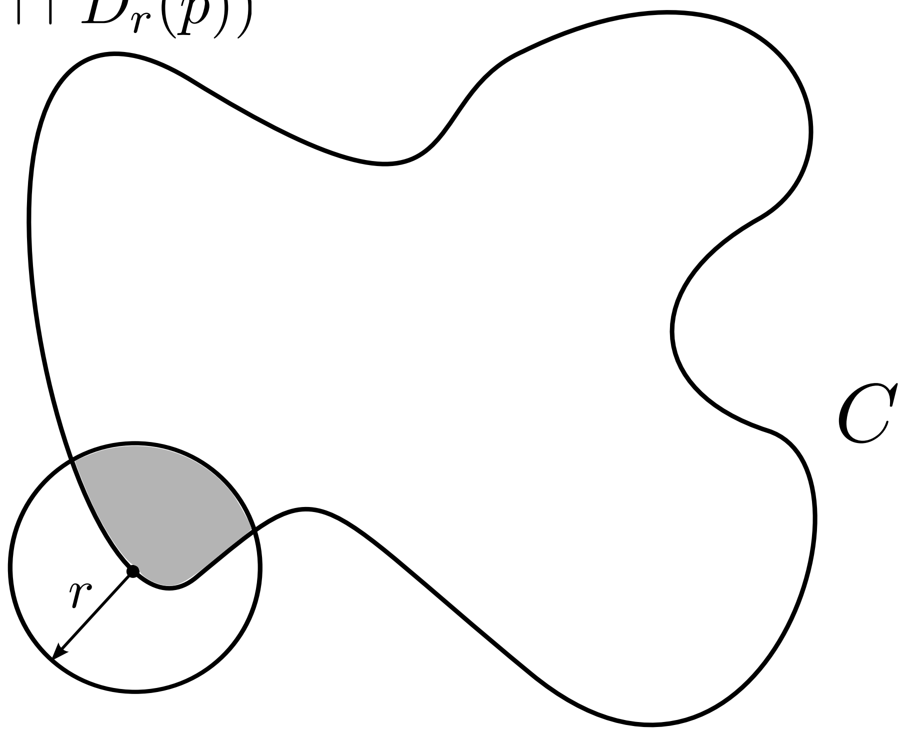


$$\frac{1}{l(C)^2}\int_C l(C\cap D_r(z(s)))\,ds$$

Brinkman, D., and Olver, P.J., Invariant histograms, Amer. Math. Monthly 119 (2012), 4-24

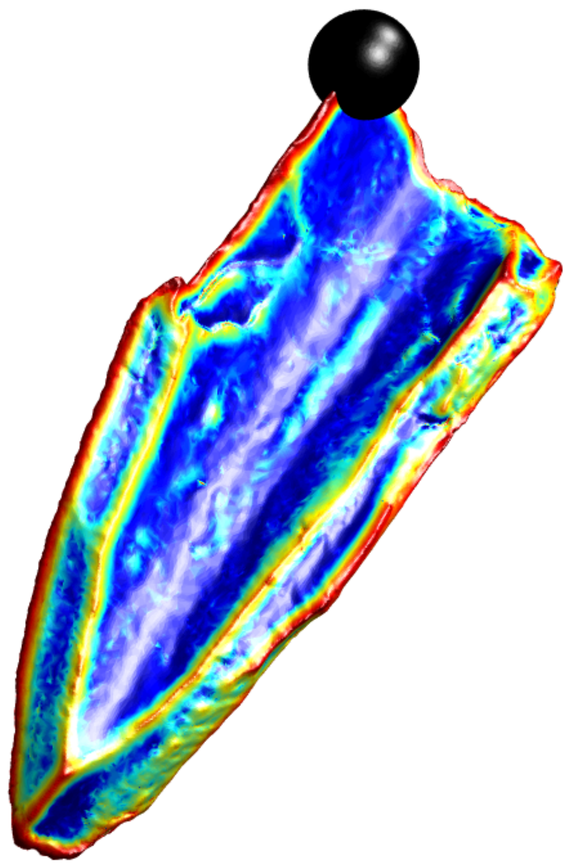
The Circular Area Invariant

$$A_{C,r}(p) = A(\text{int}\, C \cap D_r(p))$$

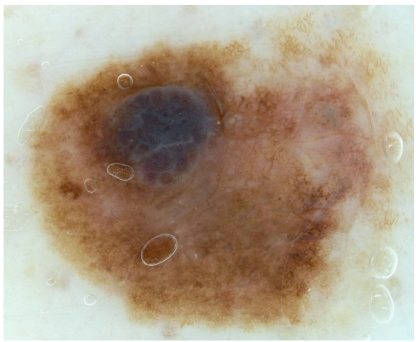
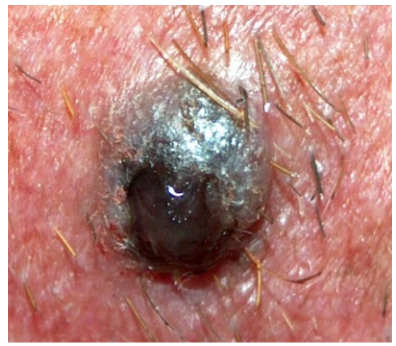


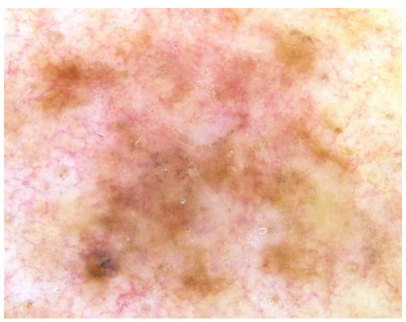
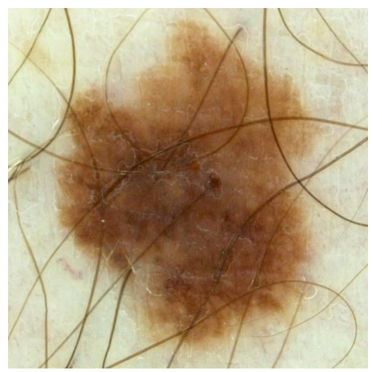
Calder and Esedoglu (2012)

The Spherical Volume Invariant



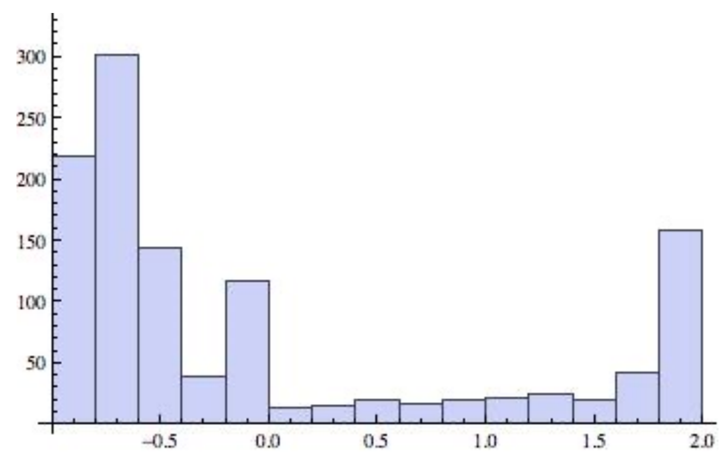
Distinguishing Moles from Melanomas

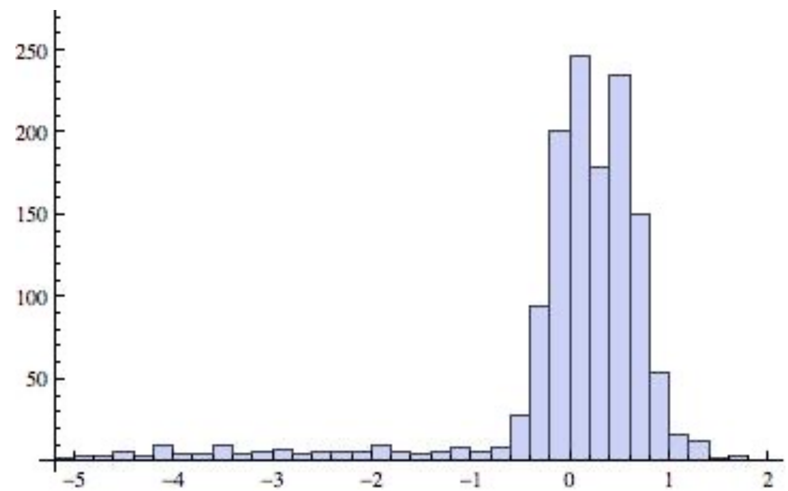




- Anna Grim and Cheri Shakiban, 2015

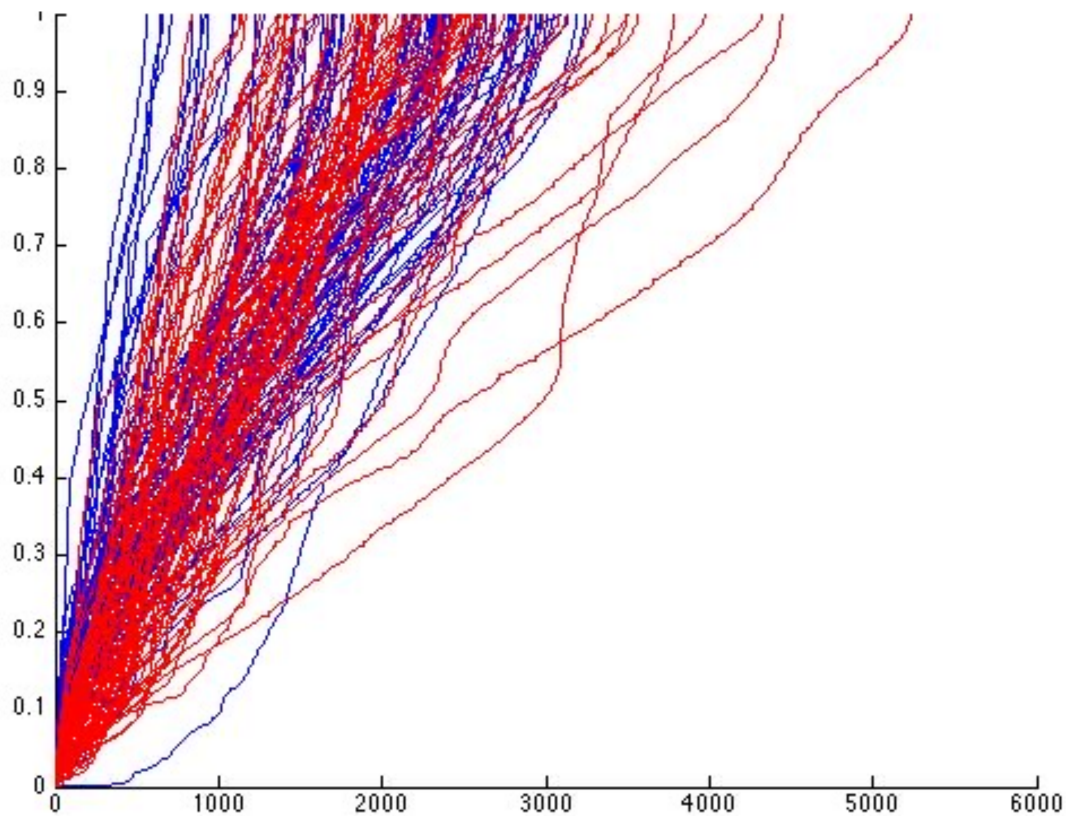
Distance Histogram — Melanoma



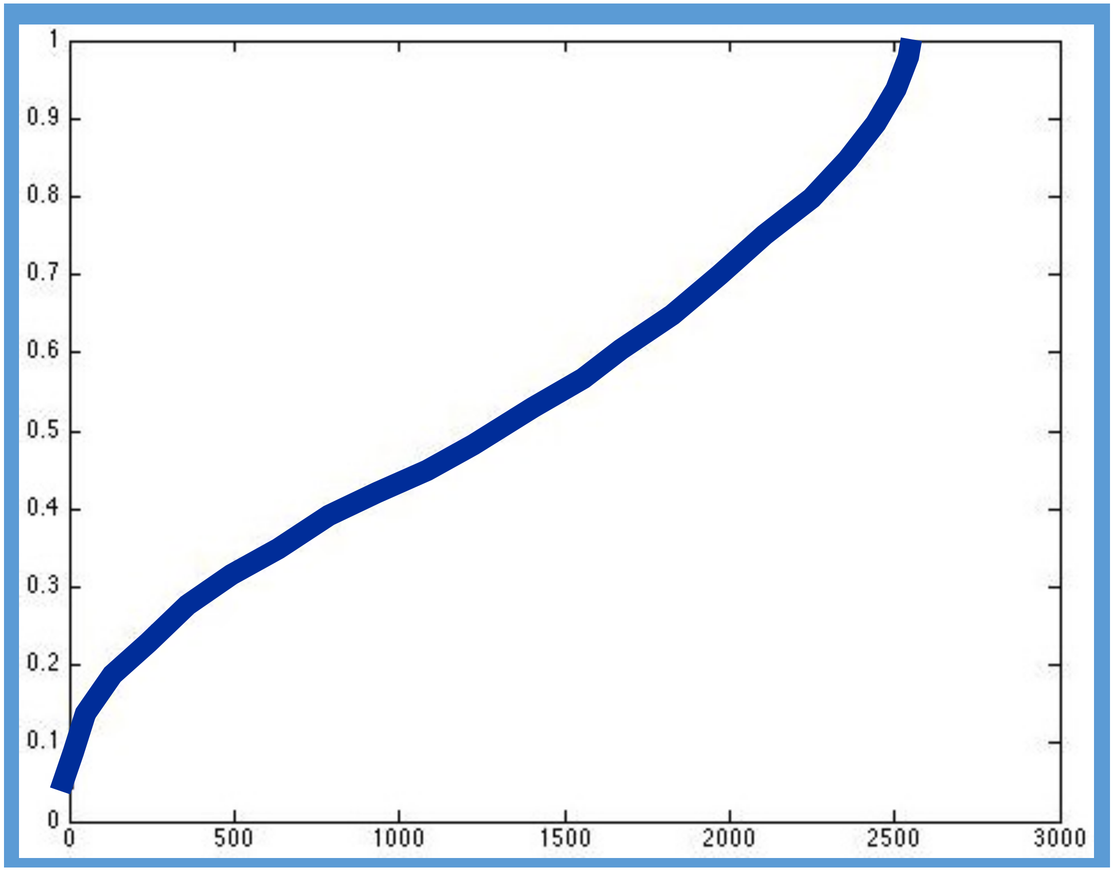
Distance Histogram — Mole



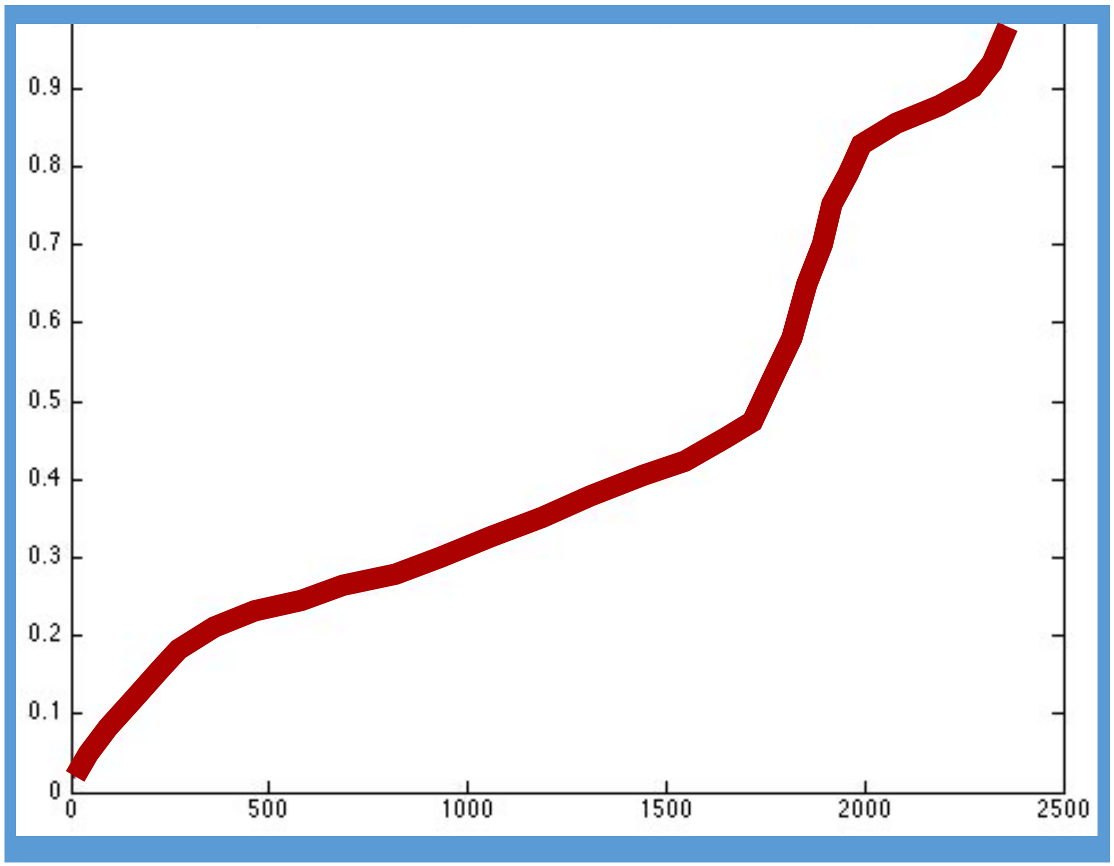
CUMULATIVE HISTOGRAM:
Mole versus Melanoma



