Fractalization and Quantization in Dispersive Systems

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Dispersion of surface waves on a pond

Dispersion

Definition. A linear partial differential equation is called dispersive if the different Fourier modes travel unaltered but at different speeds.

Substituting

$$u(t,x) = e^{\,\mathrm{i}\,(k\,x - \omega\,t)}$$

produces the dispersion relation

$$\omega = \omega(k), \qquad\qquad \omega, k \in \mathbb{R}$$

relating frequency ω and wave number k.

Phase velocity: $$c_p = \frac{\omega(k)}{k}$$

Group velocity: $$c_g = \frac{d\omega}{dk}$$ (stationary phase)

A Simple Linear Dispersive Wave Equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$\Longrightarrow$ linearized Korteweg–deVries equation

Dispersion relation: $\omega = k^3$

Phase velocity: $c_p = \dfrac{\omega}{k} = k^2$

Group velocity: $c_g = \dfrac{d\omega}{dk} = 3\,k^2$

Thus, wave packets (and energy) move *faster* (to the right) than the individual waves.

Linear Dispersion on the Line

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad u(0,x) = f(x)$$

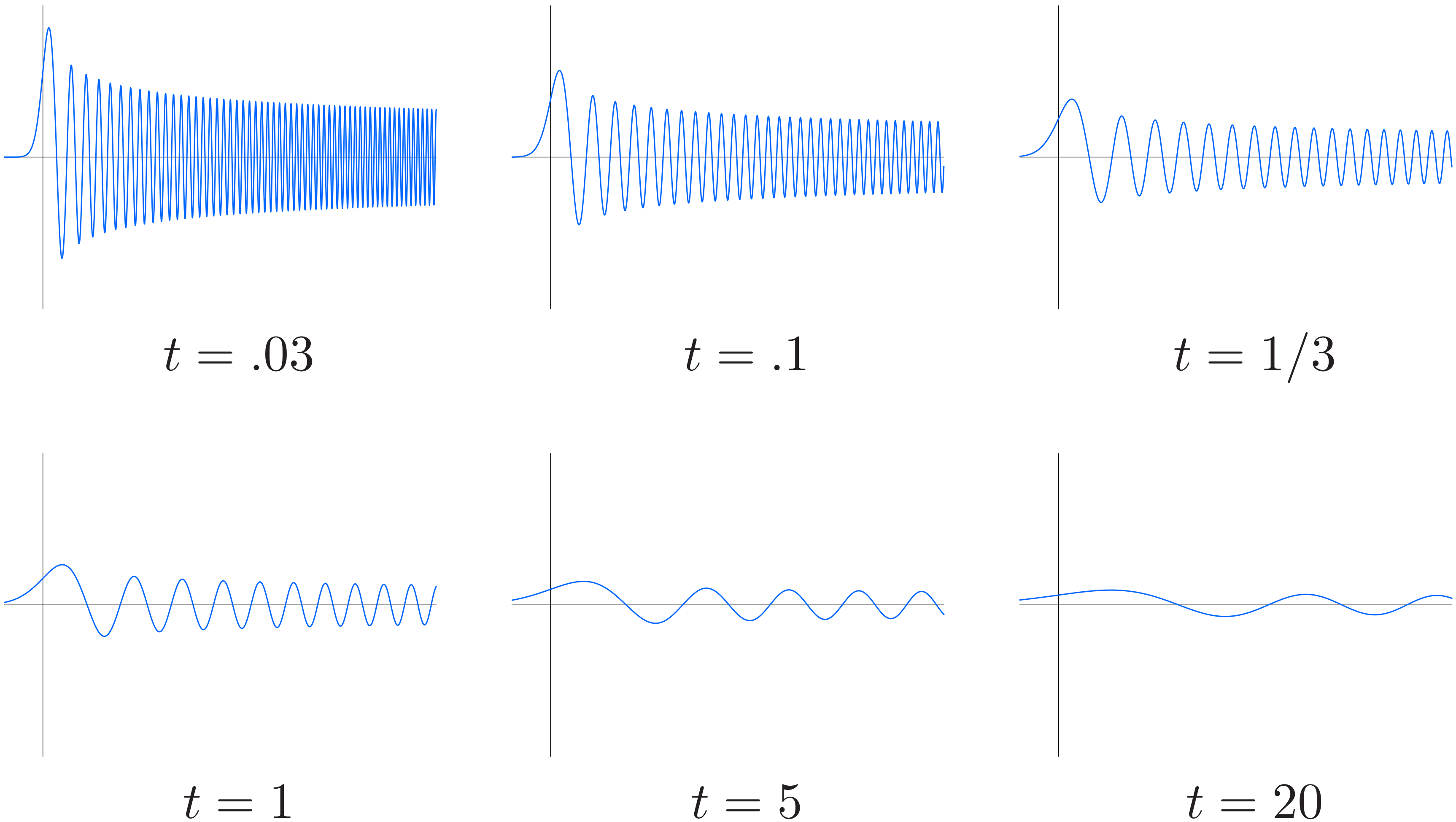
Fourier transform solution:

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{\,\mathrm{i}\,(k\,x - k^3\,t)}\, dk$$

Fundamental solution $\qquad u(0,x) = \delta(x)$

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\,\mathrm{i}\,(k\,x - k^3\,t)}\, dk = \frac{1}{\sqrt[3]{3\,t}}\ \mathrm{Ai}\left(-\frac{x}{\sqrt[3]{3\,t}} \right)$$

Fundamental solution to linearized KdV



$t = .03$
$t = .1$
$t = 1/3$
$t = 1$
$t = 5$
$t = 20$

Linear Dispersion on the Line

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad u(0,x) = f(x)$$

Superposition solution formula:

$$u(t,x) = \frac{1}{\sqrt[3]{3\,t}} \int_{-\infty}^{\infty} f(\xi)\ \mathrm{Ai}\left(\frac{\xi - x}{\sqrt[3]{3\,t}}\right) d\xi$$

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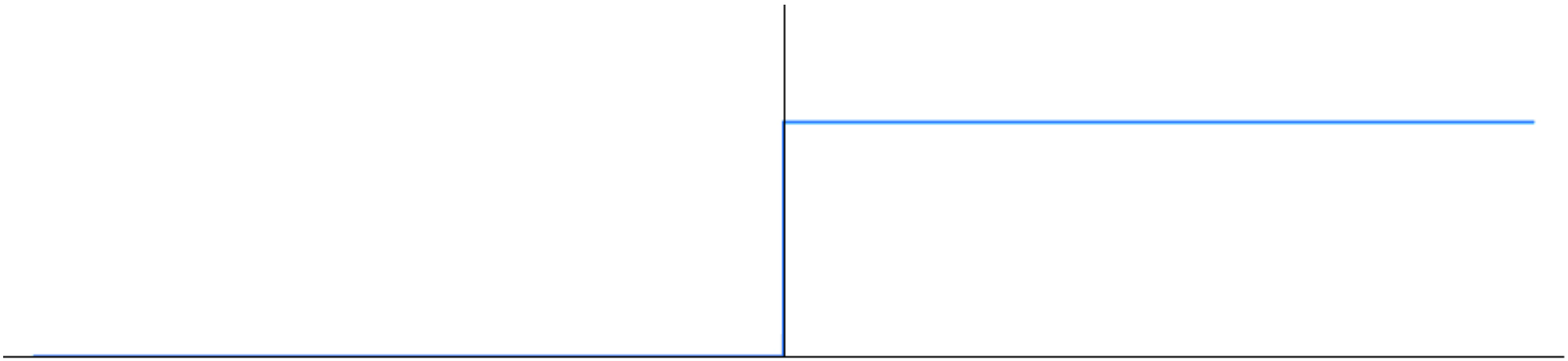
Step function initial data: $\quad u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$

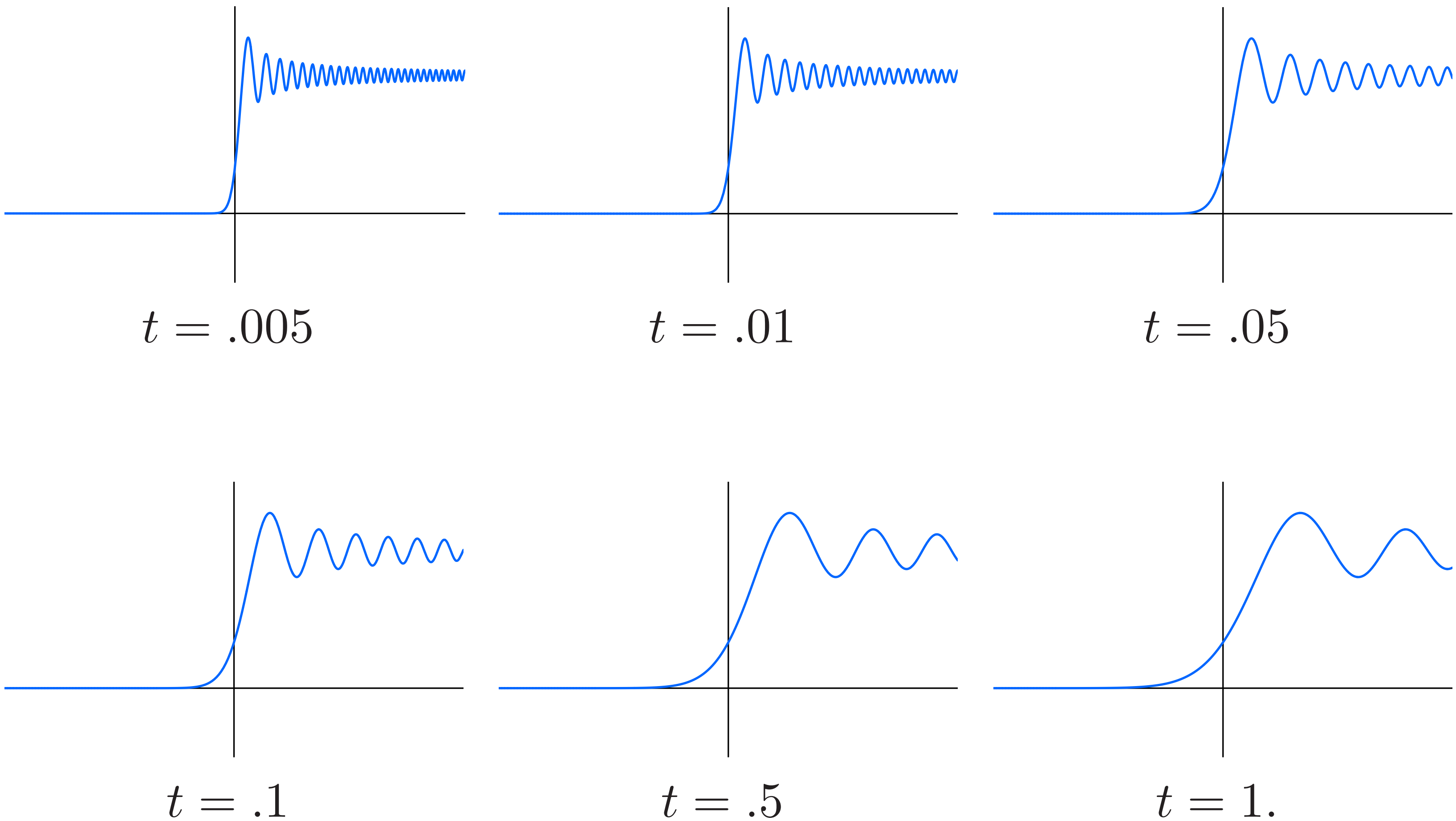
$$u(t,x) = \frac{1}{3} \ - \ H\left(-\frac{x}{\sqrt[3]{3t}} \right)$$

$$H(z) = \frac{z\,\Gamma\!\left(\frac{1}{3}\right)\,{}_1F_2\left(\frac{1}{3};\,\frac{2}{3},\,\frac{4}{3};\,\frac{1}{9}z^3\right)}{3^{5/3}\,\Gamma\!\left(\frac{2}{3}\right)\Gamma\!\left(\frac{4}{3}\right)} - \frac{z^2\,\Gamma\!\left(\frac{2}{3}\right)\,{}_1F_2\left(\frac{2}{3};\,\frac{4}{3},\,\frac{5}{3};\,\frac{1}{9}z^3\right)}{3^{7/3}\,\Gamma\!\left(\frac{4}{3}\right)\Gamma\!\left(\frac{5}{3}\right)}$$

$\Longrightarrow$ Mathematica — via Meijer G functions

Step solution to linearized KdV





$t = .005$
$t = .01$
$t = .05$
$t = .1$
$t = .5$
$t = 1.$

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi) \qquad \frac{\partial^2 u}{\partial x^2}(t,-\pi) = \frac{\partial^2 u}{\partial x^2}(t,\pi)$$

Step function initial data:

$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Periodic Linear Dispersion

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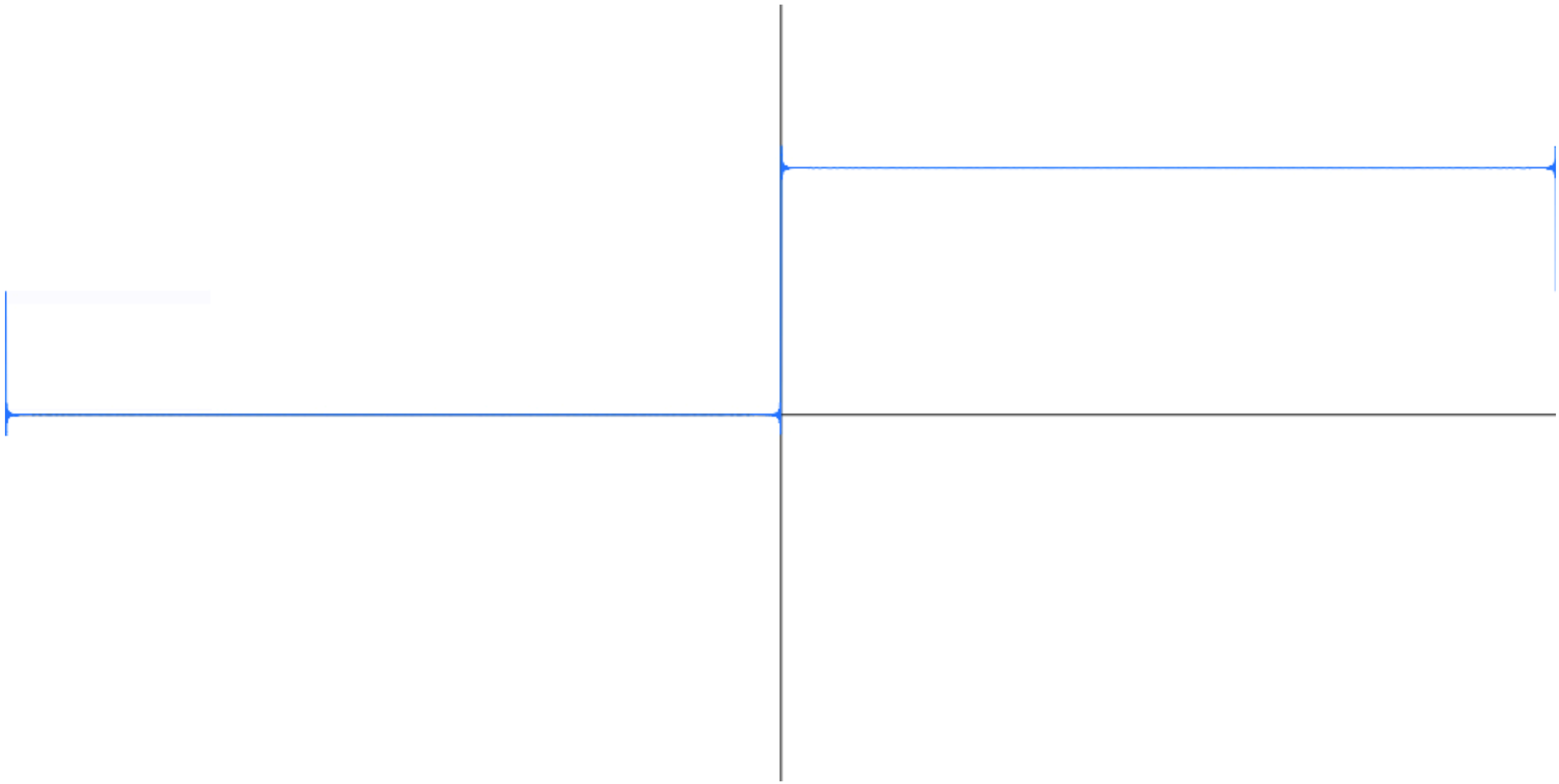
Step function initial data:

$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

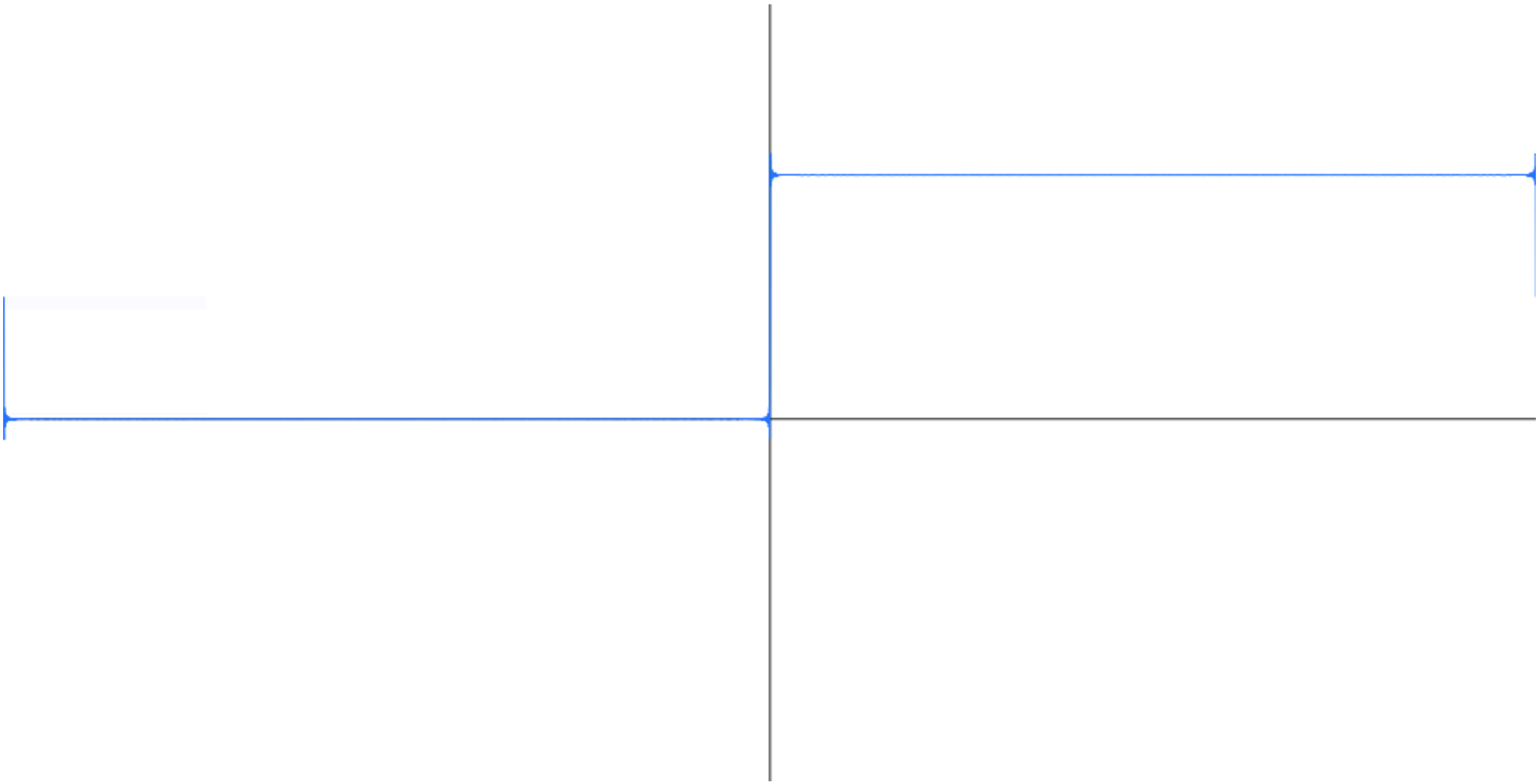
Fourier series solution formula:

$$u^\star(t,x) \;\sim\; \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin(\,(2\,j+1)\,x - (2\,j+1)^3\,t\,)}{2\,j+1}\,.$$

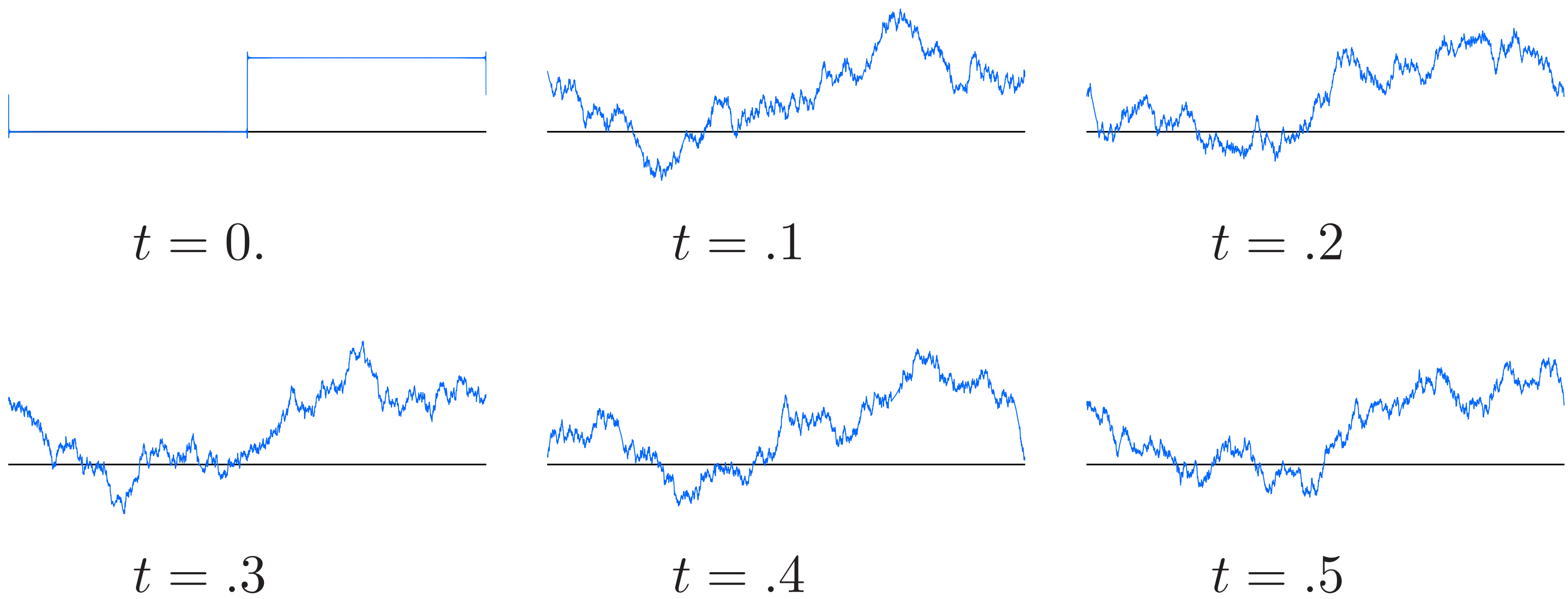
Periodic linearized KdV with $\Delta t = .01$