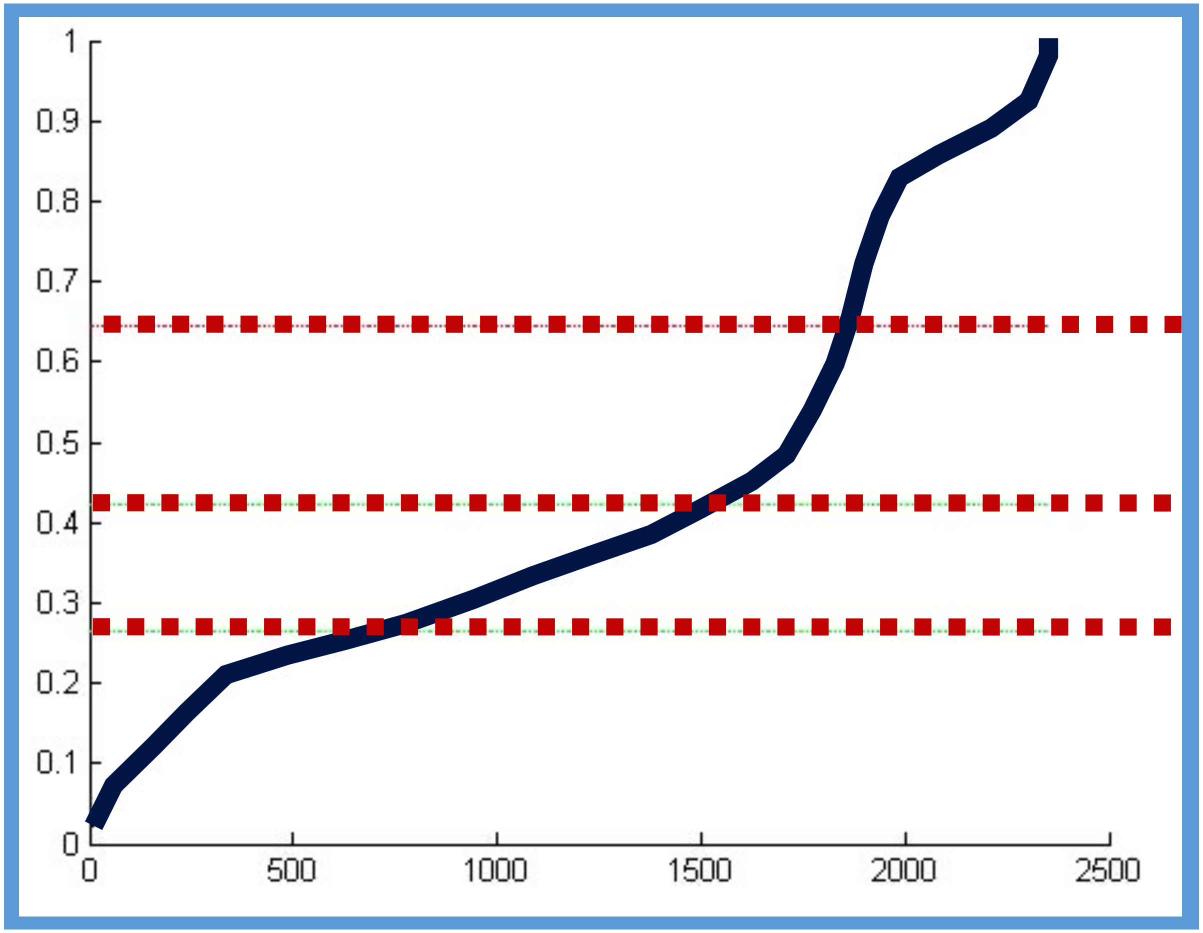
TYPICAL MOLE CUMULATIVE HISTOGRAM

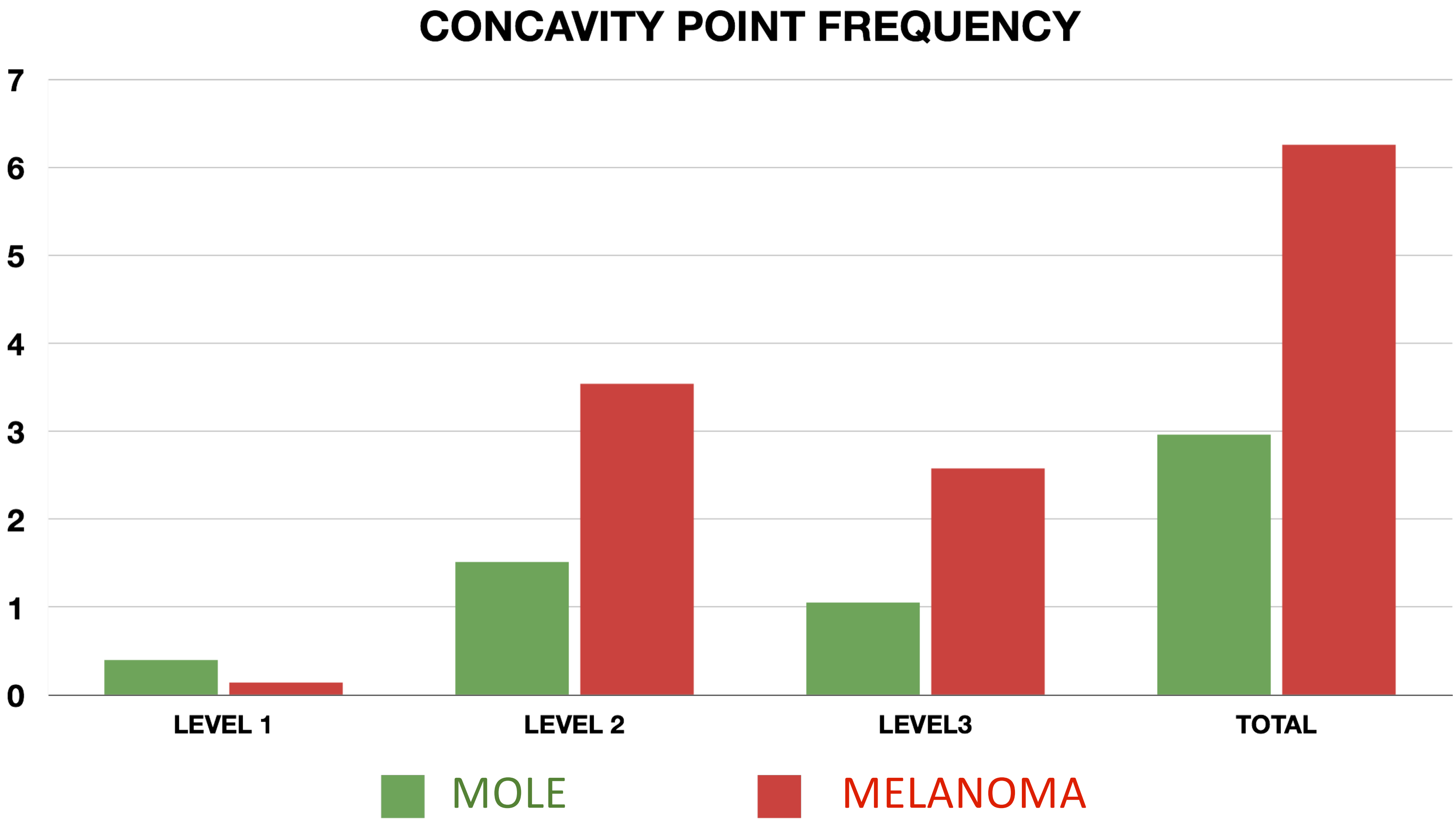


TYPICAL MELANOMA CUMULATIVE HISTOGRAM



CONCAVITY POINT ANALYSIS




CONCAVITY POINT FREQUENCY
7
6
5
4
3
2
1
0
LEVEL 1
LEVEL 2
LEVEL3
TOTAL
MOLE
MELANOMA

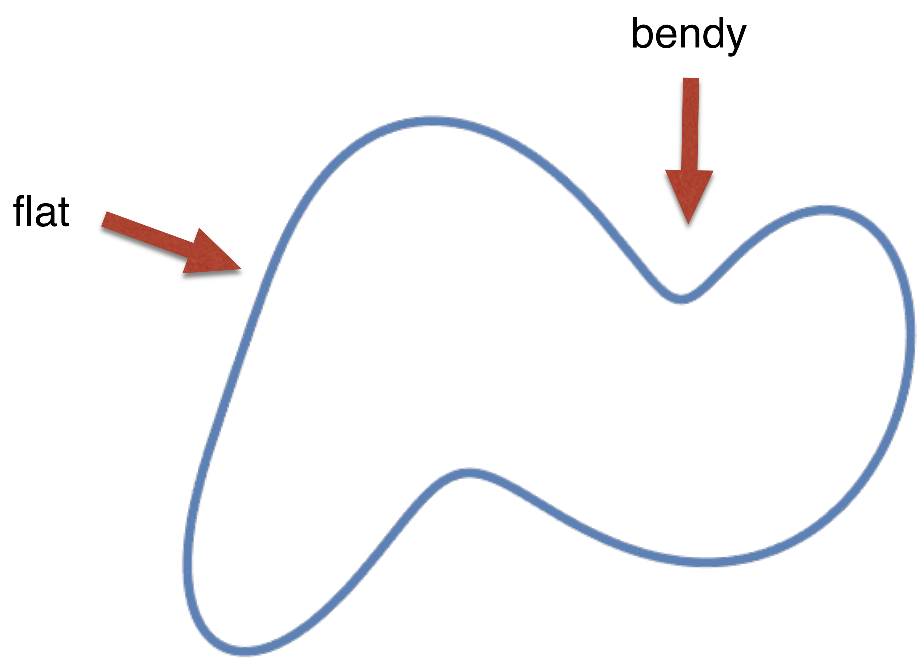
For smooth objects — curves, surfaces, etc.,

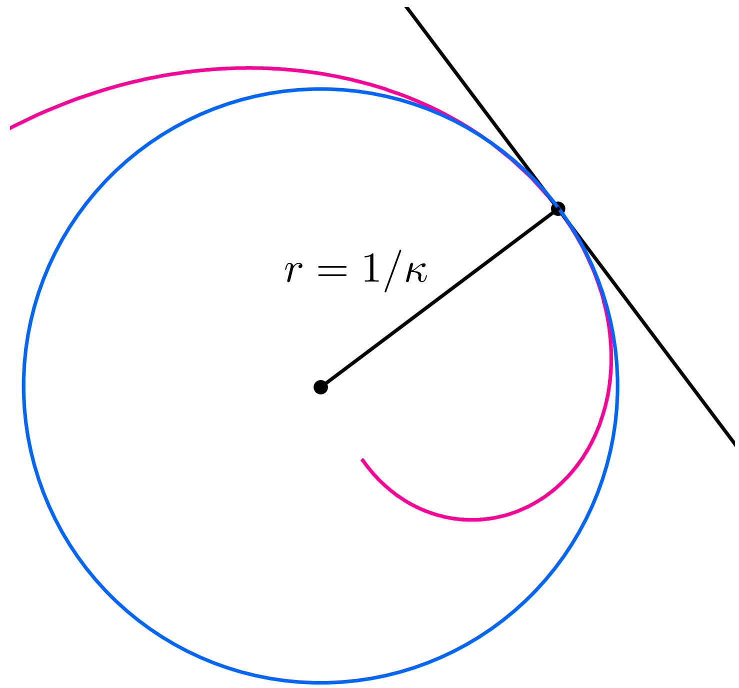
we can use calculus to construct

Differential Invariants

A Differential Invariant

Curvature is a measure of "bendiness".





Curvature = reciprocal of radius of osculating circle

Euclidean Plane Curves: $G = \mathrm{SE}(2)$

Differentiation with respect to the Euclidean-invariant arc length element ds is an invariant differential operator, meaning that it maps differential invariants to differential invariants.

Thus, starting with curvature κ, we can generate an infinite collection of higher order Euclidean differential invariants:

$$\kappa, \qquad \frac{d\kappa}{ds}, \qquad \frac{d^2\kappa}{ds^2}, \qquad \frac{d^3\kappa}{ds^3}, \qquad \cdots$$

Theorem. All Euclidean differential invariants are functions of the derivatives of curvature with respect to arc length: $\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots$

Euclidean Plane Curves: $G = \mathrm{SE}(2)$

Assume the curve $C \subset M$ is a graph: $\quad y = u(x)$

Differential invariants:

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \qquad \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}, \qquad \frac{d^2\kappa}{ds^2} = \cdots$$

Arc length (invariant one-form):

$$ds = \sqrt{1+u_x^2}\; dx, \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Similarity Plane Curves: $G = \mathrm{SE}(2) \times \mathbb{R}$

Similarity "curvature":

$$\widehat{\kappa} = \frac{\kappa_s}{\kappa^2} \qquad \widehat{\kappa}_{\hat{s}} = \cdots$$

Similarity arc length:

$$d\,\hat{s} = \kappa\,ds \qquad \frac{d}{d\hat{s}} = \frac{1}{\kappa}\,\frac{d}{ds}$$

Theorem. All similarity differential invariants are functions of the derivatives of the similarity curvature with respect to similarity arc length: $\widehat{\kappa}, \quad \widehat{\kappa}_{\hat{s}}, \quad \widehat{\kappa}_{\hat{s}\hat{s}}, \quad \cdots$

Equi-affine Plane Curves: $G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$

Equi-affine curvature:

$$\kappa = \frac{5\,u_{xx}u_{xxxx} - 3\,u_{xxx}^2}{9\,u_{xx}^{8/3}} \qquad \frac{d\kappa}{ds} = \cdots$$

Equi-affine arc length:

$$ds = \sqrt[3]{u_{xx}}\;dx \qquad \frac{d}{ds} = \frac{1}{\sqrt[3]{u_{xx}}}\,\frac{d}{dx}$$

Theorem. All equi-affine differential invariants are functions of the derivatives of equi-affine curvature with respect to equi-affine arc length: $\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots$

Projective Plane Curves: $G = \mathrm{PSL}(2)$

Projective curvature:

$$\kappa \ = K(u^{(7)},\ \cdots) \qquad \frac{d\kappa}{ds} = \ \cdots \qquad \frac{d^2\kappa}{ds^2} = \ \cdots$$

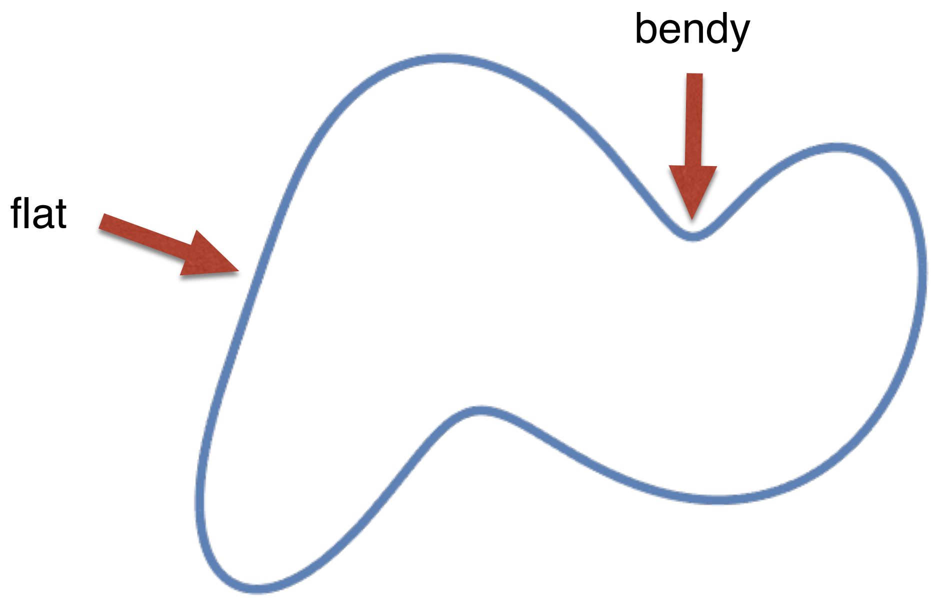
Projective arc length:

$$ds \ = \ L(u^{(5)},\ \cdots)\ dx \qquad \frac{d}{ds} = \frac{1}{L}\ \frac{d}{dx}$$

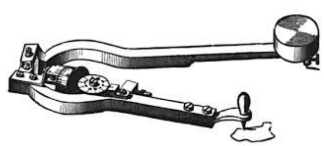
Theorem. All projective differential invariants are functions of the derivatives of projective curvature with respect to projective arc length:

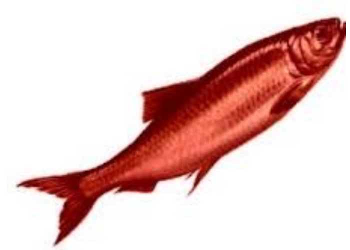
$$\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots$$

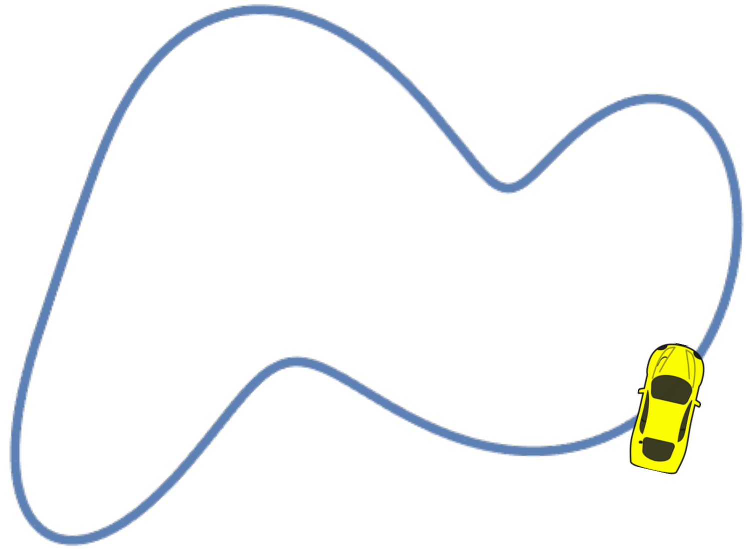
Euclidean Curvature is a measure of “bendiness”.

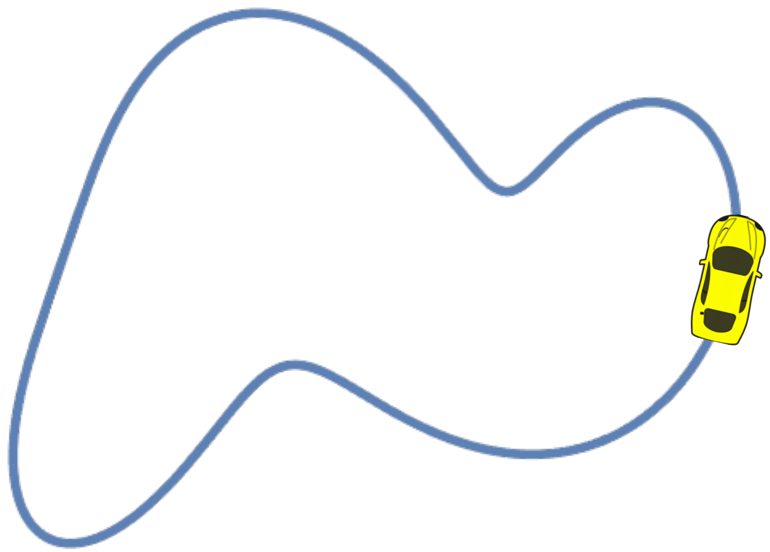


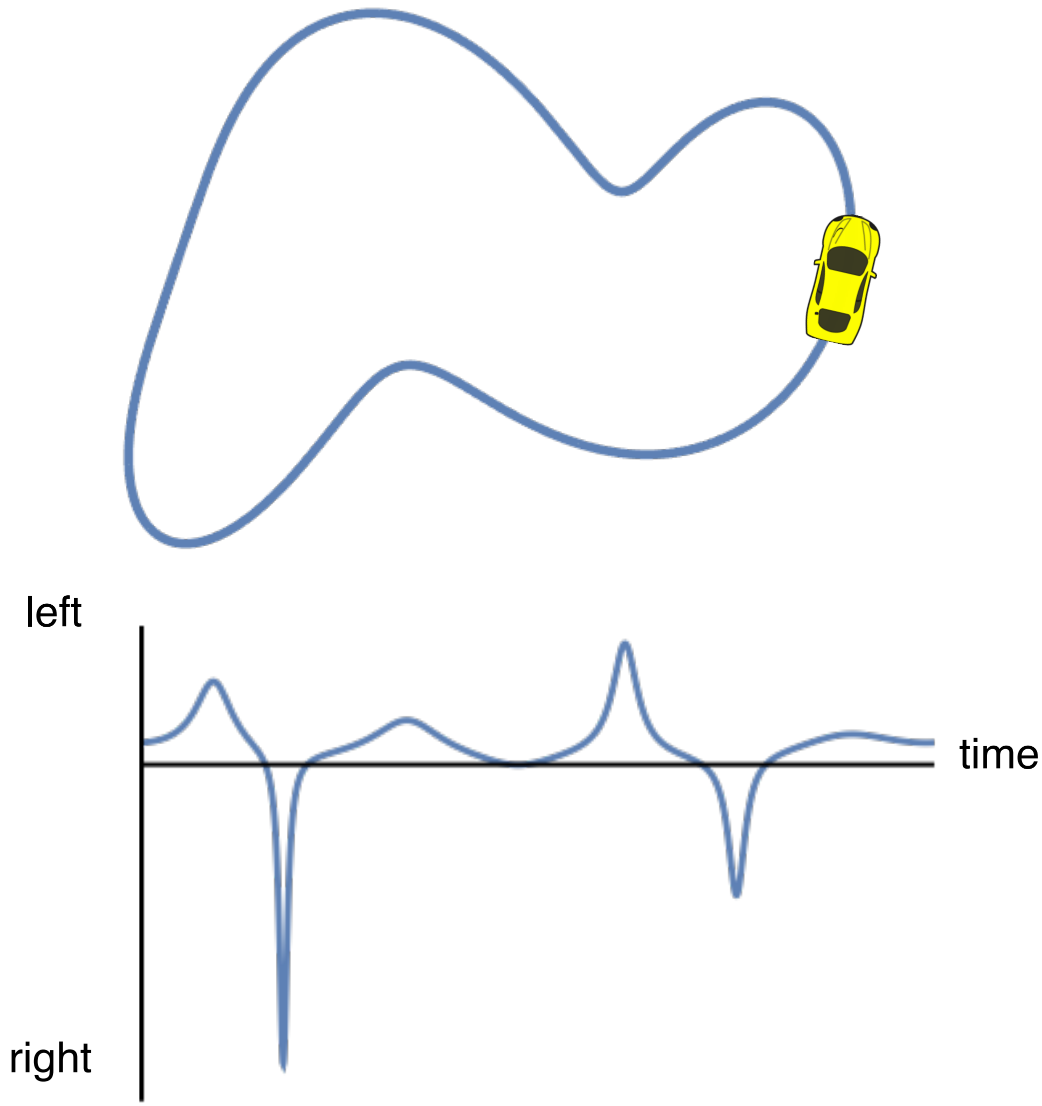
What everyday device can measure curvature?



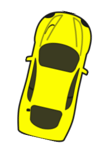


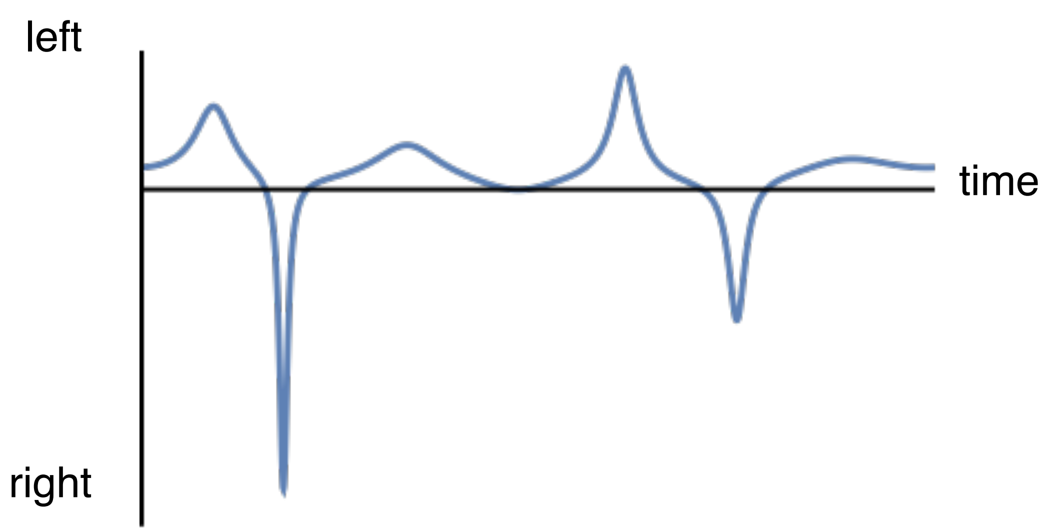




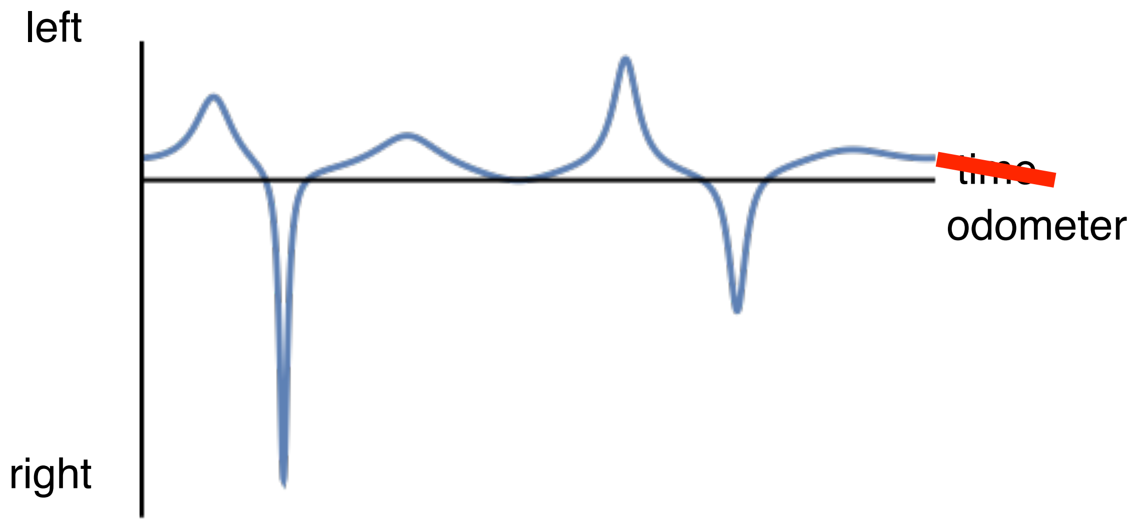


left
time
right

Can you reconstruct the racetrack?



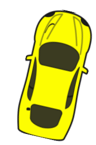


Can you reconstruct the racetrack?

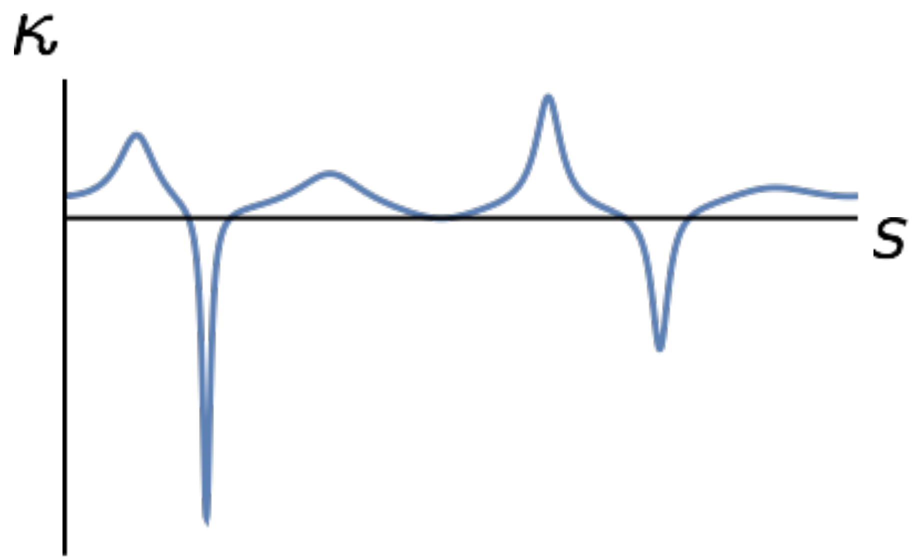


Can you reconstruct the racetrack?