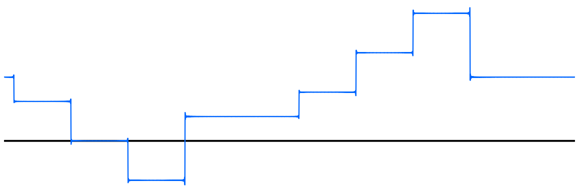
Periodic linearized KdV with $\Delta t = \pi/300$



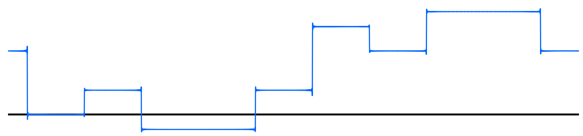
Periodic linearized KdV — irrational times



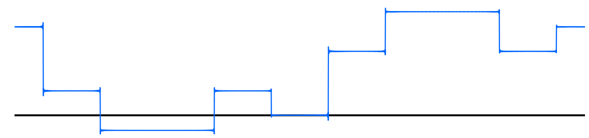
Periodic linearized KdV — rational times



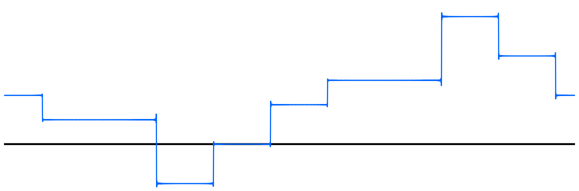
$t = \frac{1}{30}\pi$



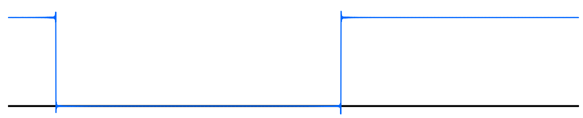
$t = \frac{1}{15}\pi$



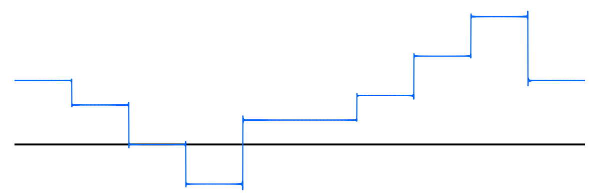
$t = \frac{1}{10}\pi$



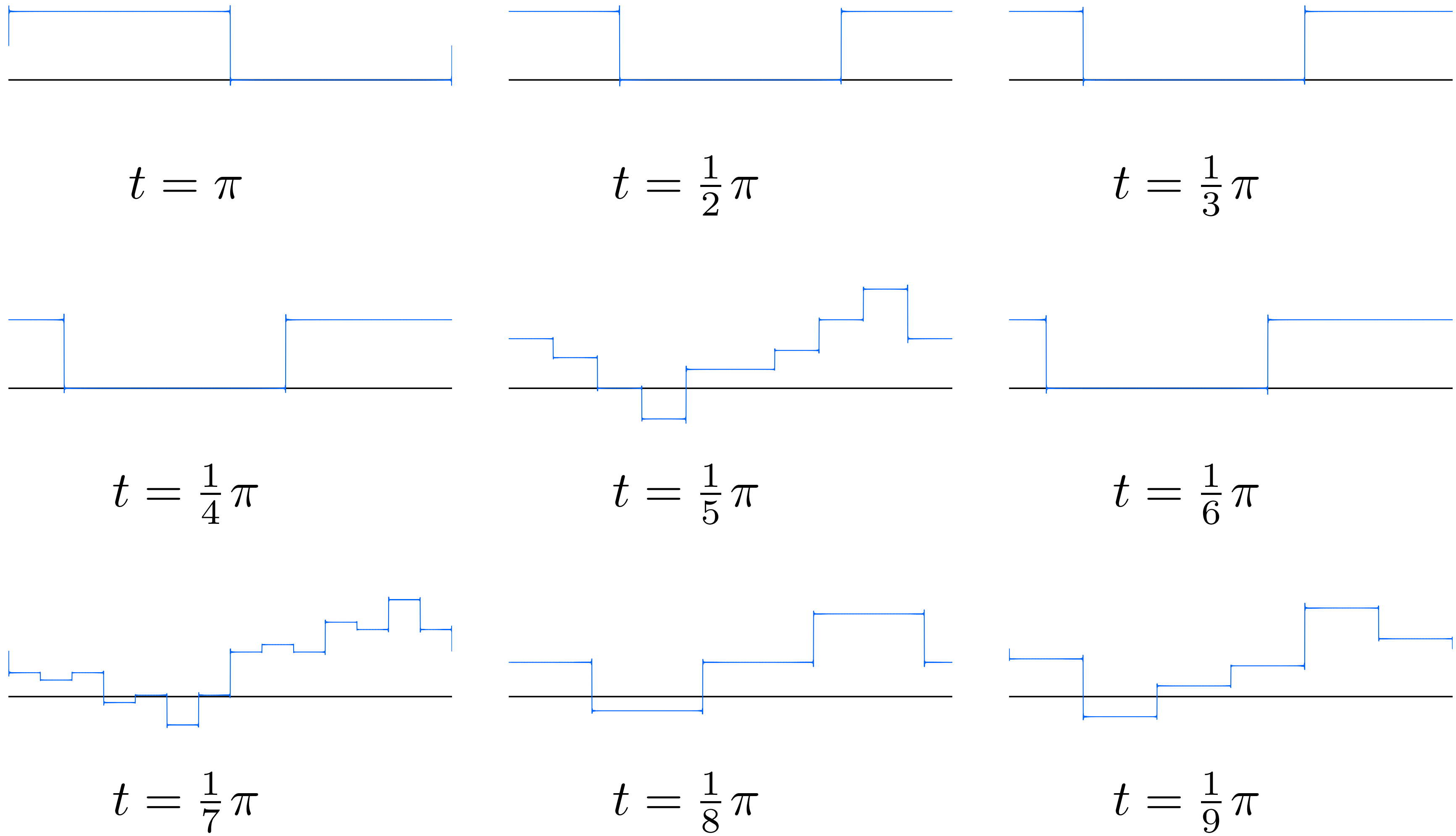
$t = \frac{2}{15}\pi$



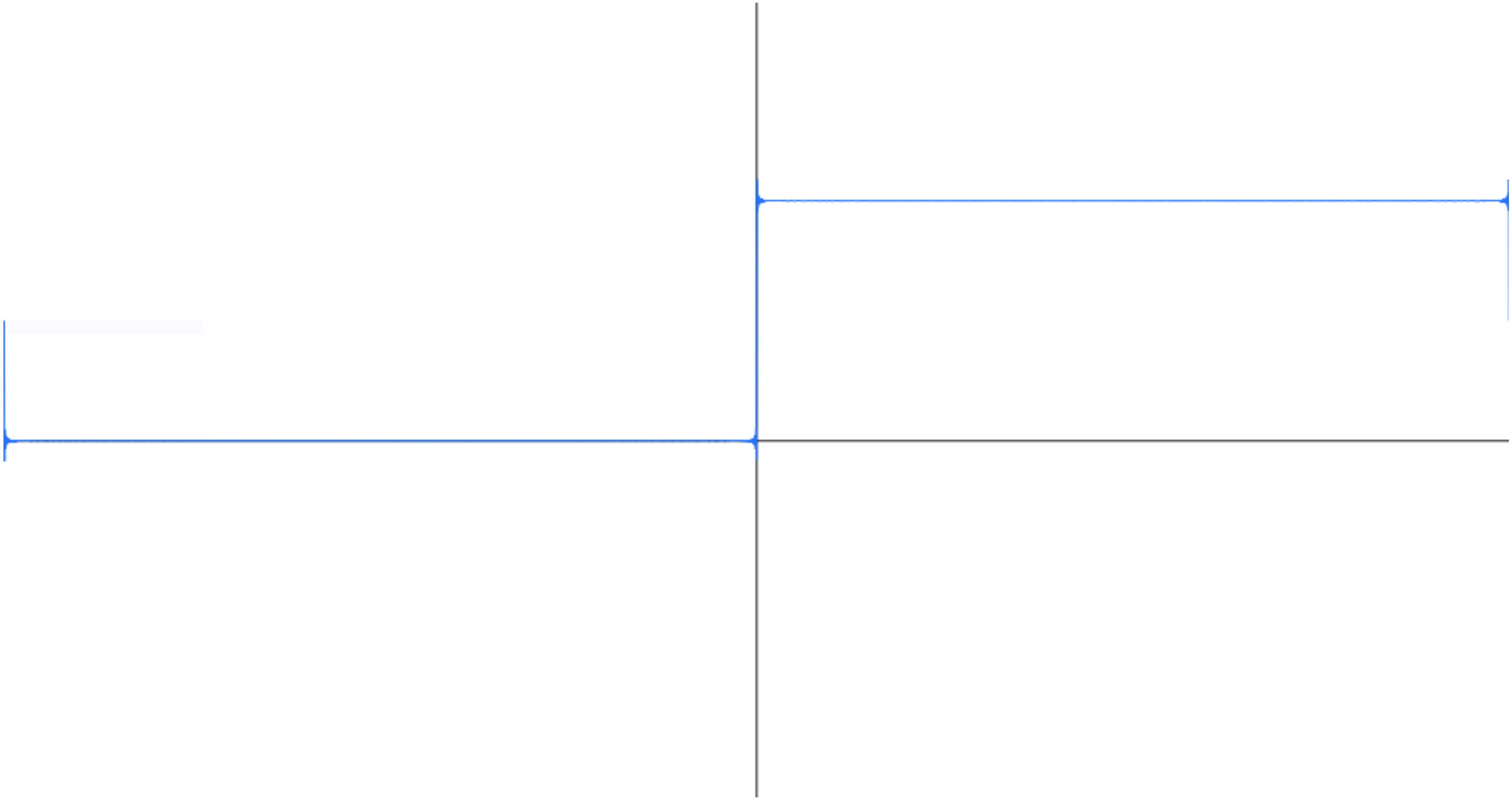
$t = \frac{1}{6}\pi$



$t = \frac{1}{5}\pi$


$t = \pi$
$t = \frac{1}{2}\pi$
$t = \frac{1}{3}\pi$
$t = \frac{1}{4}\pi$
$t = \frac{1}{5}\pi$
$t = \frac{1}{6}\pi$
$t = \frac{1}{7}\pi$
$t = \frac{1}{8}\pi$
$t = \frac{1}{9}\pi$

Periodic linearized KdV with Δt = .0001



Theorem. At rational time $t = 2\pi p/q$, the solution $u^\star(t,x)$ is constant on every subinterval $2\pi j/q < x < 2\pi(j+1)/q$. At irrational time $u^\star(t,x)$ is a non-differentiable continuous fractal function.

Lemma.

$$f(x) \;\sim\; \sum_{k=-\infty}^{\infty} c_k\, e^{\,\mathrm{i}\,k\,x}$$

is piecewise constant on intervals $2\pi\, j/q < x < 2\pi\,(j+1)/q$ if and only if

$$\widehat{c}_k = \widehat{c}_l, \quad k \equiv l \not\equiv 0 \mod q, \qquad\qquad \widehat{c}_k = 0, \quad 0 \neq k \equiv 0 \mod q.$$

where

$$\widehat{c}_k = \frac{2\,\pi\,k\,c_k}{\mathrm{i}\,q\,(e^{-2\,\mathrm{i}\,\pi\,k/q} - 1)} \qquad\qquad k \not\equiv 0 \mod q.$$

$\Longrightarrow$ DFT

The Fourier coefficients of the solution $u^\star(t,x)$ at rational time $t = 2\pi\, p/q$ are

$$c_k = b_k\, e^{-2\pi\,\mathrm{i}\; k^3\, p/q} \qquad (*)$$

where, for the step function initial data,

$$b_k = \begin{cases} -\,\mathrm{i}\,/(\pi\, k), & k \text{ odd}, \\ 1/2, & k = 0, \\ 0, & 0 \neq k \;\; \text{even}. \end{cases}$$

Crucial observation: $\boxed{\text{if } k \equiv l \mod q \text{ then } k^3 \equiv l^3 \mod q}$

which implies

$$e^{-2\pi\,\mathrm{i}\; k^3\, p/q} = e^{-2\pi\,\mathrm{i}\; l^3\, p/q}$$

and hence the Fourier coefficients $(*)$ satisfy the condition in the Lemma. *Q.E.D.*

Revival

Fundamental Solution: $F(0,x) = \delta(x)$.

Theorem. At rational time $t = 2\pi\, p/q$, the fundamental solution $F(t,x)$ is a linear combination of finitely many periodically extended delta functions, based at $2\pi\, j/q$ for integers $-\frac{1}{2}q < j \leq \frac{1}{2}q$.

Revival

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Corollary. At rational time, any solution profile $u(2\pi p/q, x)$ to the periodic initial-boundary value problem is a linear combination of $\leq q$ translates of the initial data, namely $f(x + 2\pi j/q)$, and hence its value depends on only finitely many values of the initial data.