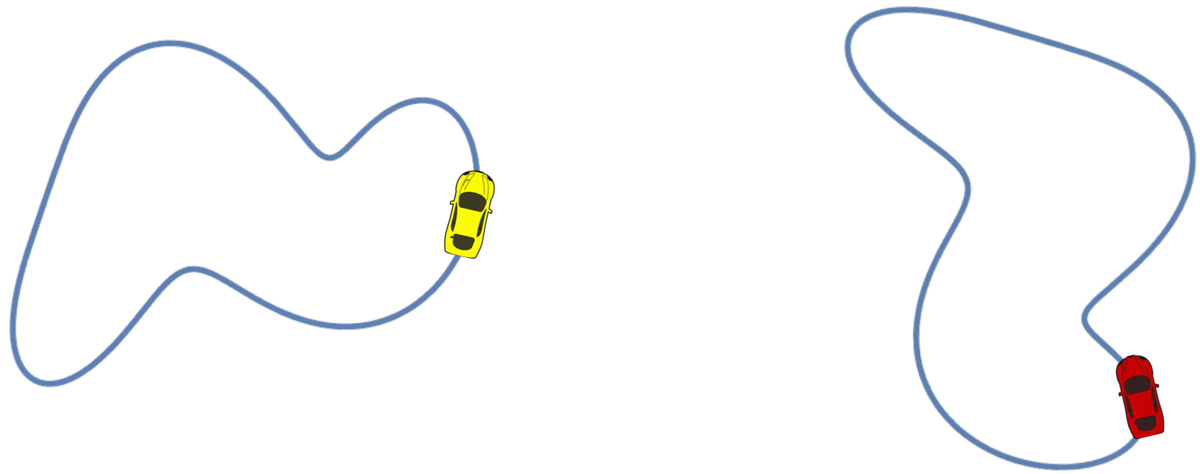
κ is (Euclidean) curvature



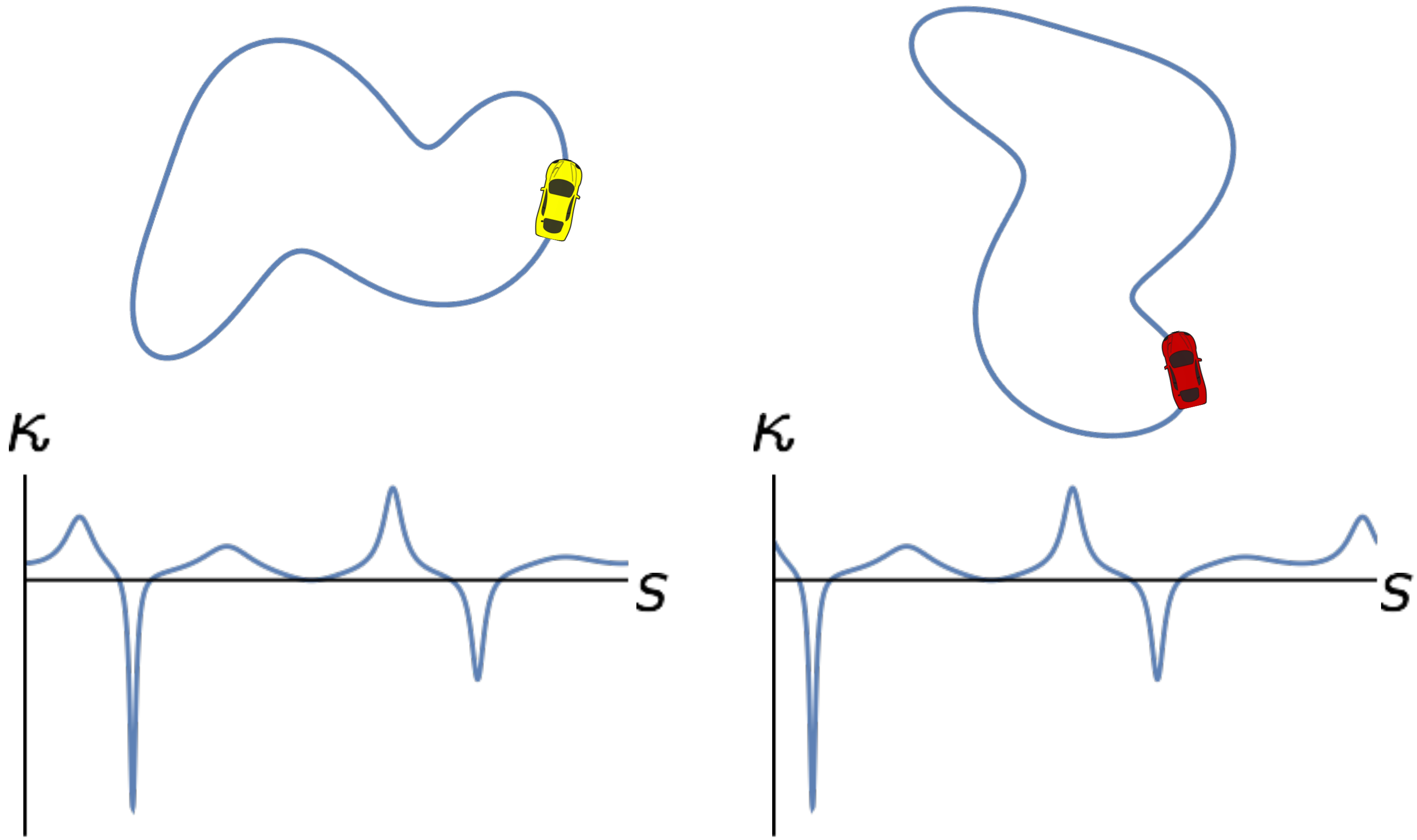
s is (Euclidean) arclength



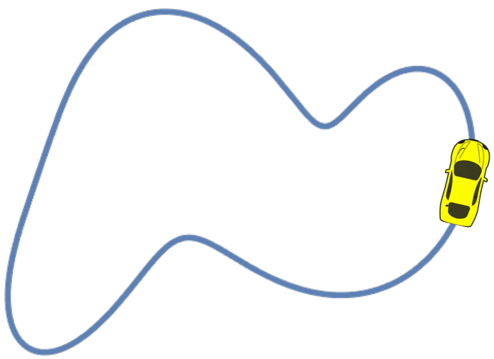
Racetrack comparison problem

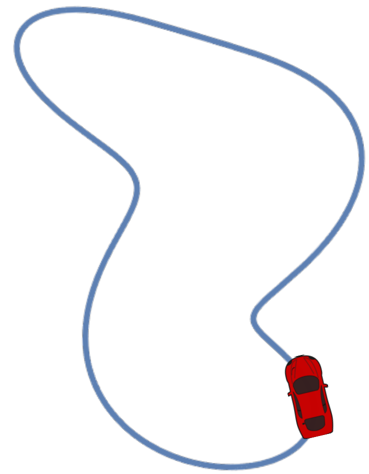


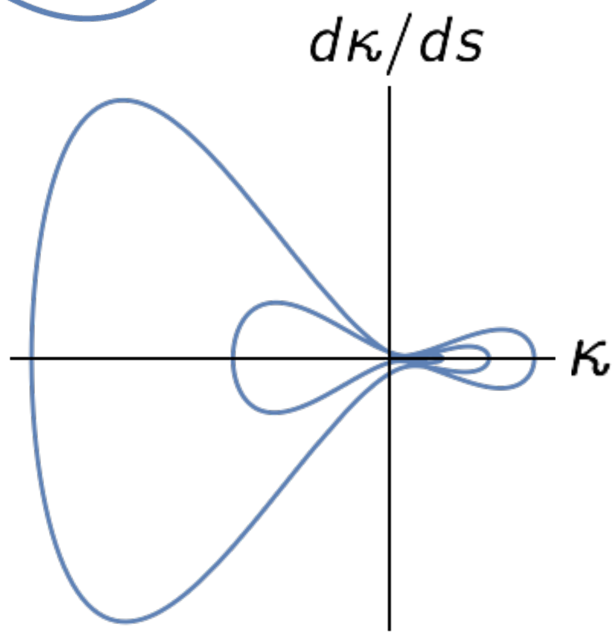
Racetrack comparison problem



Racetrack comparison problem





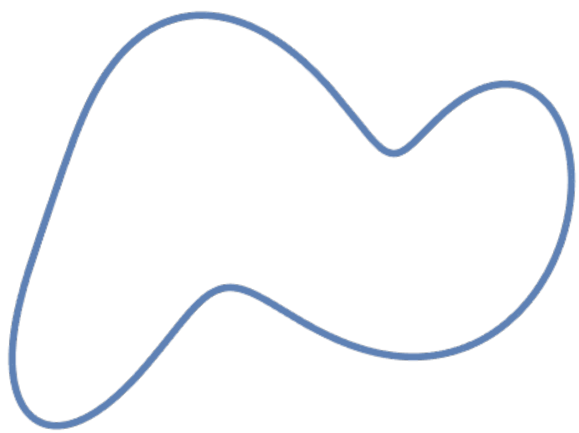


$d\kappa/ds$

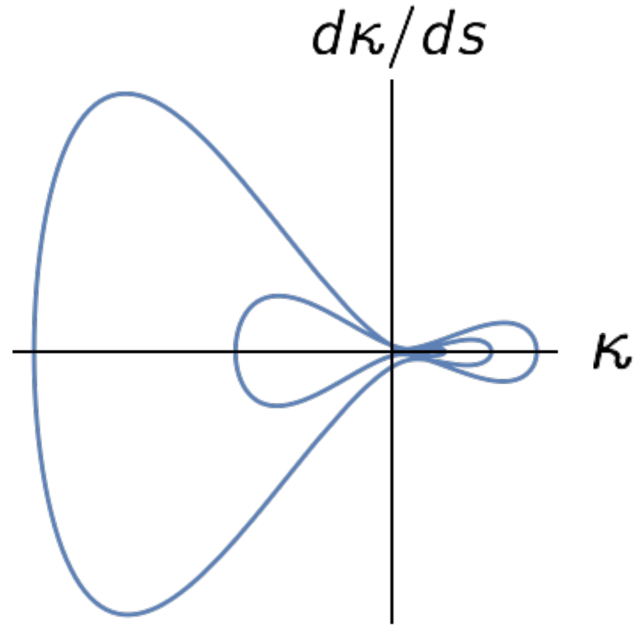
= rate of change (derivative) of curvature

The Invariant Signature

The invariant signature of a planar curve is the set traced out by curvature and the rate of change of curvature (its arclength derivative).



original curve

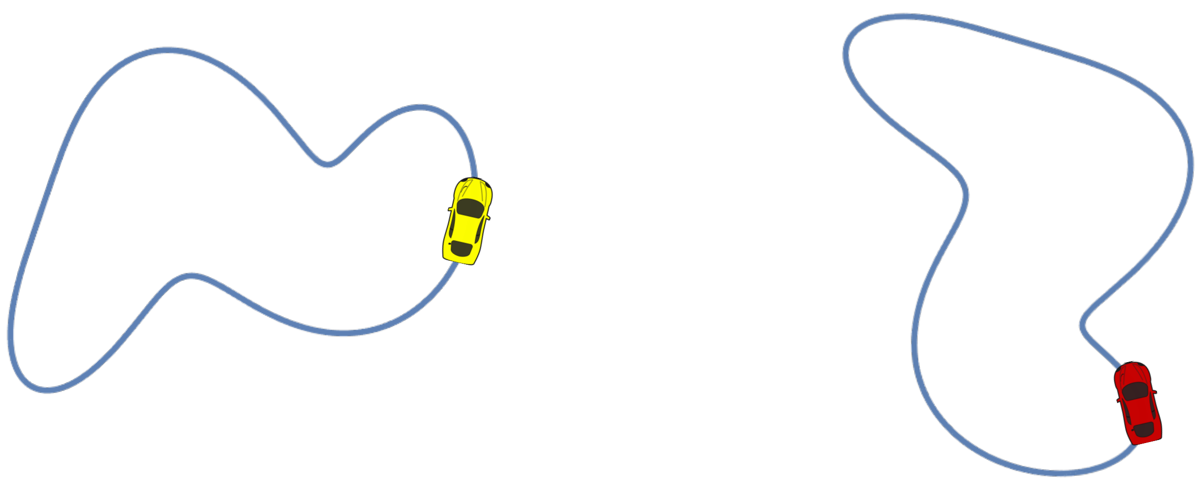


invariant signature

The invariant signature

Theorem

Two regular curves are related by a group transformation if and only if they have the same invariant signatures.



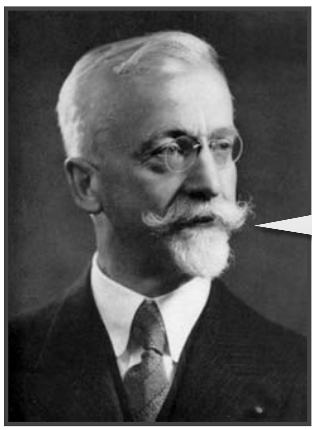
(Calabi, Haker, Olver, Shakiban, Tannenbaum 1998)

The invariant signature

Theorem

Two regular curves are related by a group transformation if and only if they have the same invariant signatures.

Proof idea



Theorem (Élie Cartan 1908)

Shapes are related if and only if they have the same relationships among their **differential invariants**.

3D Differential Invariant Signatures

Euclidean space curves: $C \subset \mathbb{R}^3$

$$\Sigma = \{\, (\, \kappa\,,\, \kappa_s\,,\, \tau\,)\,\} \;\subset\; \mathbb{R}^3$$

- κ — curvature, τ — torsion

Euclidean surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{\, \left(\, H\,,\, K\,,\, H_{,1}\,,\, H_{,2}\,,\, K_{,1}\,,\, K_{,2}\right)\,\right\} \;\subset\; \mathbb{R}^6$$

or $$\widehat{\Sigma} = \left\{\, \left(\, H\,,\, H_{,1}\,,\, H_{,2}\,,\, H_{,11}\right)\,\right\} \;\subset\; \mathbb{R}^4$$

- H — mean curvature, K — Gauss curvature

Moving Frames

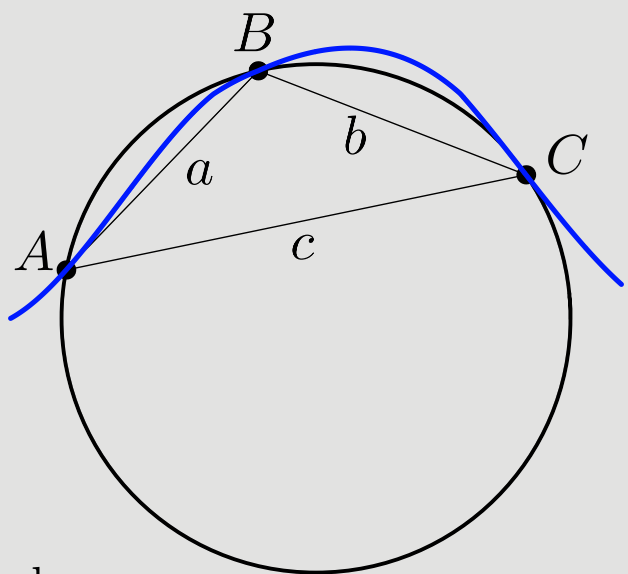
The mathematical theory is all based on the new equivariant method of moving frames (Fels+PJO, 1999) which provides a systematic and algorithmic calculus for constructing complete systems of differential invariants, joint invariants, joint differential invariants, invariant differential operators, invariant differential forms, invariant variational problems, invariant conservation laws, invariant numerical algorithms, invariant signatures, etc., etc.

Symmetry–Preserving Numerical Methods

- Invariant numerical approximations to differential invariants.
- Invariantization of numerical integration methods.

$\Longrightarrow$ Structure-preserving algorithms

Numerical approximation to curvature

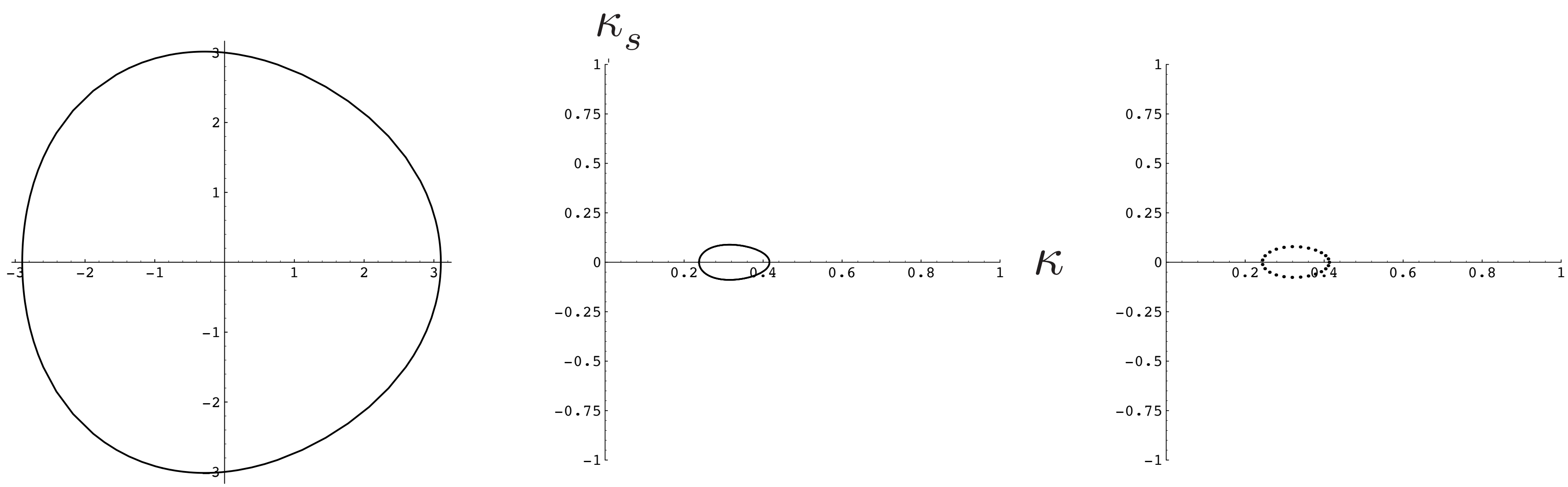


Heron's formula

$$\widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

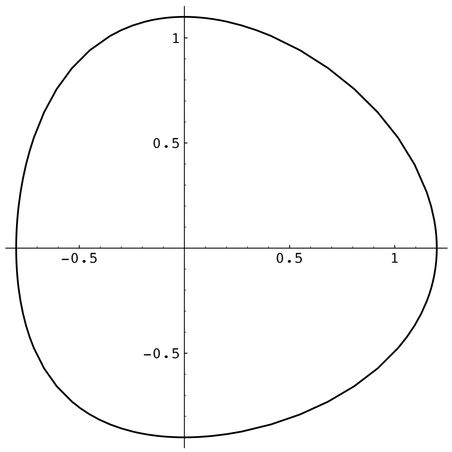
$$s = \frac{a+b+c}{2} \qquad \text{—} \qquad \text{semi-perimeter}$$

The polar curve $r = 3 + \frac{1}{10} \cos 3\theta$

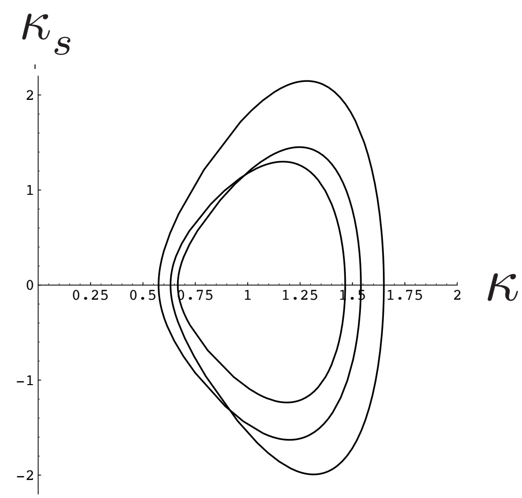


The Original Curve Euclidean Signature Numerical Signature

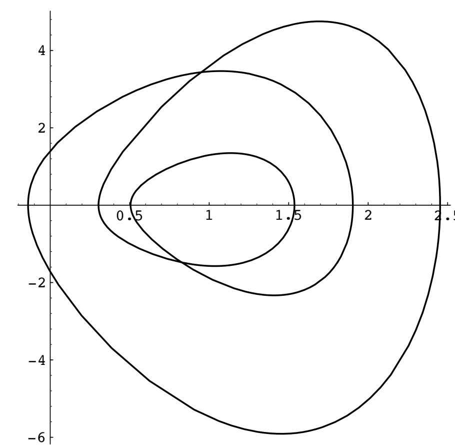
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \;\; y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

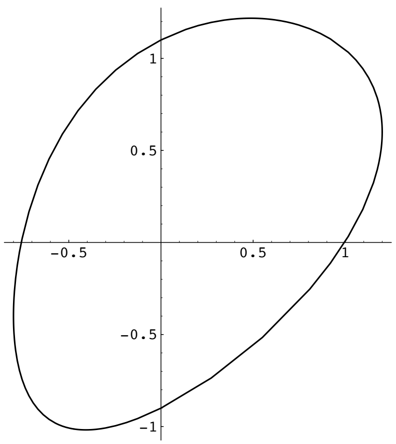


Euclidean Signature

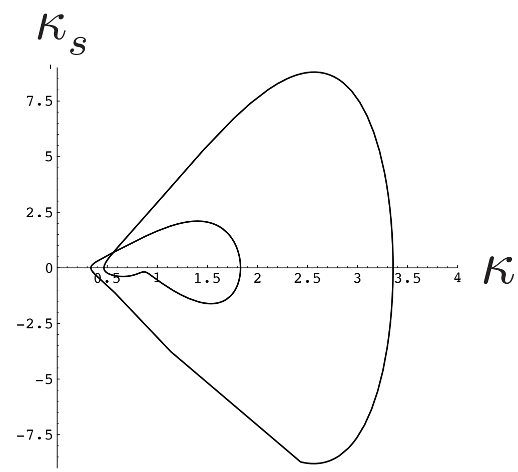


Equi-affine Signature

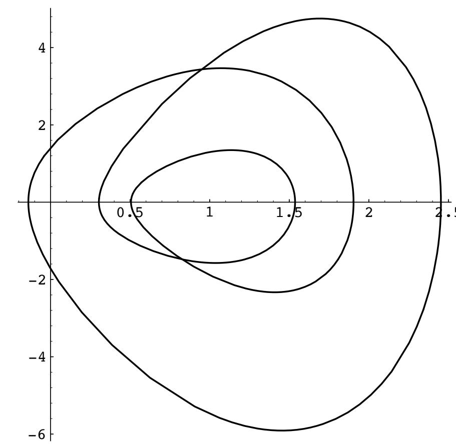
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

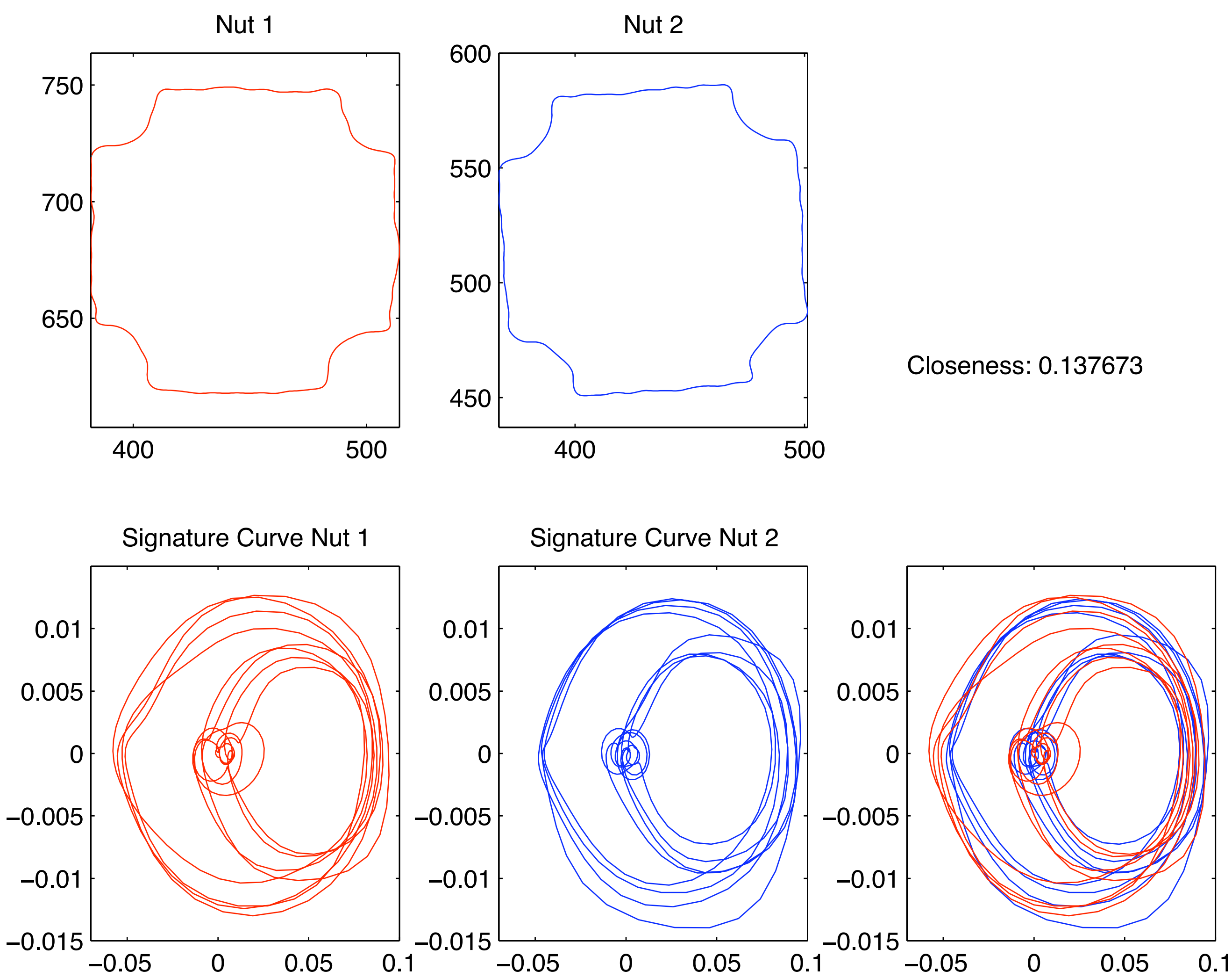


Euclidean Signature

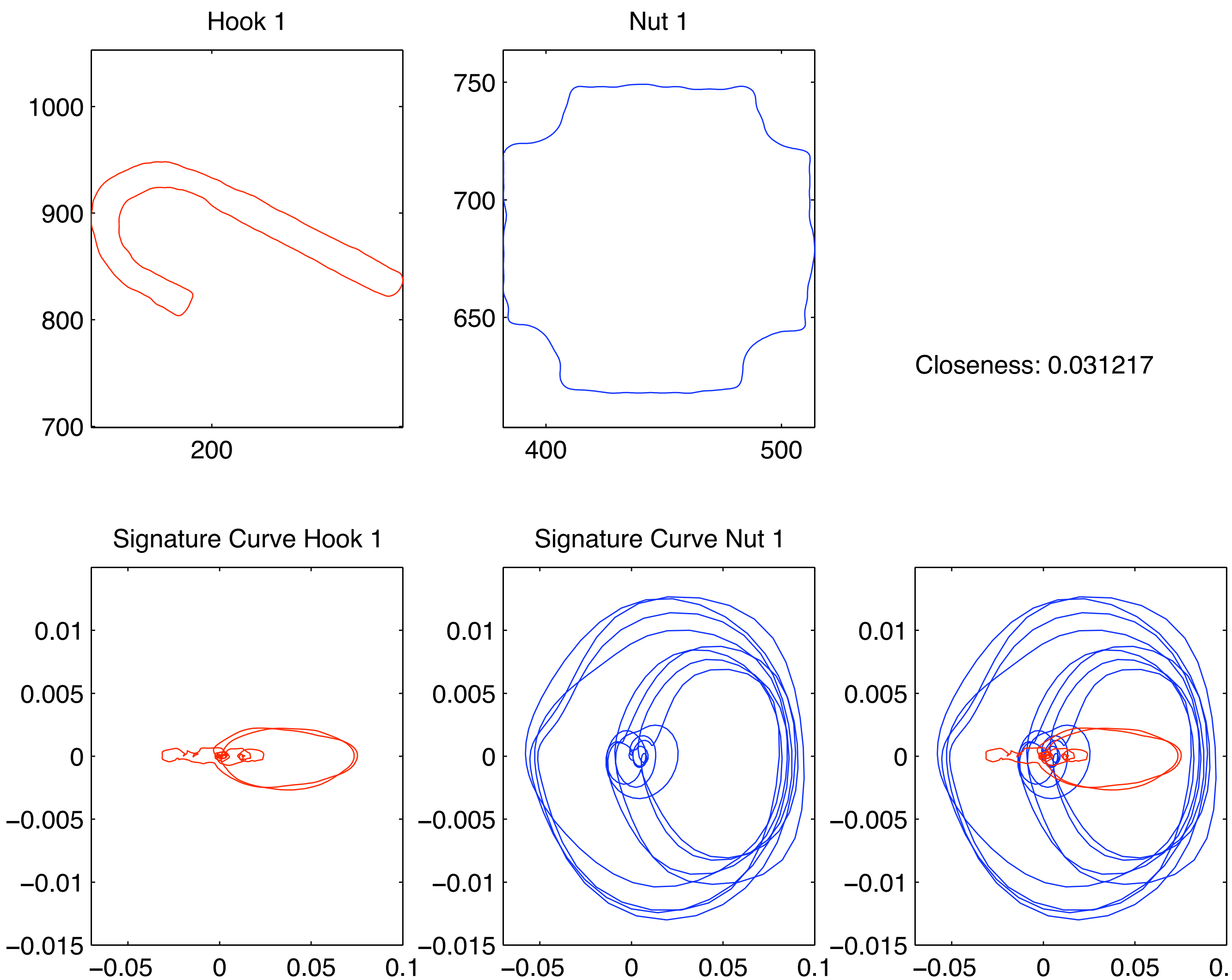


Equi-affine Signature

Object Recognition

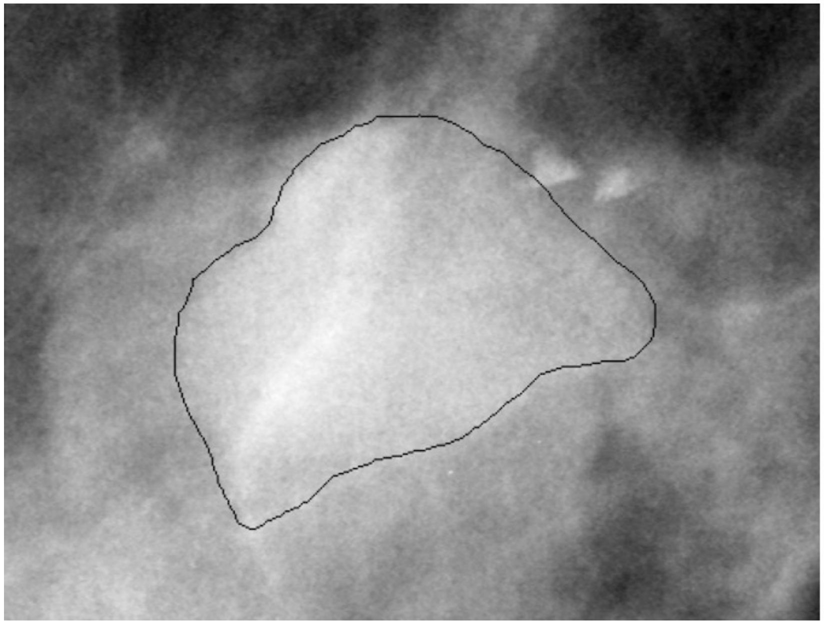


Nut 1
750
700
650
400
500
Nut 2
600
550
500
450
400
500
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

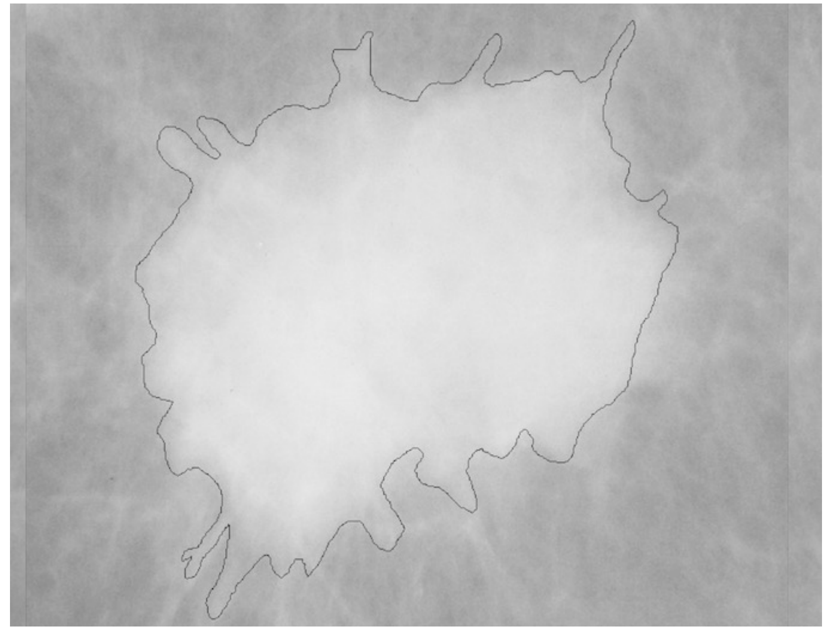


Hook 1
1000
900
800
700
200
Nut 1
750
700
650
400
500
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

Diagnosing breast tumors



Benign — cyst

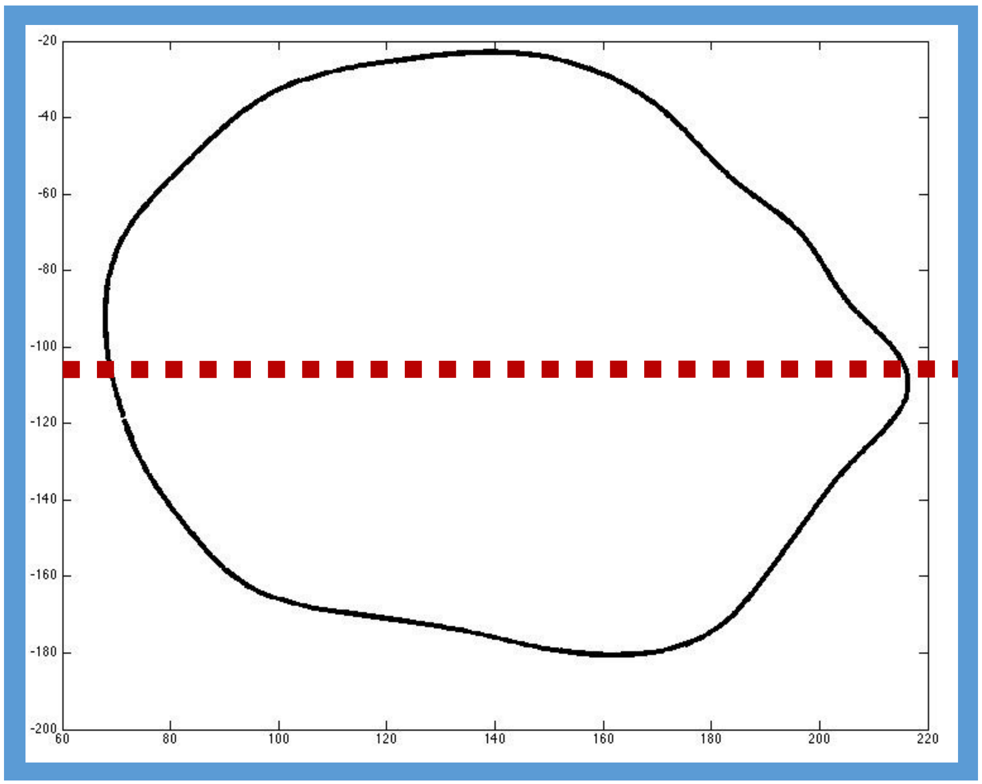


Malignant — cancerous

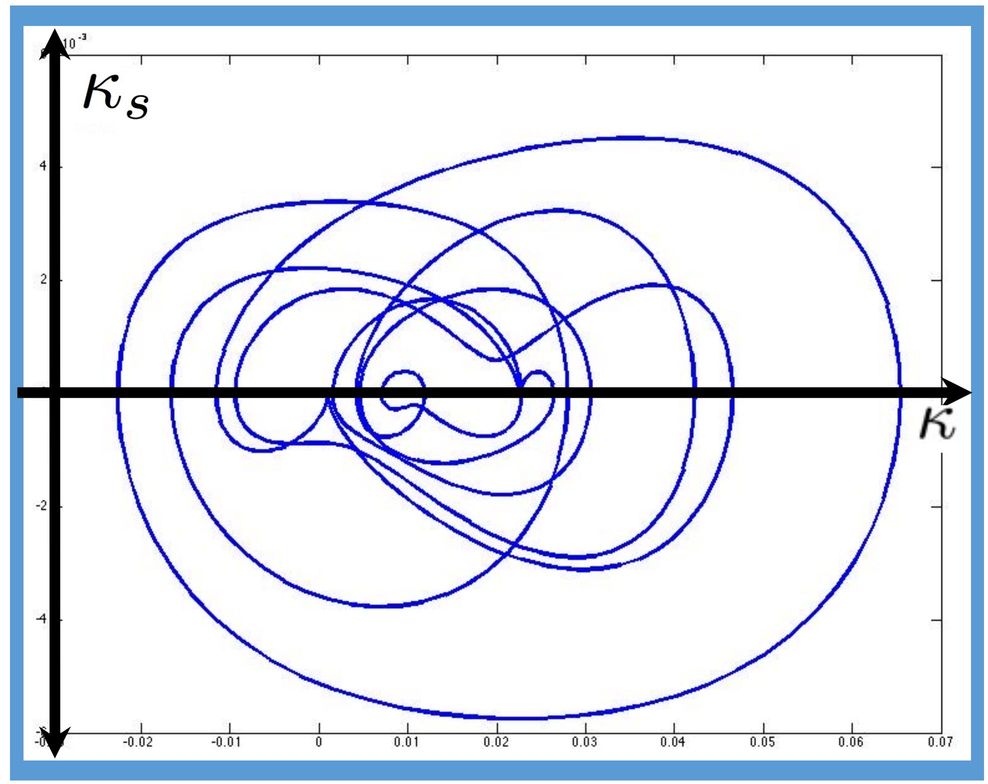
Anna Grim, Cheri Shakiban (2017)