★★ The same quantization/fractalization phenomenon appears in any linearly dispersive equation with "integral polynomial" dispersion relation:

$$\omega(k) = \sum_{m=0}^{n} c_m k^m$$

where

$$c_m = \alpha\, n_m \qquad n_m \in \mathbb{Z}$$

Linear Free-Space Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

Dispersion relation: $\omega = k^2$

Phase velocity: $c_p = \dfrac{\omega}{k} = k$

Group velocity: $c_g = \dfrac{d\omega}{dk} = 2\,k$

The Talbot Effect

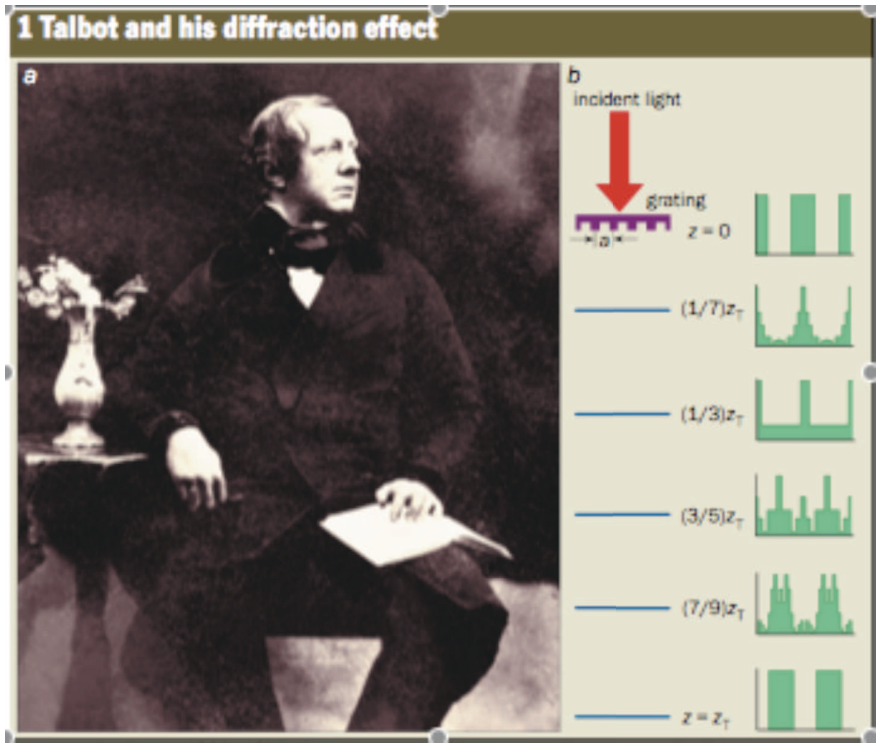
$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

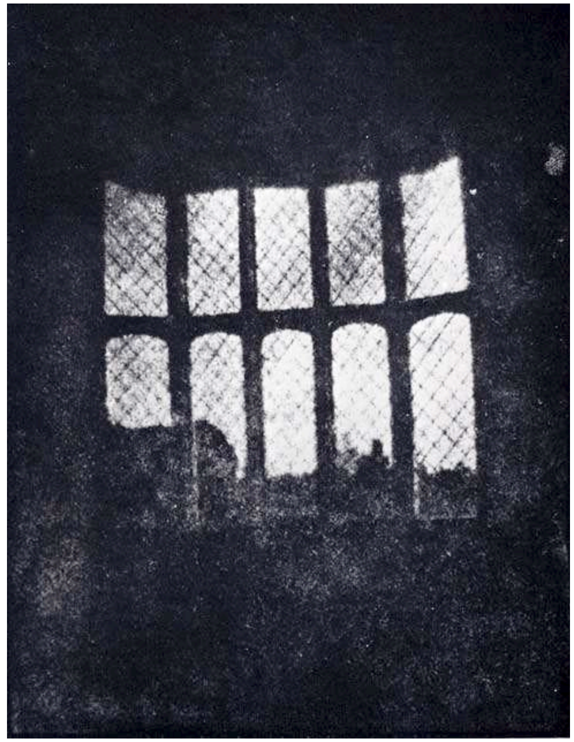
$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}\,(t,-\pi) = \frac{\partial u}{\partial x}\,(t,\pi)$$

- Michael Berry et. al.
- Oskolkov
- Kapitanski, Rodnianski

  "Does a quantum particle know the time?"
- Michael Taylor
- Bernd Thaller, *Visual Quantum Mechanics*

William Henry Fox Talbot (1800–1877)





⋆ Talbot's 1835 image of a latticed window in Lacock Abbey

$\Longrightarrow$ oldest photographic negative in existence.

A Talbot Experiment

Fresnel diffraction by periodic gratings (1836):

"It was very curious to observe that though the grating was greatly out of the focus of the lens . . . the appearance of the bands was perfectly distinct and well defined . . . the experiments are communicated in the hope that they may prove interesting to the cultivators of optical science."

— *Fox Talbot*

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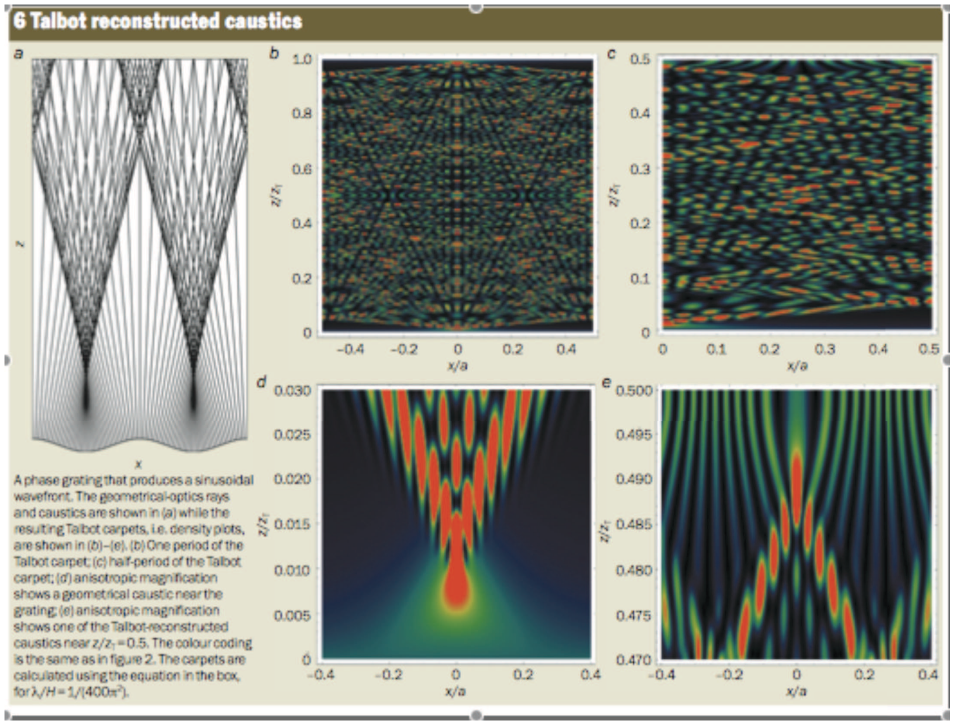
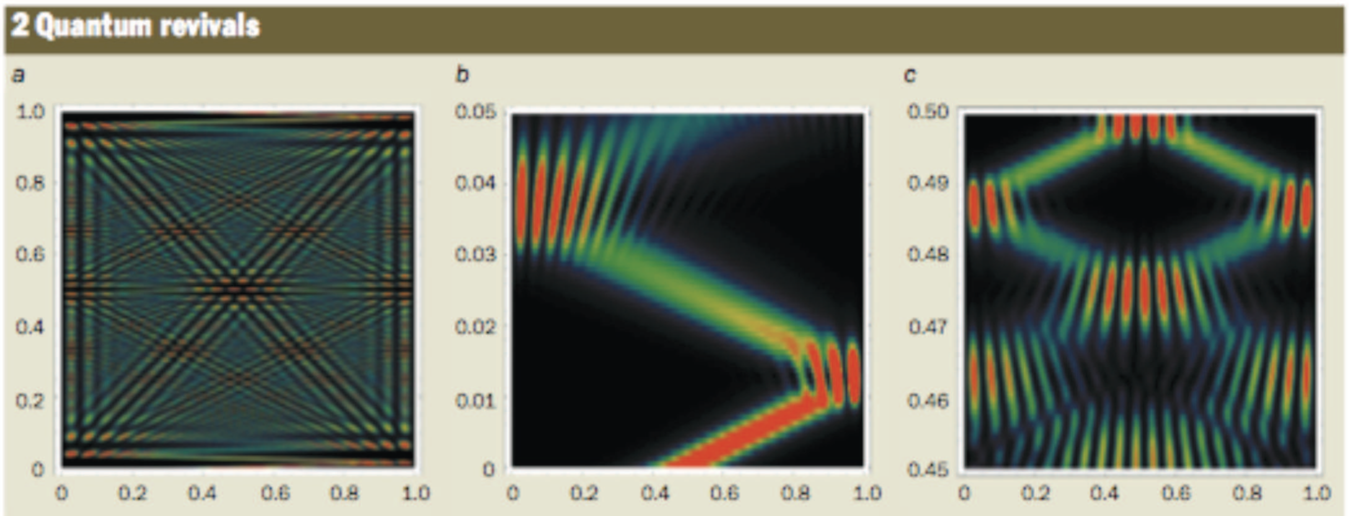
$\Longrightarrow$ Lord Rayleigh calculates the Talbot distance (1881)