A BENIGN TUMOR

Contour

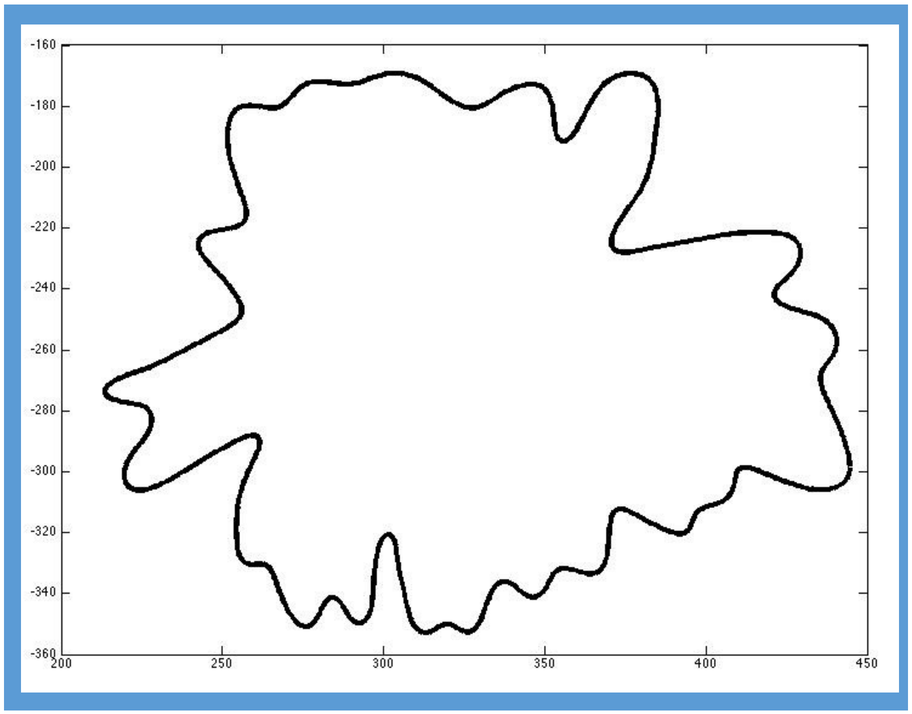


Signature Curve

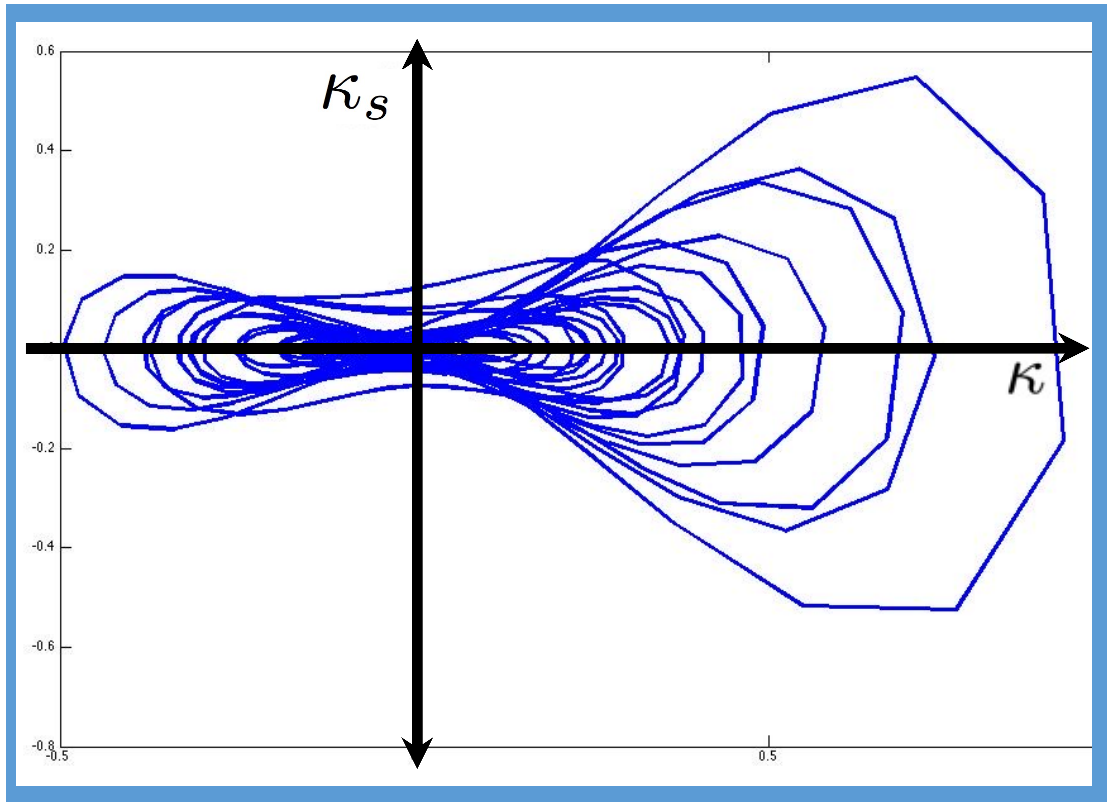


A MALIGNANT TUMOR

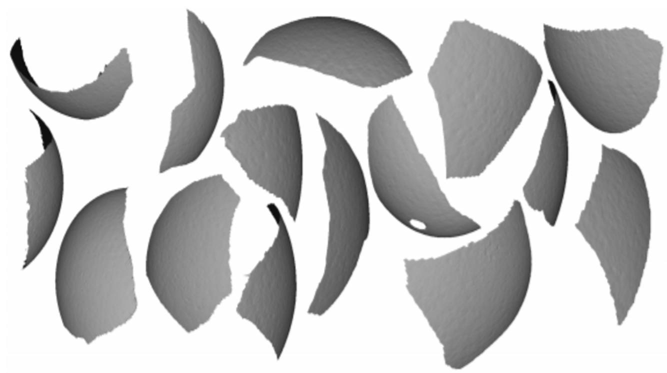
Contour



Signature Curve



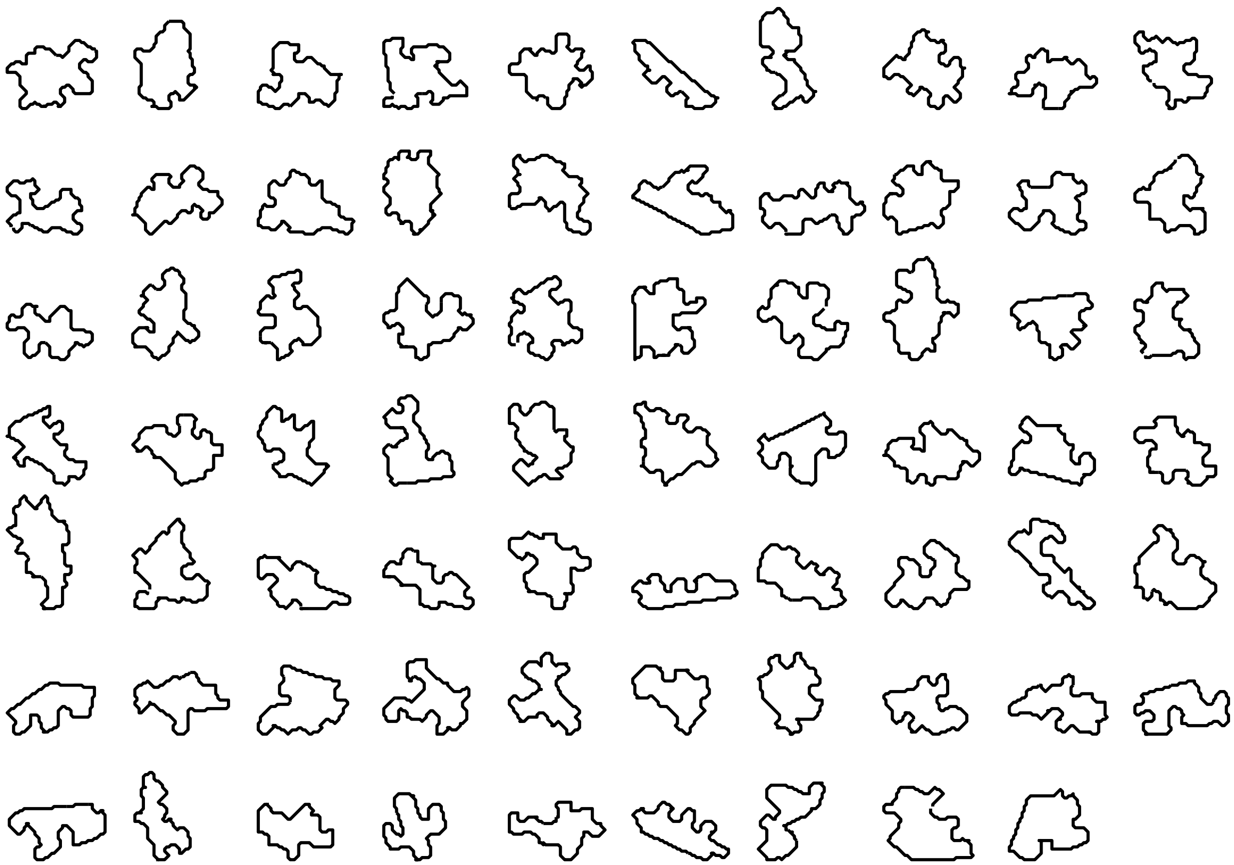
Reassembly of Broken Objects



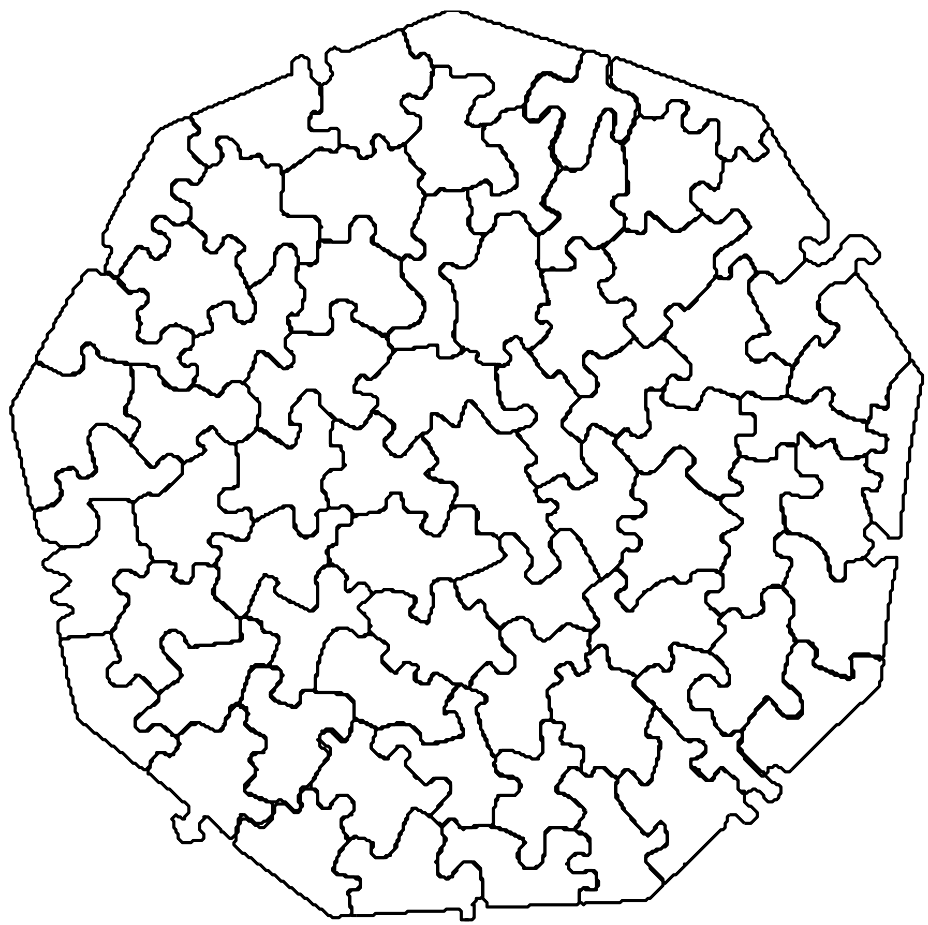
James Rizzi
1000
Ravensburger Puzzle
Ravensburger


the most unique
puzzle ever
the BAFFLER™
by CHRIS Yates
The Nonagon
67 pieces

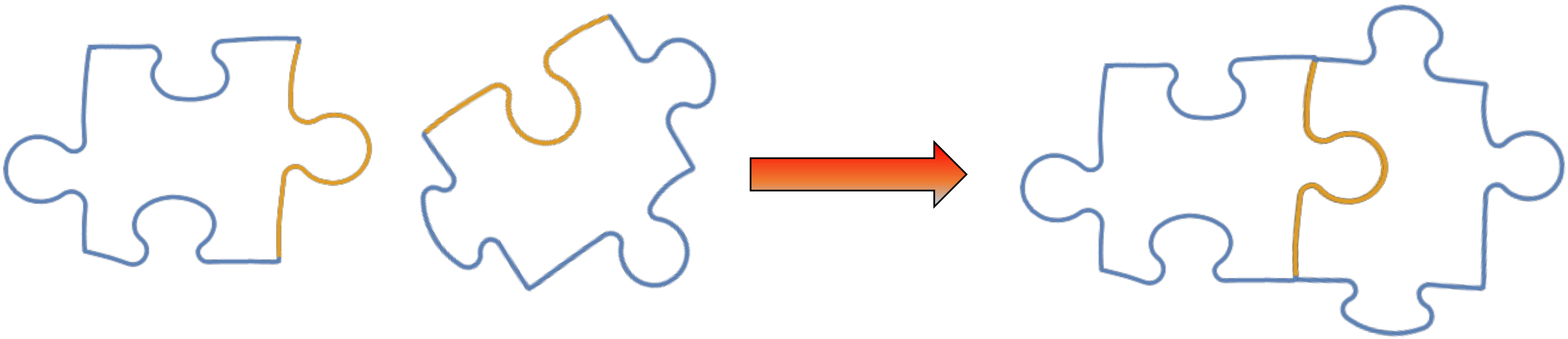
The Baffler Nonagon



The Baffler Nonagon — Solved



Automatic puzzle reassembly



Step 0. Digitally photograph and smooth the puzzle pieces.

Step 1. Numerically compute invariant signatures of (parts of) pieces.

Step 2. Compare signatures to find potential fits.

Step 3. Put them together, if they fit, as closely as possible.

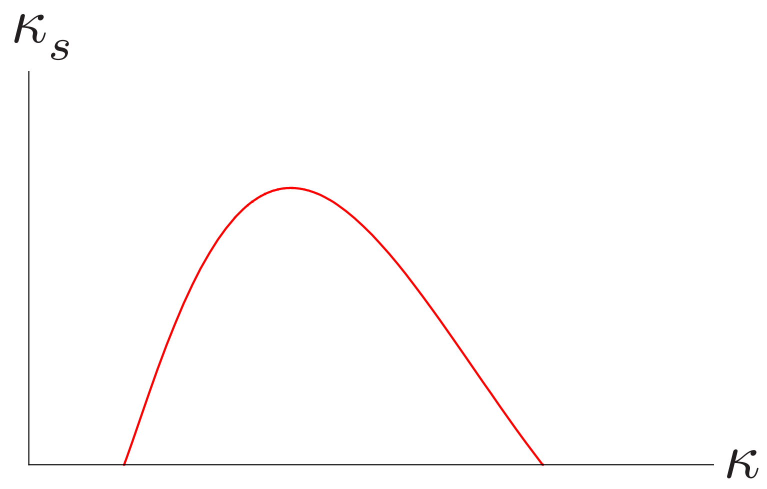
Repeat steps 1–3 until puzzle is assembled….

Localization of Signatures

Bivertex arc: $\kappa_s \neq 0$ everywhere

except $\kappa_s = 0$ at the two endpoints

The signature Σ of a bivertex arc is a single arc that starts and ends on the κ–axis.



Bivertex Decomposition

v-regular curve — finitely many generalized vertices

$$C = \bigcup_{j=1}^{m} B_j \cup \bigcup_{k=1}^{n} V_k$$

$B_1, \ldots, B_m$ — bivertex arcs

$V_1, \ldots, V_n$ — generalized vertices: $n \geq 4$

Main Idea: Compare individual bivertex arcs, and then decide whether the rigid equivalences are (approximately) the same.

D. Hoff & PJO, Extensions of invariant signatures for object recognition, *J. Math. Imaging Vision* **45** (2013), 176–185.

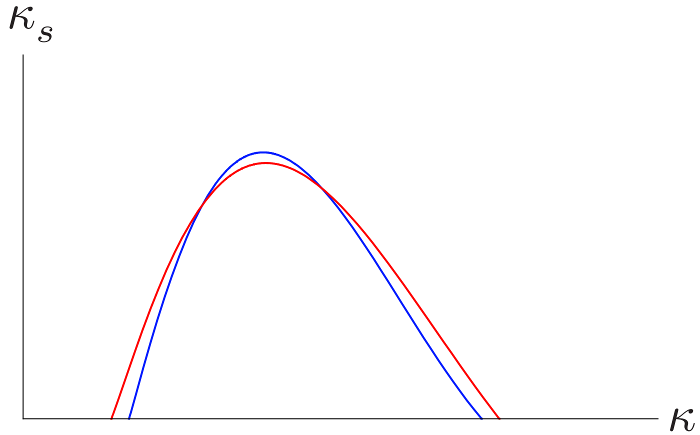
Signature Metrics

Used to compare signatures:

- Hausdorff
- Monge–Kantorovich transport
- Electrostatic/gravitational attraction
- Latent semantic analysis
- Histograms
- Geodesic distance
- Diffusion metric
- Gromov–Hausdorff & Gromov–Wasserstein

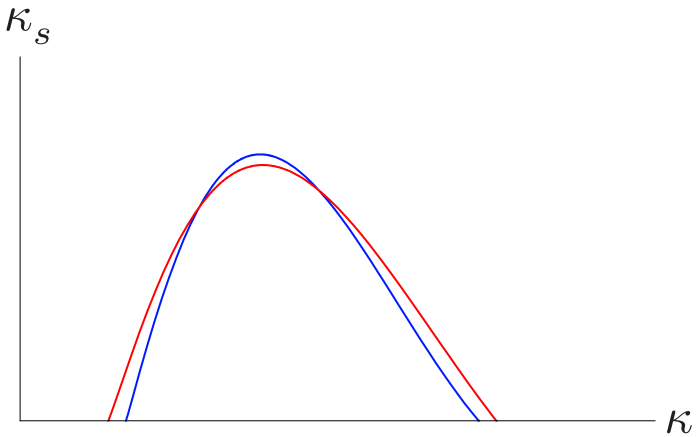
Gravitational/Electrostatic Attraction

★ Treat the two (signature) curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.

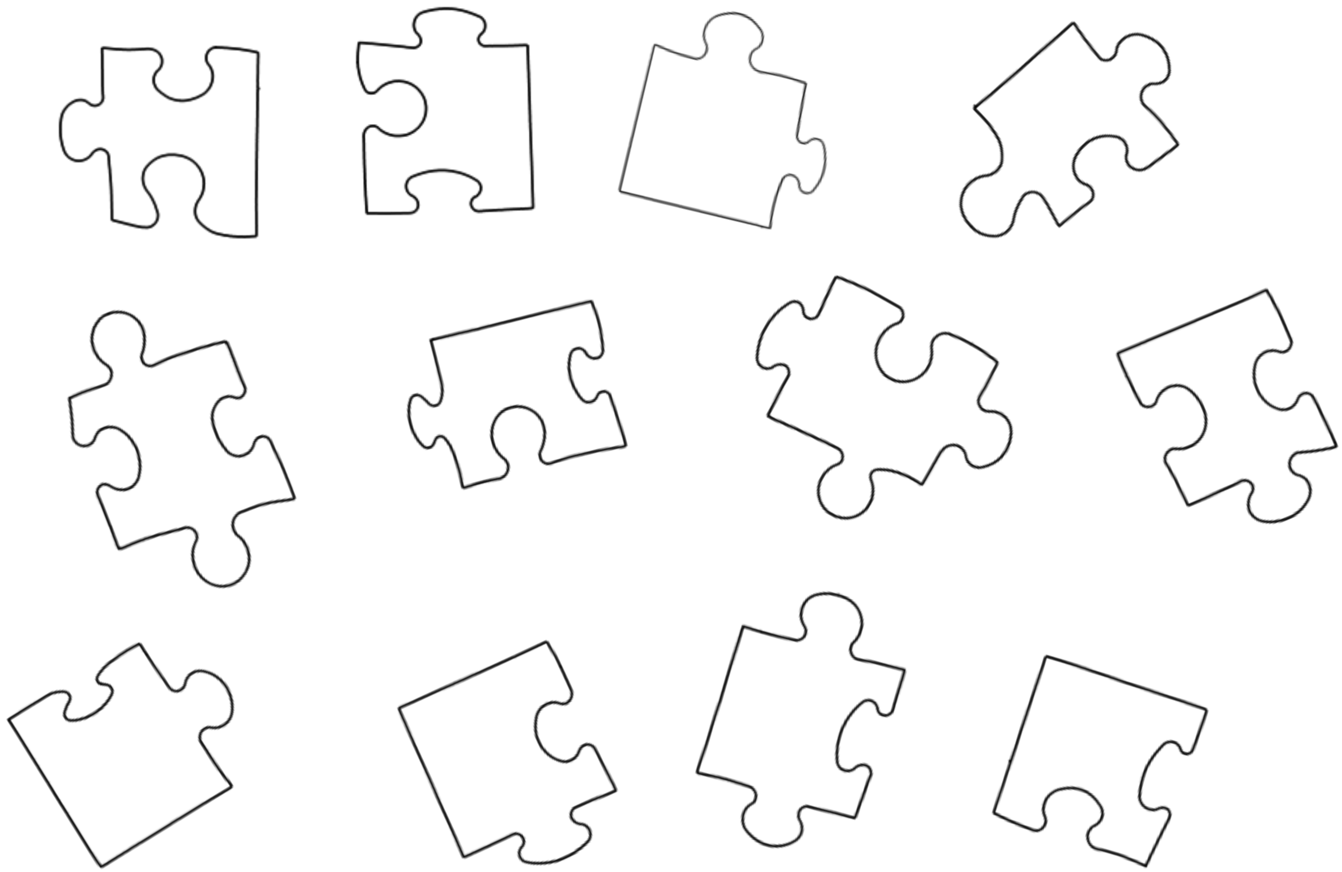


Gravitational/Electrostatic Attraction

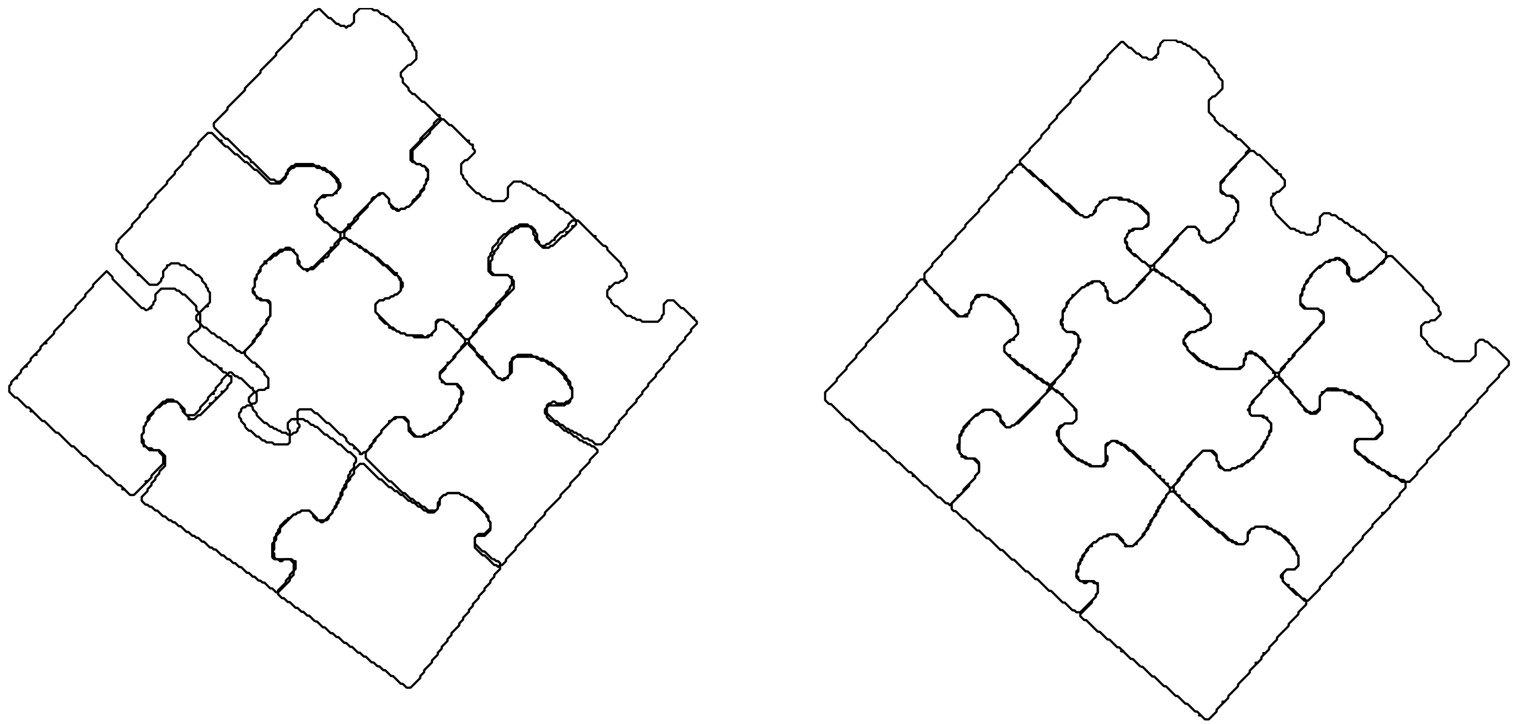
- ⋆ Treat the two (signature) curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.
- ⋆ In practice, we are dealing with discrete data (pixels) and so treat the curves and signatures as point masses/charges.



Assembling the puzzle…



Piece Locking

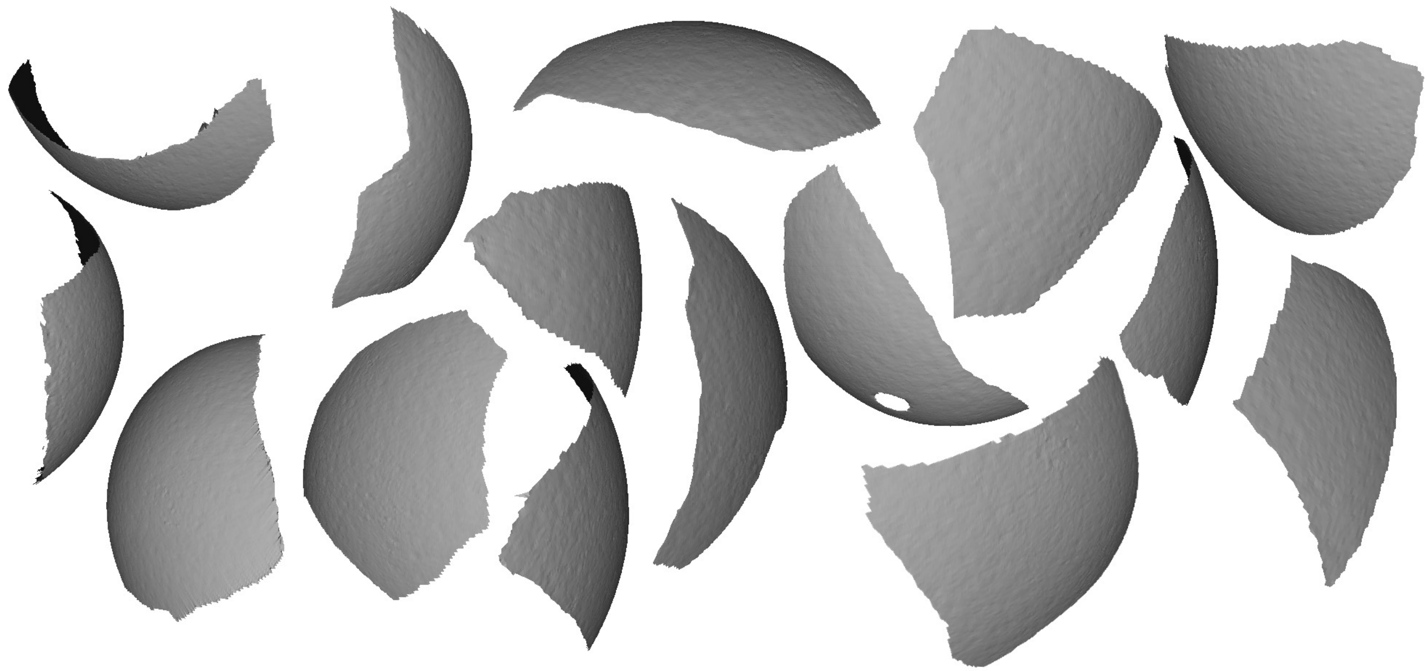


⋆⋆ Minimize force and torque based on gravitational attraction of the two matching edges.

Putting Humpty Dumpty Together Again

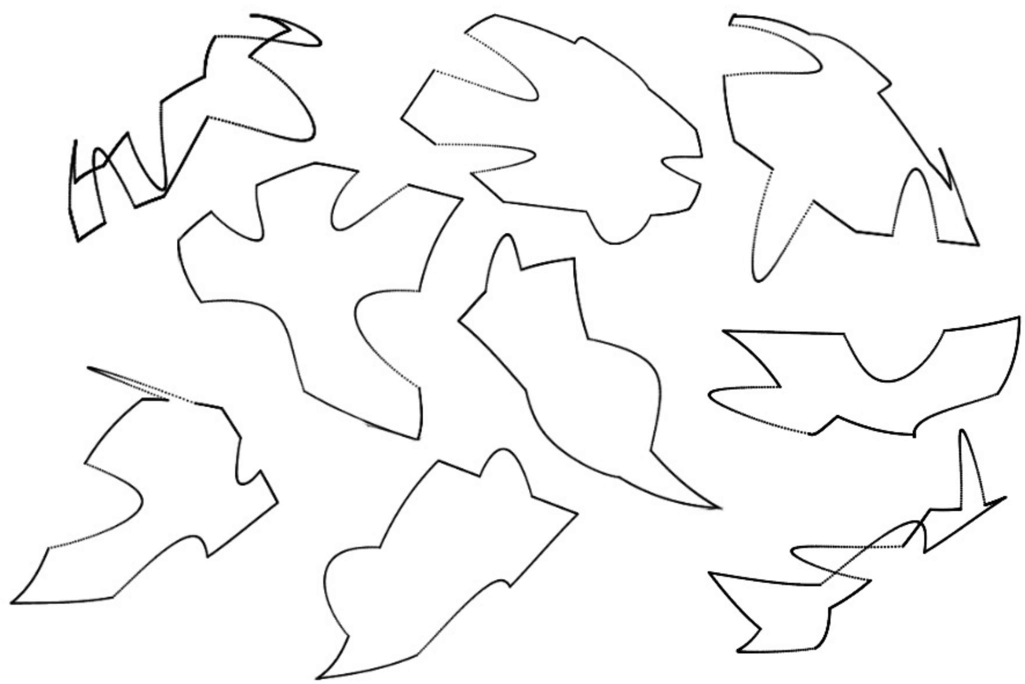
Anna Grim, Ryan Slechta, Tim O’Connor, Rob Thompson, Cheri Shakiban, Peter Olver

A broken ostrich egg

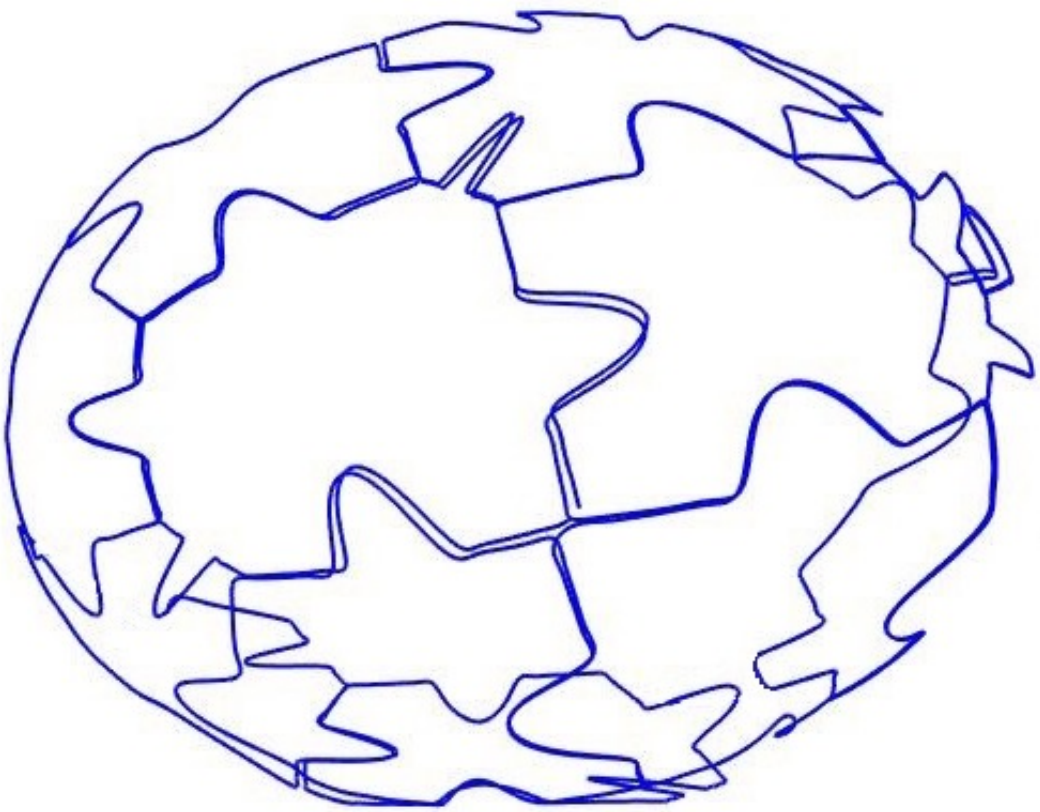


(Scanned by M. Bern, Xerox PARC)

A synthetic 3d jigsaw puzzle

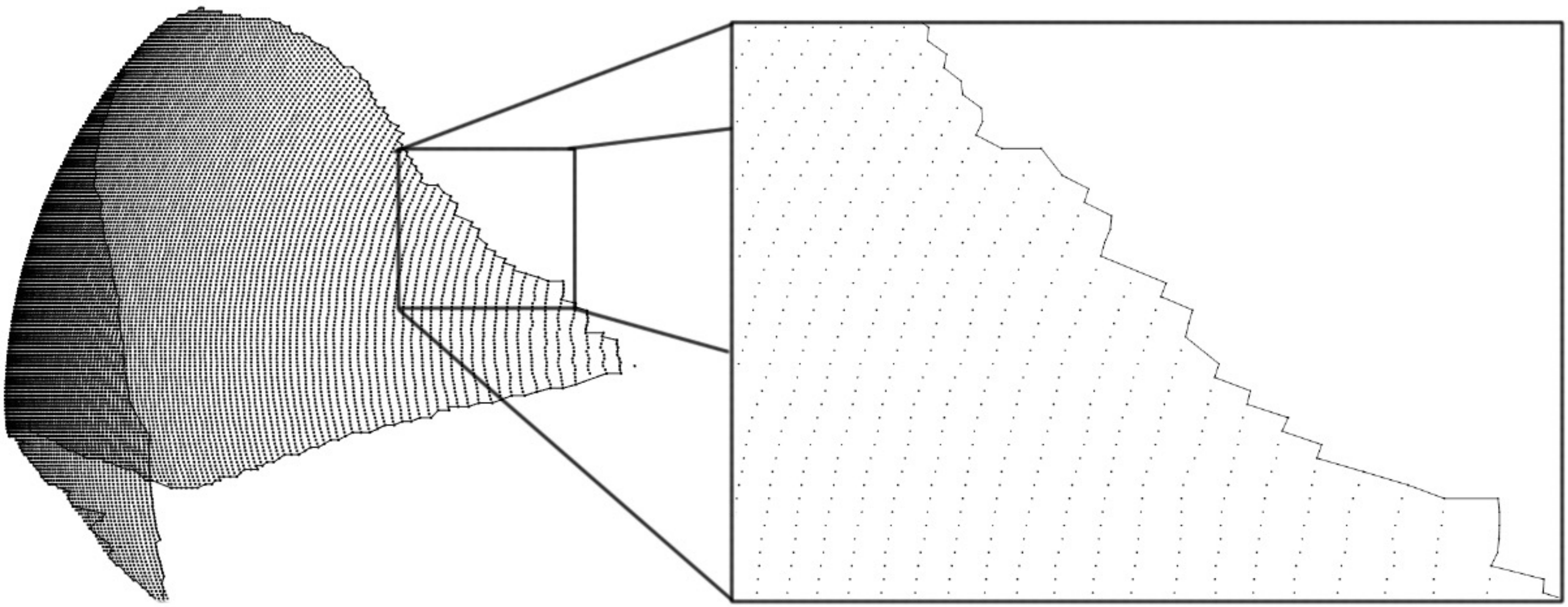


Assembly of synthetic spherical puzzle

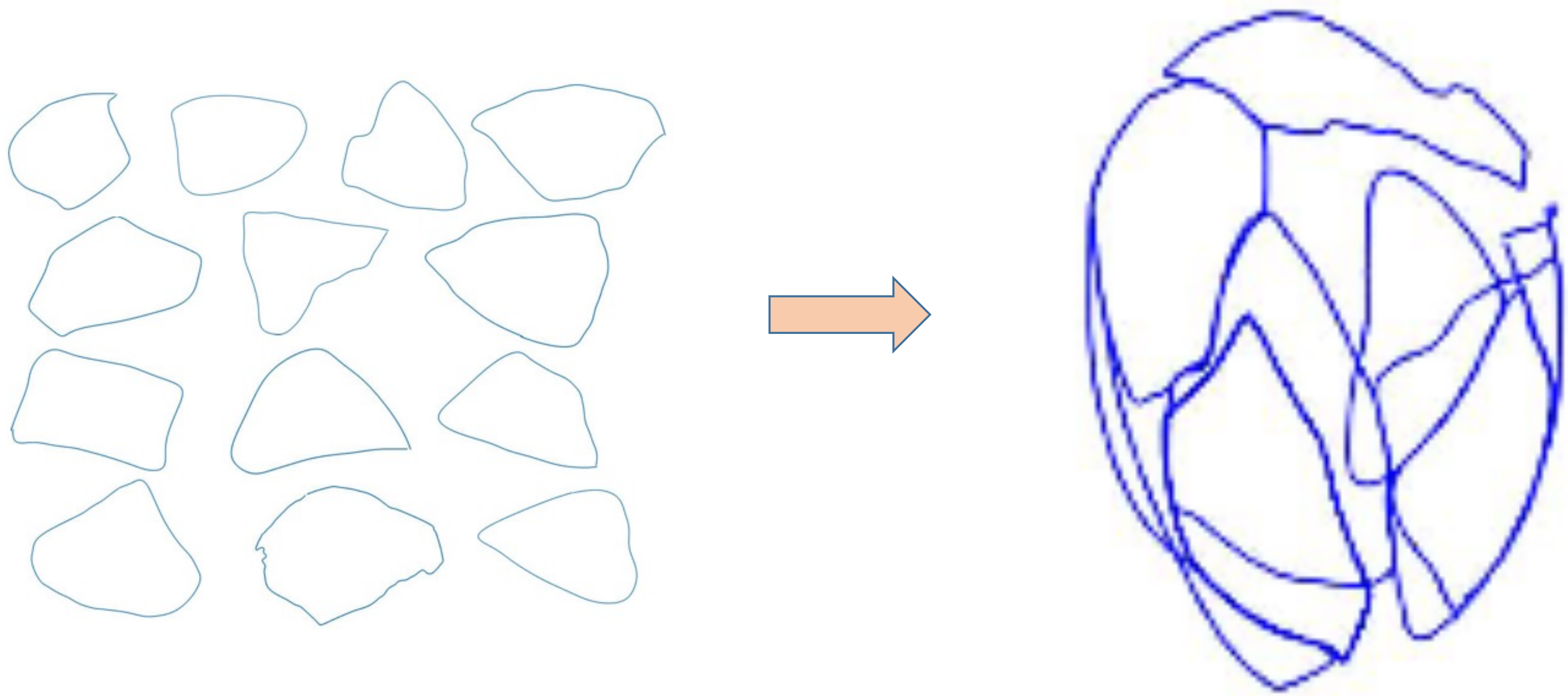


- Uses curvature and torsion invariants

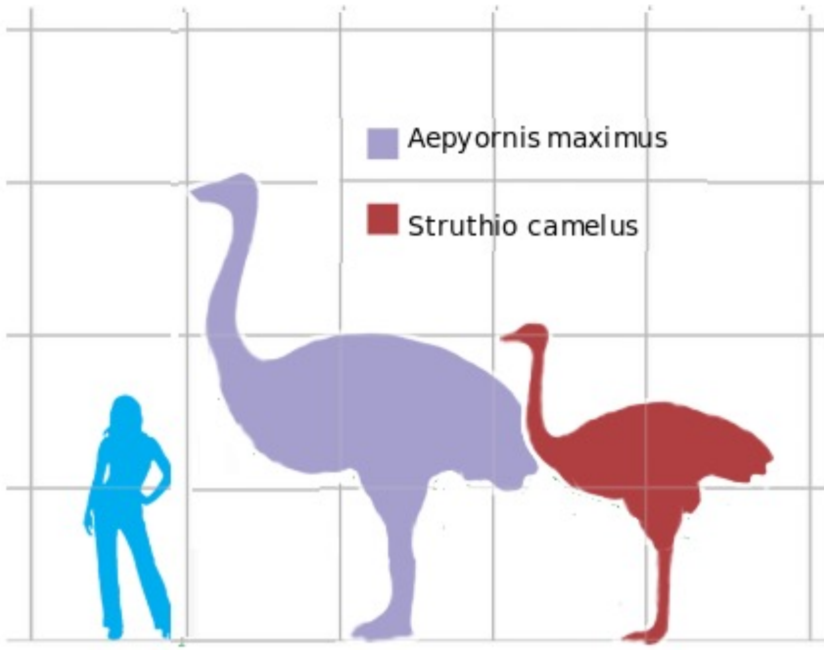
An egg piece



All the king's horses and men



The elephant bird of Madagascar



(Image from wikipedia.org)

- more than 3 meters tall
- extinct by the 1700's
- one egg could make about 160 omelets

Elephant bird egg shells

(Extract from “Zoo Quest to Madagascar”, BBC 1961)

The elephant bird of Madagascar

(Image from Tennant's Auctioneers)

- pictured egg is 70% complete
- complete egg recently sold for $100,000

Puzzles in archaeology

Puzzles in archaeology

Puzzles in surgery

Puzzles in anthropology and paleontology

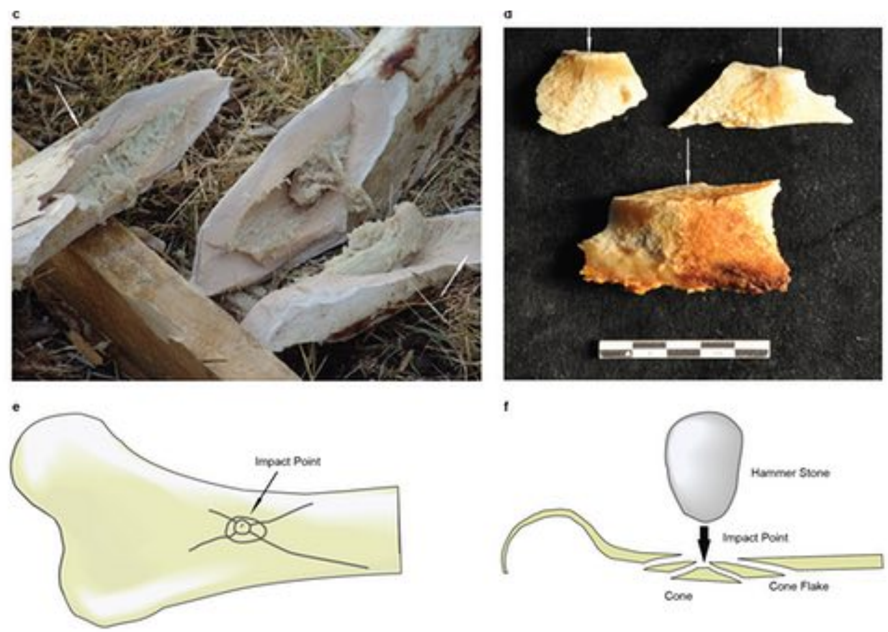
theguardian

Could history of humans in North America be rewritten by broken bones?

Smashed mastodon bones show humans arrived over 100,000 years earlier than previously thought say researchers, although other experts are sceptical

Ian Sample Science editor

Wednesday 26 April 2017 13.00 EDT

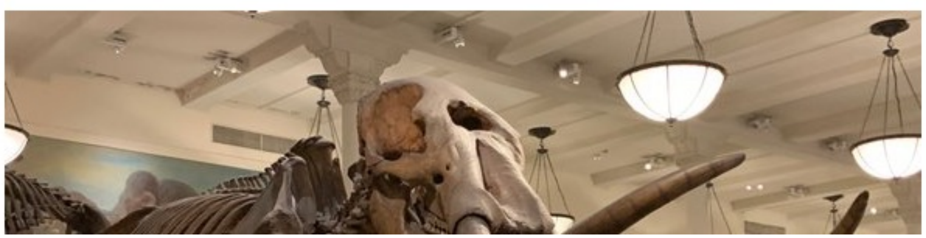


Laelaps

Busted Mastodon Is Ice Age Roadkill

A mastodon said to be pulverized by Ice Age humans was probably busted up by roadwork

By Brian Switek on April 10, 2019

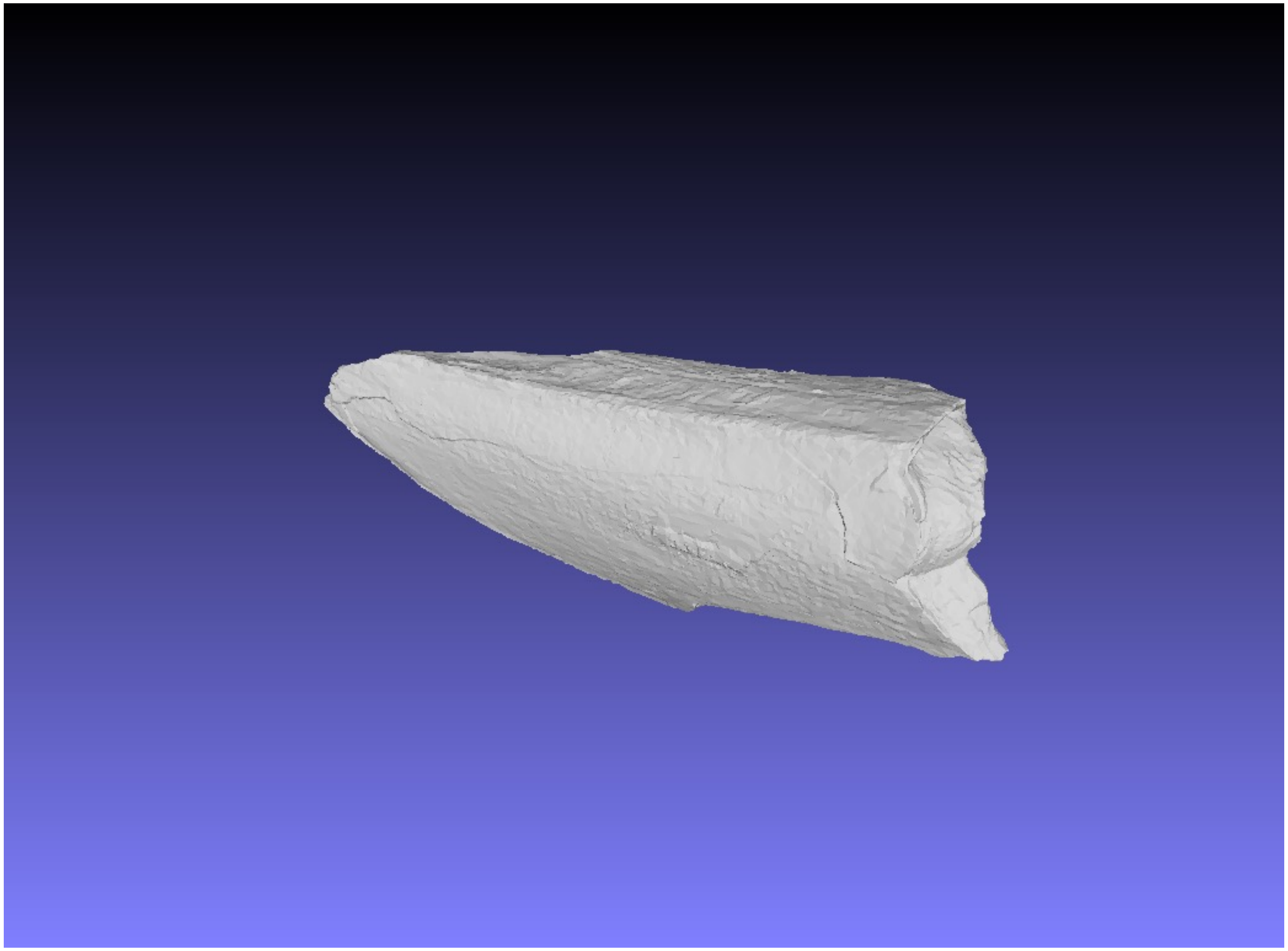


Anthropological Implications

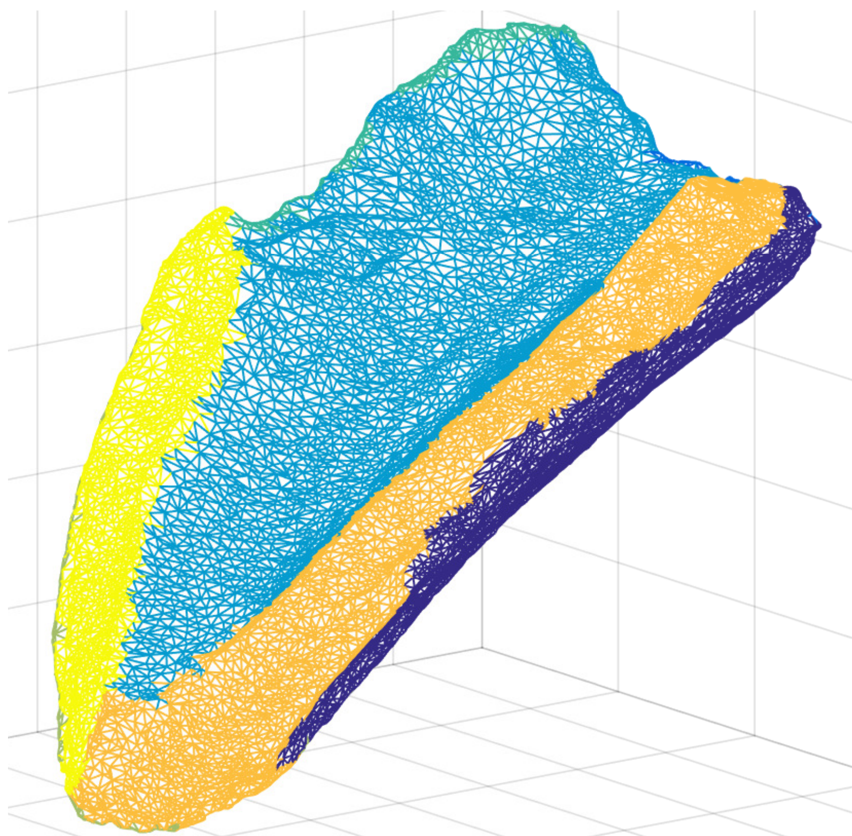
- Meat eater vs. vegetarian
- Brain development
- Scavenging vs. hunting
- Food sharing
- Social structures
- Cooperative behavior
- Home bases/central places
- Carcass transport
- Butchering behavior

OR
?