The Quantized/Fractal Talbot Effect



- Optical experiments — Berry & Klein
- Diffraction of matter waves (helium atoms) — Nowak et. al.

Quantum Revival



- Electrons in potassium ions — Yeazell & Stroud
- Vibrations of bromine molecules — Vrakking, Villeneuve, Stolow

Periodic Linear Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}\,(t,-\pi) = \frac{\partial u}{\partial x}\,(t,\pi)$$

Integrated fundamental solution:

$$u(t,x) = \frac{1}{2\,\pi} \sum_{0 \neq k = -\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x - k^2 t)}}{k}\,.$$

For $x/t \in \mathbb{Q}$, this is known as a Gauss sum (or, more generally, k^n, a Weyl sum), of great importance in number theory

$\Longrightarrow$ Hardy, Littlewood, Weil, I. Vinogradov, etc.

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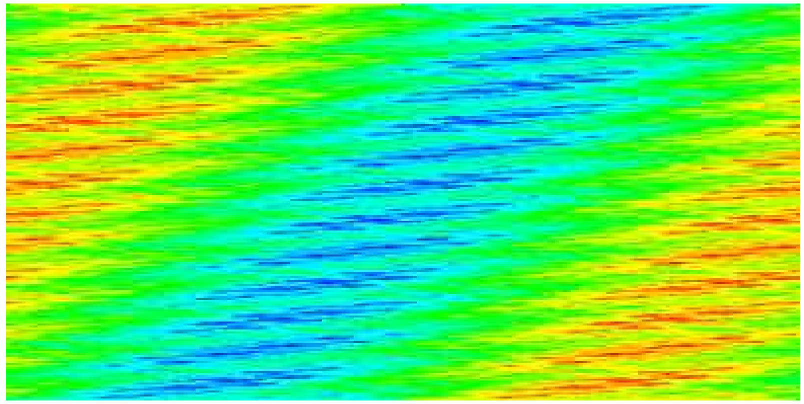
$\star\star$ The Riemann Hypothesis!

Integrated fundamental solution:

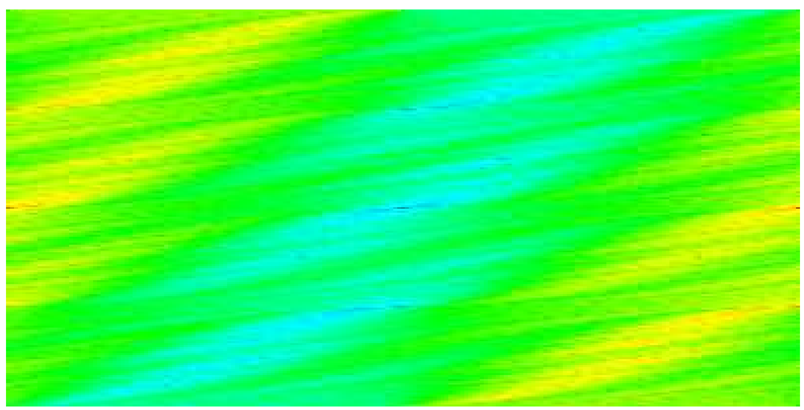
$$u(t,x) = \frac{1}{2\pi} \sum_{0 \neq k = -\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x - k^2 t)}}{k}\,.$$

Theorem.

- The fundamental solution $\partial u/\partial x$ is a Jacobi theta function. At rational times $t = 2\pi\,p/q$, it linear combination of delta functions concentrated at rational nodes $x_j = 2\pi\,j/q$.
- At irrational times t, the integrated fundamental solution is a continuous but nowhere differentiable function.



Dispersive Carpet



Schrödinger Carpet

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = L(D_x)\, u, \qquad u(t, x + 2\pi) = u(t, x)$$

Dispersion relation:

$$u(t,x) = e^{\,\mathrm{i}\,(k\,x - \omega\,t)} \quad \Longrightarrow \quad \omega(k) = -\,\mathrm{i}\, L(-\,\mathrm{i}\, k) \qquad \text{assumed real}$$

Riemann problem: step function initial data

$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Solution:

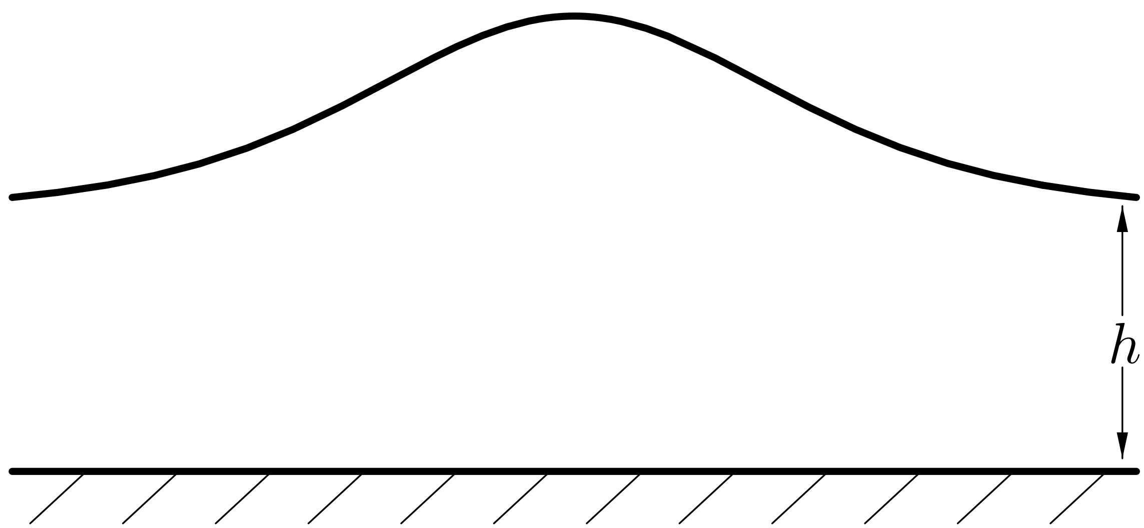
$$u(t,x) \;\sim\; \frac{1}{2} \;+\; \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin[\,(2\,j+1)\,x - \omega(2\,j+1)\,t\,]}{2\,j+1}\,.$$