Bone fragment



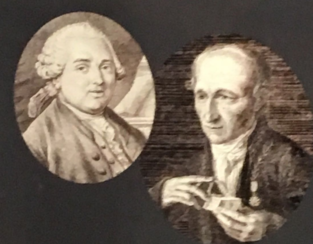
Segmentation




Goniomètre
(rapporteur)
dit « mesure-angle »
Instrument de mesure pour
les angles des cristaux
Laiton et argent (XIXe siècle)

XVIIIe siècle

Les débuts de la **cristallographie** moderne

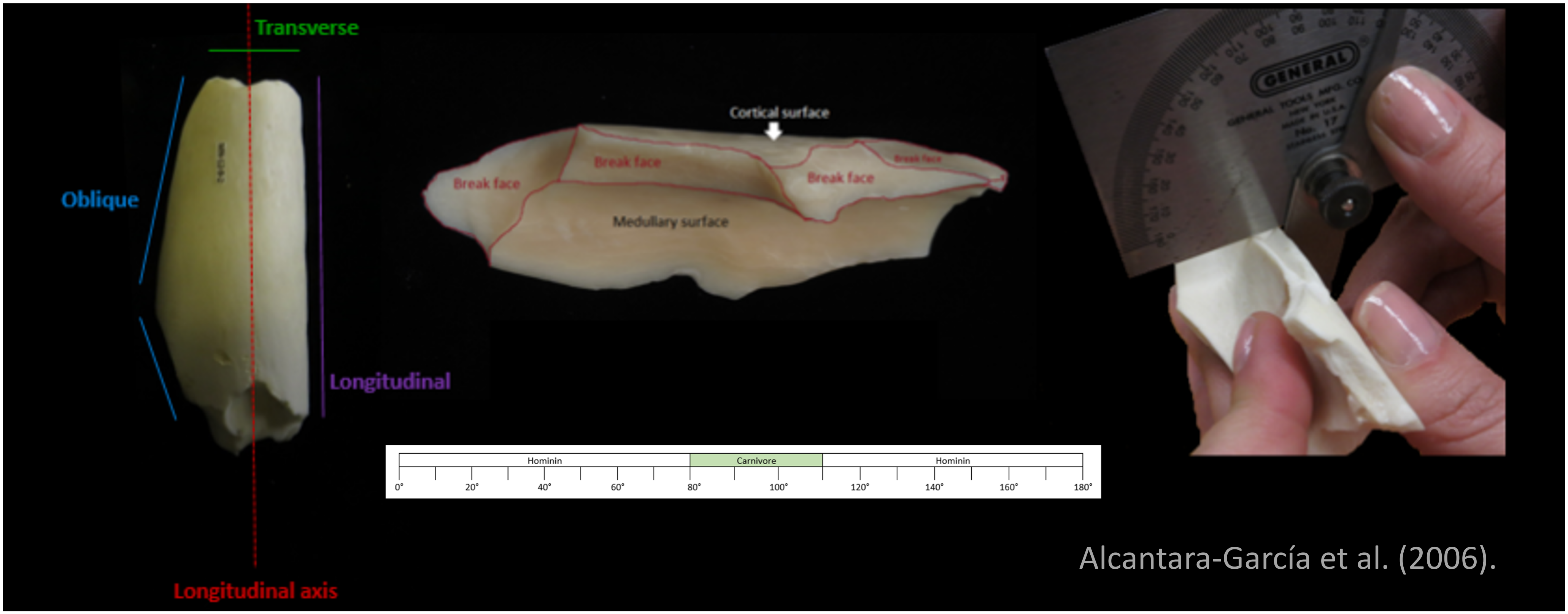


J.-B. Romé de l'Isle (1736-1790)
et R. J. Haüy (1743-1822)

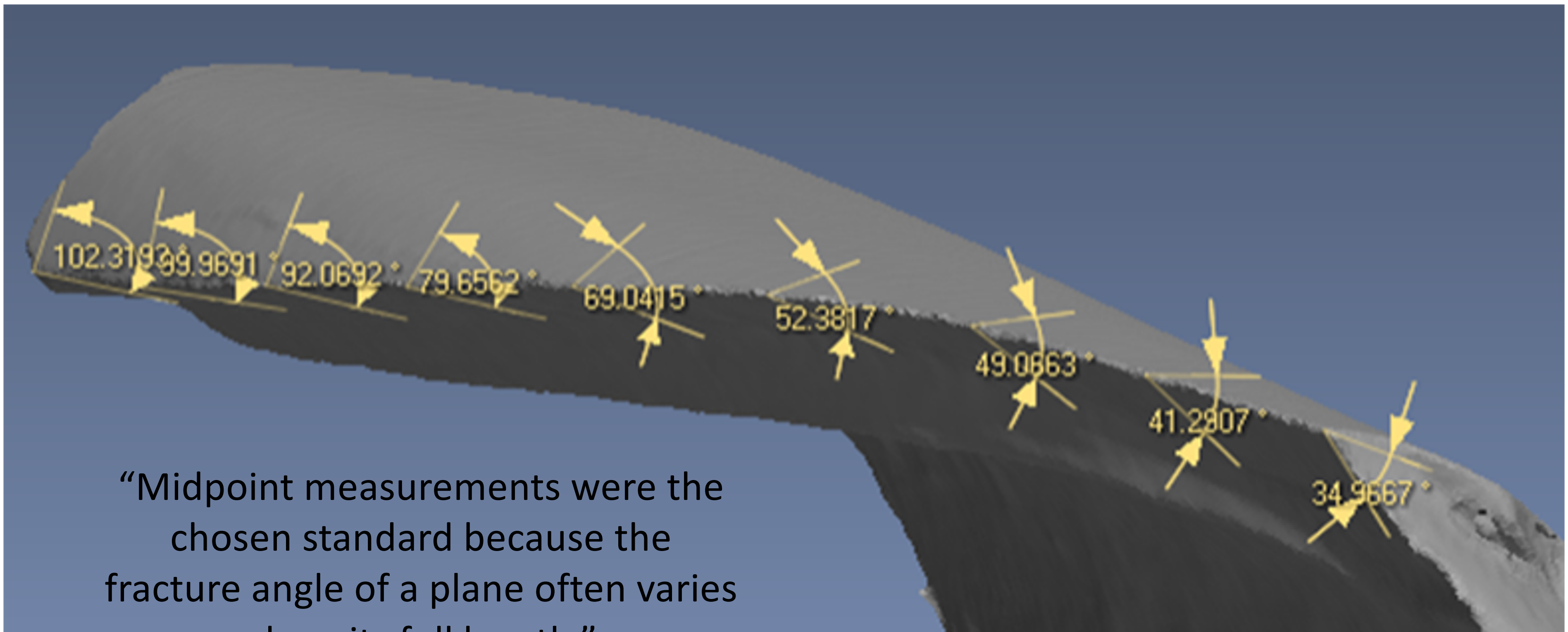
Jean-Baptiste Romé de l'Isle et René Just Haüy sont les deux fondateurs de la cristallographie moderne dans la seconde moitié du XVIIIe siècle. Ils ont utilisé ces gabarits, ce goniomètre et ces modèles en bois pour modéliser et démontrer leurs théories.


Système cubique
Système orthorhombique
Système quadratique
Système trigonal
Système monoclinique
Système hexagonal
Système triclinique

Fracture Angles — goniometer measurements



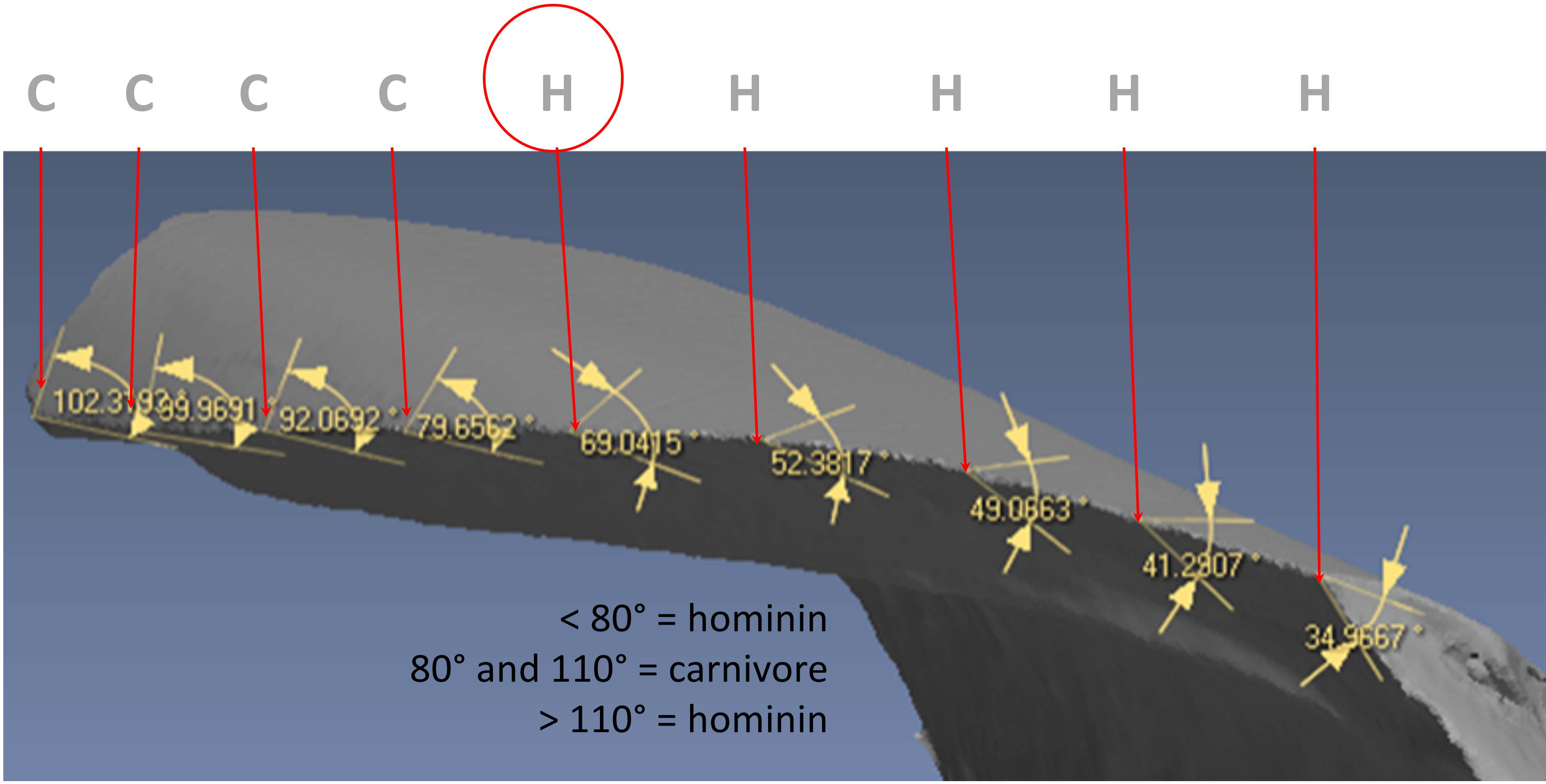
Fracture Angles: Methods



"Midpoint measurements were the chosen standard because the fracture angle of a plane often varies along its full length."
(Pickering et al., 2015:251)

Carnivore Created Fragment

Average = 66°



Fracture Angles

- Not fully tested
  - Limited experimental studies
  - Different taxa tested in each
    - Different results related to taxon and element
    - No independent testing of the same taxon

TRABAJOS DE PREHISTORIA
63, Nº 1, Enero-Junio 2006, pp. 37-45. ISSN 0082-5638

DETERMINACIÓN DE PROCESOS DE FRACTURA SOBRE HUESOS FRESCOS: UN SISTEMA DE ANÁLISIS DE LOS ÁNGULOS DE LOS PLANOS DE FRACTURACIÓN COMO DISCRIMINADOR DE AGENTES BIÓTICOS

DETERMINATION OF THE FRACTURE PROCESSES OF FRESH BONE: AN ANALYTICAL SYSTEM OF THE ANGLES OF FRACTURE PLANES AS AN INDICATOR OF BIOTIC AGENTS

VIRGINIA ALC¡NTARA GARCÕA, REBECA BARBA EGIDO, JOS… MARÕA BARRAL DEL PINO, ANA BEL…N CRESPO RUIZ, ARCO IRIS EIRIZ VIDAL, ¡LVARO FALQUINA APARICIO, SILVIA HERRERO CALLEJA, ANA IBARRA JIM…NEZ, MARTA MEGÕAS GONZ¡LEZ, MAITE P…REZ GIL, VICTORIA P…REZ TELLO, JORGE ROLLAND CALVO, JOS… YRAVEDRA S¡INZ DE LOS TERREROS, AIXA VIDAL Y MANUEL DOMÕNGUEZ-RODRIGO (*)

archaeometry

Archaeometry 53, 5 (2011) 996–1011 doi: 10.1111/j.1475-4754.2010.00576.x

TESTING ANALOGICAL TAPHONOMIC SIGNATURES IN BONE BREAKING: A COMPARISON BETWEEN HAMMERSTONE-BROKEN EQUID AND BOVID BONES*

S. DE JUANA and M. DOMÍNGUEZ-RODRIGO†

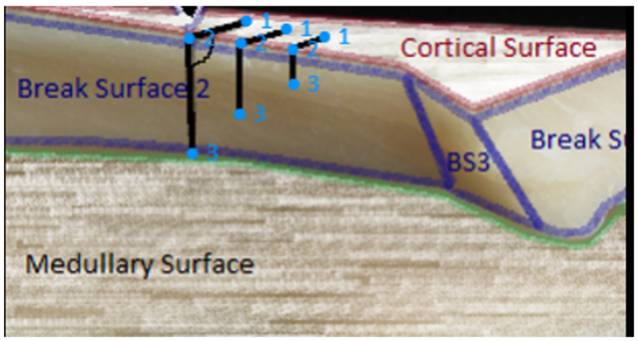
archaeometry

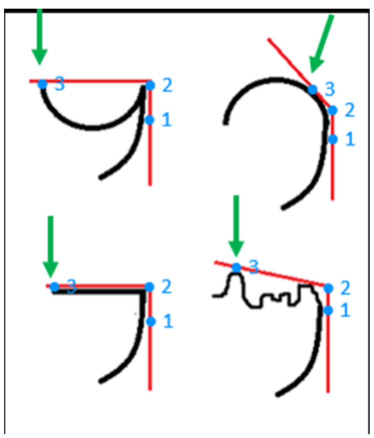
Archaeometry ••, •• (2017) ••–•• doi: 10.1111/arcm.12285

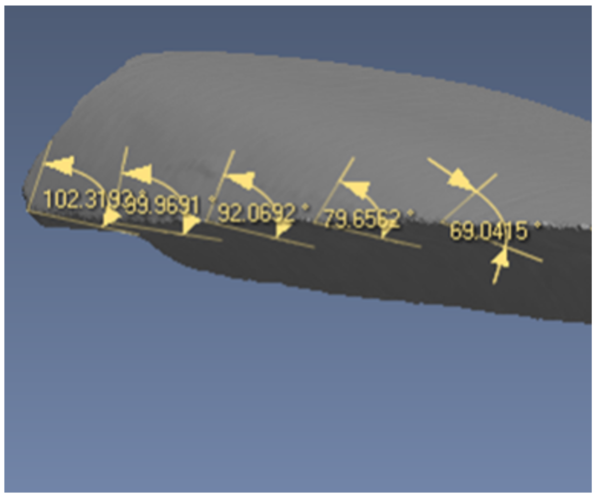
NEW ANALYTICAL METHODS FOR COMPARING BONE FRACTURE ANGLES: A CONTROLLED STUDY OF HAMMERSTONE AND HYENA (*Crocuta crocuta*) LONG BONE BREAKAGE*

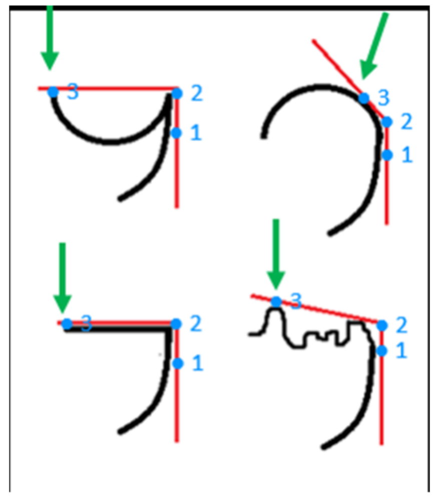
R. COIL,[1,2]† M. TAPPEN [2] and K. YEZZI-WOODLEY[2]

Fracture Angles: Methods



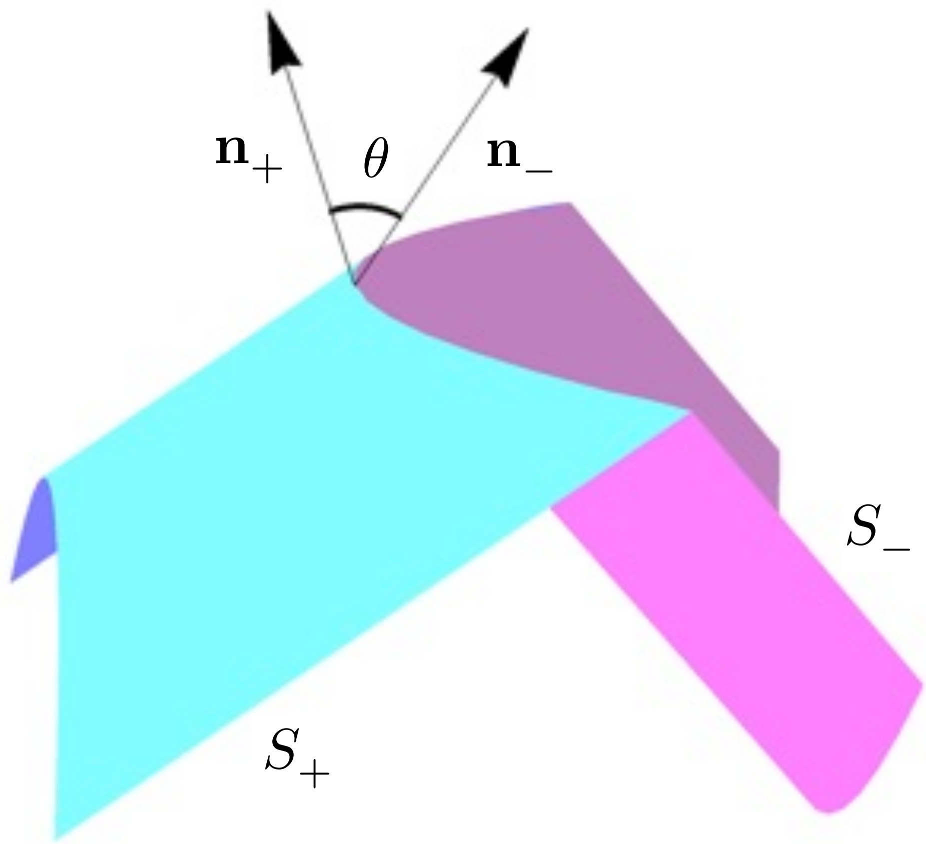


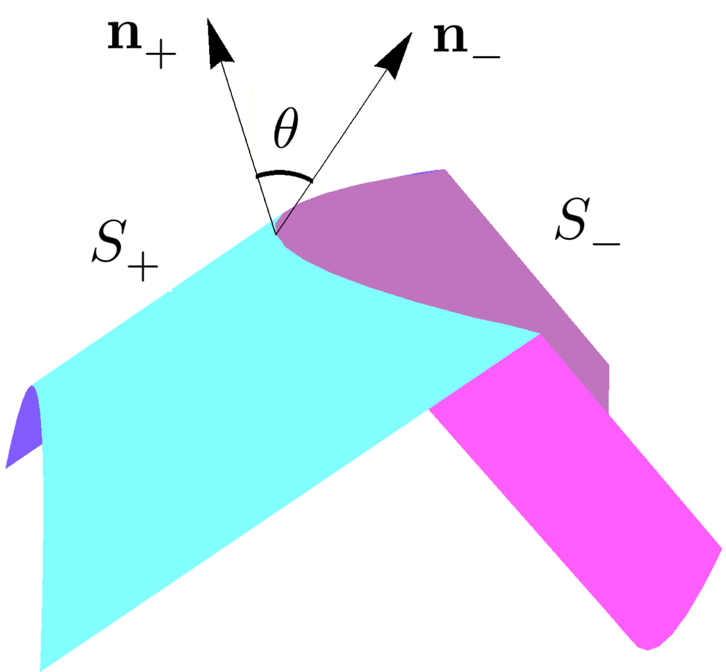

102.3193 °
99.9691 °
92.0692 °
79.6562 °
69.0415 °


3
2
1
3
2
1
3
2
1
3
2
1


GENERAL TOOLS

Virtual Goniometer

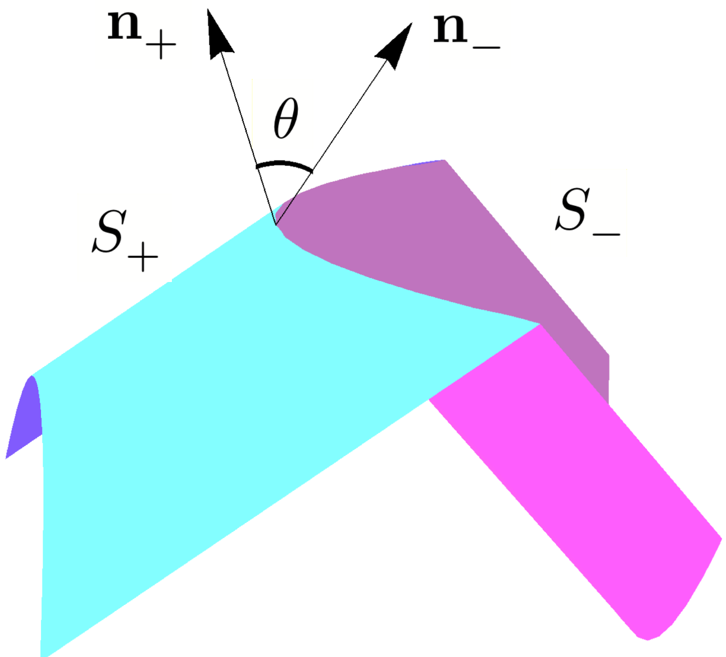




Virtual Goniometer Data

Hominins vs. Hyena via Break Angle (Humerus)

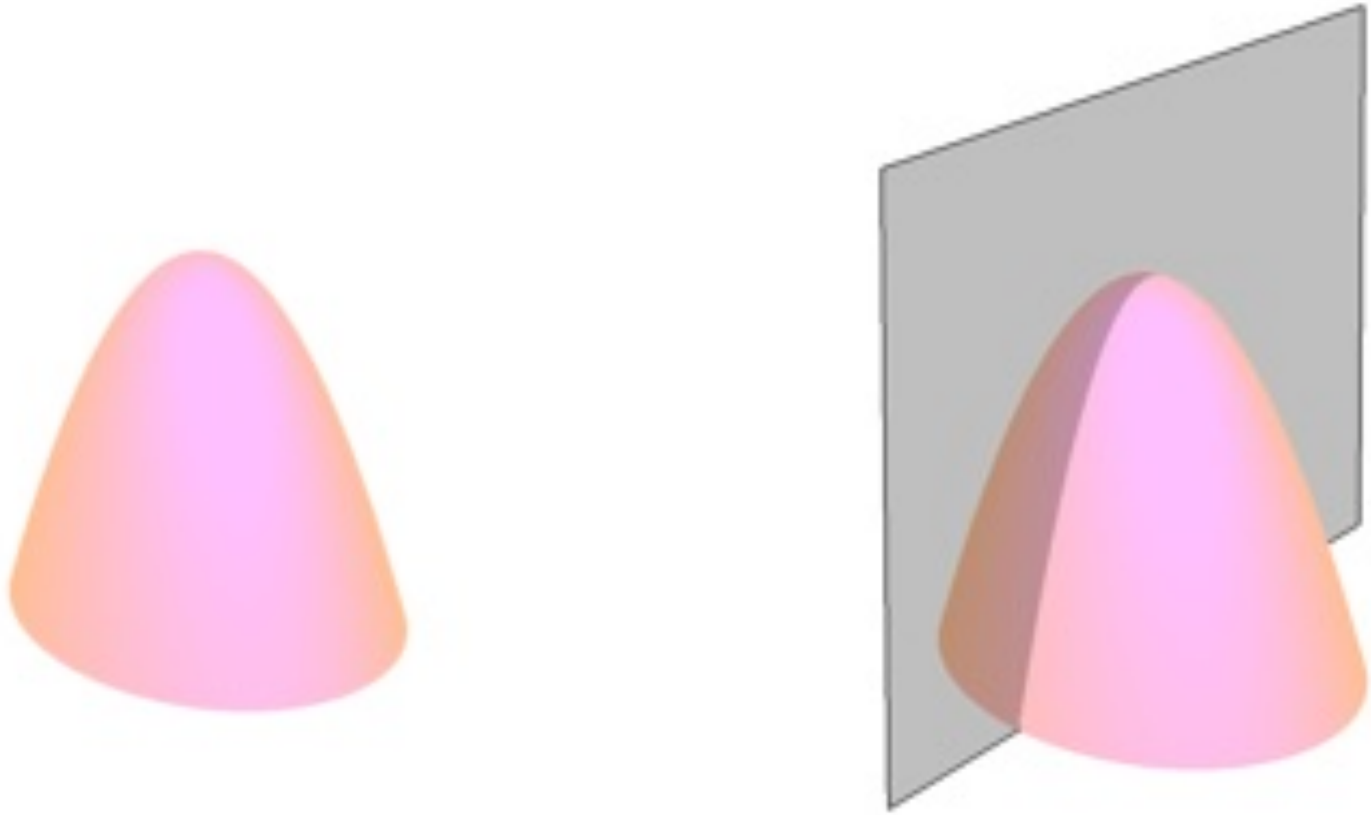
Yes category	yes Size	No category	no Size	Training percentage	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
hominin humerus	779	hyena humerus	512	75	195	0.959	0.152	0.53071	0.78756	0.041	0.85
hyena humerus	512	hominin humerus	779	75	128	0.959	0.094	0.51421	0.6963	0.041	0.91
hominin humerus	779	hyena humerus	512	65	273	0.959	0.154	0.5313	0.78974	0.041	0.85
hyena humerus	512	hominin humerus	779	65	180	0.934	0.121	0.51517	0.64706	0.066	0.88
hominin humerus	779	hyena humerus	512	50	390	0.957	0.163	0.53344	0.79126	0.043	0.84
hyena humerus	512	hominin humerus	779	50	256	0.961	0.125	0.52342	0.7622	0.039	0.88
hominin humerus	779	hyena humerus	512	40	468	0.958	0.139	0.52666	0.76796	0.042	0.86
hyena humerus	512	hominin humerus	779	40	308	0.95	0.123	0.51998	0.71098	0.05	0.88



Hominins vs. Hyena via Break Angle (Femur)

Yes category	yes Size	No category	no Size	Training percentage	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
hominin femur	1565	hyena femur	897	75	392	0.941	0.268	0.56246	0.81957	0.059	0.73
hyena femur	897	hominin femur	1565	75	225	0.959	0.139	0.52692	0.77222	0.041	0.86
hominin femur	1565	hyena femur	897	65	548	0.958	0.365	0.60138	0.89681	0.042	0.64
hyena femur	897	hominin femur	1565	65	314	0.949	0.197	0.54167	0.79435	0.051	0.8
hominin femur	1565	hyena femur	897	50	783	0.949	0.428	0.62393	0.89353	0.051	0.57
hyena femur	897	hominin femur	1565	50	449	0.942	0.233	0.5512	0.80069	0.058	0.77
hominin femur	1565	hyena femur	897	40	897	0.96	0.371	0.60415	0.90268	0.04	0.63
hyena femur	897	hominin femur	1565	40	539	0.958	0.198	0.54432	0.825	0.042	0.8

Principal Curvatures



Surface Curvatures

- Principal curvatures: κ_1, κ_2
- Gauss curvature: $K = \kappa_1 \, \kappa_2$ — intrinsic
- Mean curvature: $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ — extrinsic
- Curvature difference: $\Delta = |\, \kappa_1 - \kappa_2 \,|$

Sample Size (Manual Data)

Number of breaks per element and actor of breakage

	Femur	Humerus	Radius-Ulna	Tibia	Total
Crocuta	411	120	0	64	595
Hominin	363	291	287	333	1274
Rockfall	0	85	105	0	190
Total	774	496	392	397	2059

Number of breaks per element and actor for which no goniometer measurement could be taken

	Femur	Humerus	Radius-Ulna	Tibia	Total
Crocuta	234 (57%)	32 (27%)	-	13 (20%)	279 (47%)
Hominin	102 (28%)	51 (18%)	64 (22%)	153 (46%)	370 (29%)
Rockfall	-	21 (25%)	31 (30%)	-	52 (27%)
Total	336 (43%)	104 (21%)	95 (24%)	166 (42%)	701 (34%)

Number of breaks per element and method of breakage

	Femur	Humerus	Radius-Ulna	Tibia	Total
Batting	159	144	130	186	619
Crocuta	411	120	-	64	595
Rockfall	-	85	105	-	190
Hammerstone & Anvil	175	137	122	147	581
Hammerstone only	-	10	-	-	10
Hominin mixed method	29	-	35	-	64
Total	774	496	392	397	2059

Number of breaks per element and method for which no goniometer measurement could be taken

	Femur	Humerus	Radius-Ulna	Tibia	Total
Batting	41 (26%)	29 (20%)	22 (17%)	95 (51%)	187 (30%)
Crocuta	234 (57%)	32 (27%)	-	13 (20%)	279 (47%)
Rockfall	-	21 (25%)	31 (30%)	-	52 (27%)
Hammerstone & Anvil	57 (33%)	19 (14%)	35 (29%)	58 (39%)	169 (29%)
Hammerstone only	-	3 (30%)	-	-	3 (30%)
Hominin mixed method	4 (14%)	-	7 (20%)	-	11 (17%)
Total	336 (43%)	104 (21%)	95 (24%)	166 (42%)	701 (34%)

Sample Size (Digital Data)

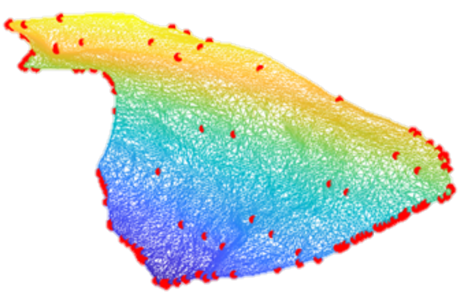
Manual Data

- 457 fragments
- 2,059 breaks
- 1,358 measurements

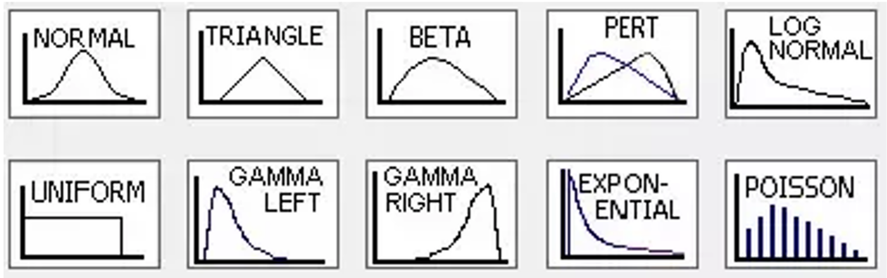
Digital Data

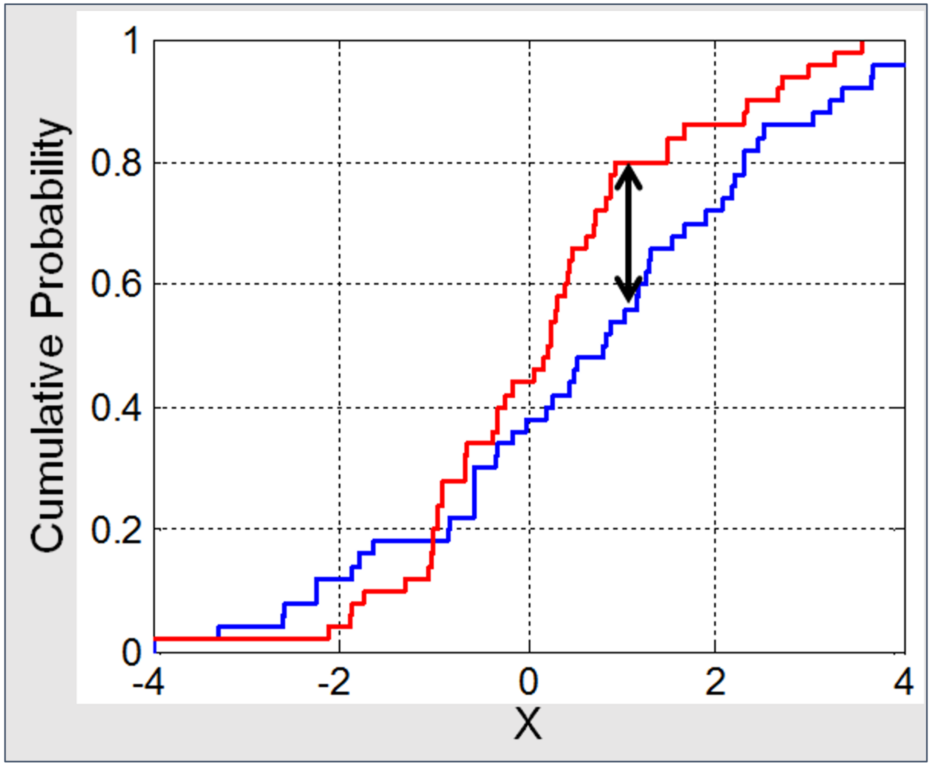
- 82 fragments
- 1,376,900 measurements
- 1% = 13,769

	Femur	Humerus	Tibia	Radius-Ulna	Total
Batting	1,758	606	1,878	1,531	5,773
Crocuta	1,824	780	-	-	2,604
Hammerstone & Anvil	1,485	1,003	1,291	1,613	5,392
Total	5,067	2,389	3,169	3,144	13,769



First Stages





Kolmogorov-Smirnov test

Hominins vs. hyena (femur) – principal curvature differences

Yes category	yes Size	No category	no Size	Training percentage	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
hominin (femur)	3243	hyena (femur)	1824	75	811	0.942	1	1	0.94518	0.058	0
hyena (femur)	1824	hominin (femur)	3243	75	456	0.95	1	1	0.95238	0.05	0
hominin (femur)	3243	hyena (femur)	1824	65	1136	0.947	1	1	0.94967	0.053	0
hyena (femur)	1824	hominin (femur)	3243	65	639	0.939	1	1	0.94251	0.061	0
hominin (femur)	3243	hyena (femur)	1824	50	1622	0.949	1	1	0.95147	0.051	0
hyena (femur)	1824	hominin (femur)	3243	50	912	0.946	1	1	0.94877	0.054	0
hominin (femur)	3243	hyena (femur)	1824	40	1824	0.946	1	1	0.94877	0.054	0
hyena (femur)	1824	hominin (femur)	3243	40	1095	0.938	1	1	0.94162	0.062	0

Hominins vs. hyena (humerus) – principal curvature differences

Yes category	yes Size	No category	no Size	Training percentage	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
hominin (humerus)	1609	hyena (humerus)	780	75	403	0.954	1	1	0.95602	0.046	0
hyena (humerus)	780	hominin (humerus)	1609	75	195	0.941	1	1	0.94429	0.059	0
hominin (humerus)	1609	hyena (humerus)	780	65	564	0.947	1	1	0.94967	0.053	0
hyena (humerus)	780	hominin (humerus)	1609	65	273	0.933	1	1	0.93721	0.067	0
hominin (humerus)	1609	hyena (humerus)	780	50	780	0.96	1	1	0.96154	0.04	0
hyena (humerus)	780	hominin (humerus)	1609	50	390	0.95	1	1	0.95238	0.05	0
hominin (humerus)	1609	hyena (humerus)	780	40	780	0.95	1	1	0.95238	0.05	0
hyena (humerus)	780	hominin (humerus)	1609	40	468	0.949	1	1	0.95147	0.051	0

Hammerstone vs. batting (femur) – principal curvature differences

Yes category	yes Size	No category	no Size	Training percentage	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
Batting femur	1758	HS & Anv femur	1485	75	440	0.951	1	1	0.95329	0.049	0
HS & Anv femur	1485	Batting femur	1758	75	372	0.956	1	1	0.95785	0.044	0
Batting femur	1758	HS & Anv femur	1485	65	616	0.938	1	1	0.94162	0.062	0
HS & Anv femur	1485	Batting femur	1758	65	520	0.948	1	1	0.95057	0.052	0
Batting femur	1758	HS & Anv femur	1485	50	879	0.942	1	1	0.94518	0.058	0
HS & Anv femur	1485	Batting femur	1758	50	743	0.957	1	1	0.95877	0.043	0
Batting femur	1758	HS & Anv femur	1485	40	1055	0.954	1	1	0.95602	0.046	0
HS & Anv femur	1485	Batting femur	1758	40	891	0.951	1	1	0.95329	0.049	0

HS & Anv vs. batting (humerus) – surface curvature

Yes category	yes Size	No category	no Size	Training percentage	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
anv humerus	606	hsanv humerus	1003	75	152	0.947	1	1	0.94967	0.053	0
hsanv humerus	1003	anv humerus	606	75	251	0.952	1	1	0.9542	0.048	0
anv humerus	606	hsanv humerus	1003	65	213	0.948	1	1	0.95057	0.052	0
hsanv humerus	1003	anv humerus	606	65	352	0.951	1	1	0.95329	0.049	0
anv humerus	606	hsanv humerus	1003	50	303	0.965	1	1	0.96618	0.035	0
hsanv humerus	1003	anv humerus	606	50	502	0.961	1	1	0.96246	0.039	0
anv humerus	606	hsanv humerus	1003	40	364	0.941	1	1	0.94429	0.059	0
hsanv humerus	1003	anv humerus	606	40	602	0.946	1	1	0.94877	0.054	0

HS & Anv vs. batting (tibia) – surface curvature

Yes category	yes Size	No category	no Size	Training percentage	Training Size	Sensitivit y	Specificit y	Precisio n	Negative Predictive Rate	Miss Rate	Fall out
anv tibia	1878	hsanv tibia	1291	75	470	0.945	1	1	0.94787	0.055	0
hsanv tibia	1291	anv tibia	1878	75	323	0.943	1	1	0.94607	0.057	0
anv tibia	1878	hsanv tibia	1291	65	658	0.94	1	1	0.9434	0.06	0
hsanv tibia	1291	anv tibia	1878	65	452	0.954	1	1	0.95602	0.046	0
anv tibia	1878	hsanv tibia	1291	50	939	0.946	1	1	0.94877	0.054	0
hsanv tibia	1291	anv tibia	1878	50	646	0.947	1	1	0.94967	0.053	0
anv tibia	1878	hsanv tibia	1291	40	1127	0.941	1	1	0.94429	0.059	0
hsanv tibia	1291	anv tibia	1878	40	775	0.945	1	1	0.94787	0.055	0

HS & Anv vs. batting (rad-uln) – surface curvature

Yes category	yes Size	No category	no Size	Training percentage	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
Batting raduln	1878	HS & Anv raduln	1291	75	470	0.962	1	1	0.96339	0.038	0
HS & Anv raduln	1291	Batting raduln	1878	75	323	0.957	1	1	0.95877	0.043	0
Batting raduln	1878	HS & Anv raduln	1291	65	658	0.948	1	1	0.95057	0.052	0
HS & Anv raduln	1291	Batting raduln	1878	65	452	0.95	1	1	0.95238	0.05	0
Batting raduln	1878	HS & Anv raduln	1291	50	939	0.954	1	1	0.95602	0.046	0
HS & Anv raduln	1291	Batting raduln	1878	50	646	0.953	1	1	0.95511	0.047	0
Batting raduln	1878	HS & Anv raduln	1291	40	1127	0.946	1	1	0.94877	0.054	0
HS & Anv raduln	1291	Batting raduln	1878	40	775	0.956	1	1	0.95785	0.044	0

Hominins vs. hyena (femur) – manual goniometer data

Yes category	yes Size	No category	no Size	Training percentage	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
hominin femur	261	hyena femur	177	75	66	0.956	0.368	0.60202	0.8932	0.044	0.632
hyena femur	177	hominin femur	261	75	45	0.957	0.222	0.55159	0.83774	0.043	0.778
hominin femur	261	hyena femur	177	65	92	0.959	0.502	0.6582	0.92449	0.041	0.498
hyena femur	177	hominin femur	261	65	62	0.966	0.294	0.57775	0.89634	0.034	0.706
hominin femur	261	hyena femur	177	50	131	0.963	0.561	0.68688	0.93813	0.037	0.439
hyena femur	177	hominin femur	261	50	89	0.966	0.299	0.57948	0.8979	0.034	0.701
hominin femur	261	hyena femur	177	40	157	0.949	0.494	0.65223	0.90642	0.051	0.506
hyena femur	177	hominin femur	261	40	107	0.956	0.327	0.58686	0.8814	0.044	0.673

Hominins vs. hyena (humerus) — virtual goniometer

Yes category	yes Size	No category	no Size	Training %	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
hominin humerus	779	hyena humerus	512	75	195	0.959	0.152	0.53071	0.78756	0.041	0.85
hyena humerus	512	hominin humerus	779	75	128	0.959	0.094	0.51421	0.69630	0.041	0.91
hominin humerus	779	hyena humerus	512	65	273	0.959	0.154	0.53130	0.78974	0.041	0.85
hyena humerus	512	hominin humerus	779	65	180	0.934	0.121	0.51517	0.64706	0.066	0.88
hominin humerus	779	hyena humerus	512	50	390	0.957	0.163	0.53344	0.79126	0.043	0.84
hyena humerus	512	hominin humerus	779	50	256	0.961	0.125	0.52342	0.76220	0.039	0.88
hominin humerus	779	hyena humerus	512	40	468	0.958	0.139	0.52666	0.76796	0.042	0.86
hyena humerus	512	hominin humerus	779	40	308	0.950	0.123	0.51998	0.71098	0.05	0.88

Hominins vs. hyena (femur) — virtual goniometer

Yes category	yes Size	No category	no Size	Training %	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
hominin femur	1565	hyena femur	897	75	392	0.941	0.268	0.56246	0.81957	0.059	0.73
hyena femur	897	hominin femur	1565	75	225	0.959	0.139	0.52692	0.77222	0.041	0.86
hominin femur	1565	hyena femur	897	65	548	0.958	0.365	0.60138	0.89681	0.042	0.64
hyena femur	897	hominin femur	1565	65	314	0.949	0.197	0.54167	0.79435	0.051	0.80
hominin femur	1565	hyena femur	897	50	783	0.949	0.428	0.62393	0.89353	0.051	0.57
hyena femur	897	hominin femur	1565	50	449	0.942	0.233	0.55120	0.80069	0.058	0.77
hominin femur	1565	hyena femur	897	40	897	0.960	0.371	0.60415	0.90268	0.04	0.63
hyena femur	897	hominin femur	1565	40	539	0.958	0.198	0.54432	0.82500	0.042	0.80

Hominins vs. hyena (humerus) – manual data

Yes category	yes Size	No category	no Size	Training percentage	Training Size	Sensitivity	Specificity	Precision	Negative Predictive Rate	Miss Rate	Fall out
hominin humerus	240	hyena humerus	88	75	60	0.958	0.069	0.50715	0.62162	0.042	0.931
hyena humerus	88	hominin humerus	240	75	22	0.956	0.055	0.50289	0.55556	0.044	0.945
hominin humerus	240	hyena humerus	88	65	84	0.953	0.019	0.49276	0.28788	0.047	0.981
hyena humerus	88	hominin humerus	240	65	31	0.955	0.069	0.50636	0.60526	0.045	0.931
hominin humerus	240	hyena humerus	88	50	88	0.96	0.035	0.4987	0.46667	0.04	0.965
hyena humerus	88	hominin humerus	240	50	44	0.964	0.066	0.5079	0.64706	0.036	0.934
hominin humerus	240	hyena humerus	88	40	88	0.954	0.055	0.50237	0.54455	0.046	0.945
hyena humerus	88	hominin humerus	240	40	53	0.958	0.067	0.50661	0.61468	0.042	0.933

Moving forward

- More taxa
  - *Bos*
  - *Ovis/Capra*
  - *Equus*
- All appendicular long bones
- Archaeological collections
- Factor in rock fall
- More geometric methods
  - Volume, surface areas (total/faces)
  - Mean, variance, PCA
  - Higher moments
  - Digital measures of break angles at break curves using surface normals
  - Break curve geometric invariants: curvature, torsion, etc.
  - Surface curvatures (principal, Gauss, mean, total)

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