★★ $\omega(-\,k) = -\,\omega(k)$ odd

Polynomial dispersion, rational t $\Longrightarrow$ Weyl exponential sums

Water Waves

2D Water Waves



$y = h + \eta(t, x)$ surface elevation

$\phi(t, x, y)$ velocity potential

2D Water Waves

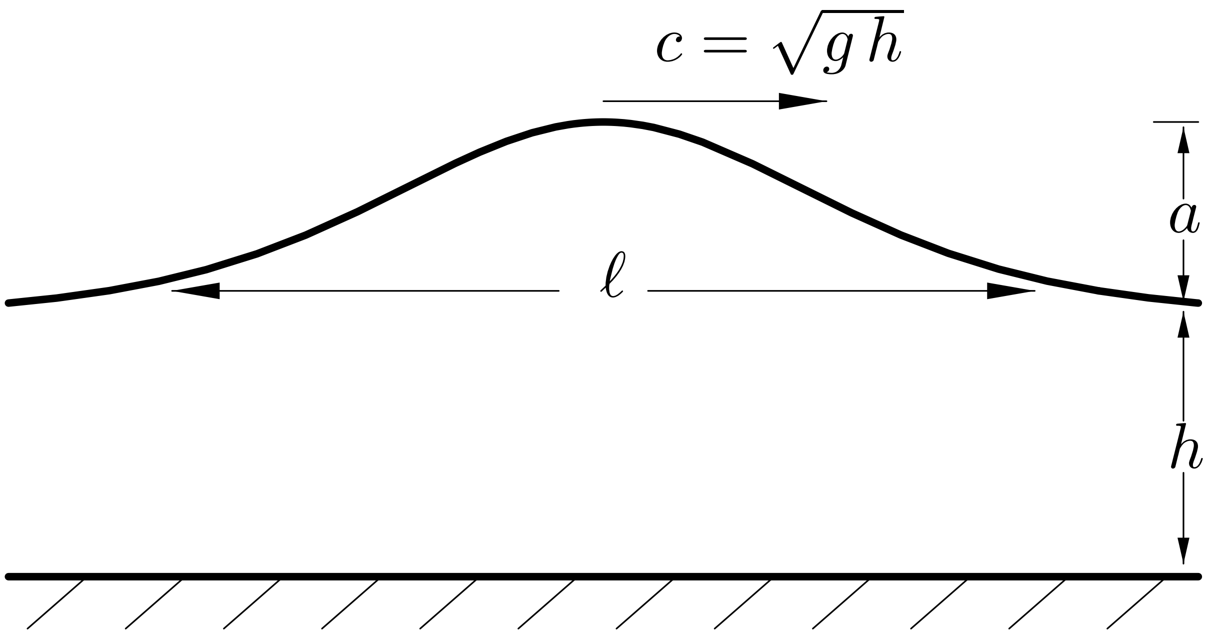
- Incompressible, irrotational fluid.
- No surface tension

$$\left.\begin{aligned} \phi_t + \tfrac{1}{2}\,\phi_x^2 + \tfrac{1}{2}\,\phi_y^2 + g\,\eta = 0 \\ \eta_t = \phi_y - \eta_x \phi_x \end{aligned}\right\} \qquad y = h + \eta(t,x)$$

$$\phi_{xx} + \phi_{yy} = 0 \qquad\qquad 0 < y < h + \eta(t,x)$$

$$\phi_y = 0 \qquad\qquad y = 0$$

- Wave speed (maximum group velocity): $c = \sqrt{g\,h}$
- Dispersion relation: $\sqrt{g\,k\,\tanh(h\,k)} = c\,k - \frac{1}{6}\,c\,h^2 k^3 + \cdots$



Small parameters — long waves in shallow water (KdV regime)

$$\alpha = \frac{a}{h} \qquad \beta = \frac{h^2}{\ell^2} = O(\alpha)$$

Rescale:

$$x \longmapsto \ell\, x \qquad y \longmapsto h\, y \qquad t \longmapsto \frac{\ell\, t}{c}$$

$$\eta \longmapsto a\, \eta \qquad \phi \longmapsto \frac{g\, a\, \ell\, \phi}{c} \qquad c = \sqrt{g\, h}$$

Rescaled water wave system:

$$\left.\begin{aligned} \phi_t + \frac{\alpha}{2}\, \phi_x^2 + \frac{\alpha}{2\, \beta}\, \phi_y^2 + \eta = 0 \\ \eta_t = \frac{1}{\beta}\, \phi_y - \alpha\, \eta_x\, \phi_x \end{aligned}\right\} \qquad y = 1 + \alpha\, \eta$$

$$\beta\, \phi_{xx} + \phi_{yy} = 0 \qquad 0 < y < 1 + \alpha\, \eta$$

$$\phi_y = 0 \qquad y = 0$$

Boussinesq expansion

Set

$$\psi(t,x) = \phi(t,x,0) \qquad u(t,x) = \phi_x(t,x,\theta) \qquad 0 \leq \theta \leq 1$$

Solve Laplace equation:

$$\phi(t,x,y) = \psi(t,x) - \tfrac{1}{2}\,\beta^2\, y^2\, \psi_{xx} + \tfrac{1}{4!}\,\beta^4\, y^4\, \psi_{xxxx} + \ \cdots$$

Plug expansion into free surface conditions: To first order

$$\psi_t + \eta + \tfrac{1}{2}\,\alpha\,\psi_x^2 - \tfrac{1}{2}\,\beta\,\psi_{xxt} = 0$$

$$\eta_t + \psi_x + \alpha\,(\eta\psi_x)_x - \tfrac{1}{6}\,\beta\,\psi_{xxxx} = 0$$

Bidirectional Boussinesq systems:

$$u_t + \eta_x + \alpha\, u\, u_x - \tfrac{1}{2}\, \beta\, (\theta^2 - 1)\, u_{xxt} = 0$$

$$\eta_t + u_x + \alpha\, (\eta\, u)_x - \tfrac{1}{6}\, \beta\, (3\,\theta^2 - 1)\, u_{xxx} = 0$$

★ ★ at $\theta = 1$ this system is *integrable* (tri-Hamiltonian)
but *ill-posed* (!?!)

Bidirectional Boussinesq systems:

$$u_t + \eta_x + \alpha\, u\, u_x - \tfrac{1}{2}\,\beta\,(\theta^2 - 1)\, u_{xxt} = 0$$

$$\eta_t + u_x + \alpha\,(\eta\, u)_x - \tfrac{1}{6}\,\beta\,(3\,\theta^2 - 1)\, u_{xxx} = 0$$

$\star\star$ at $\theta = 1$ this system is *integrable* (tri-Hamiltonian)
but *ill-posed* (!?!)

Boussinesq equation

$$u_{tt} = u_{xx} + \tfrac{1}{2}\,\alpha\,(u^2)_{xx} - \tfrac{1}{6}\,\beta\, u_{xxxx}$$

Regularized Boussinesq equation

$$u_{tt} = u_{xx} + \tfrac{1}{2}\,\alpha\,(u^2)_{xx} + \tfrac{1}{6}\,\beta\, u_{xxtt}$$

$\Longrightarrow$ DNA dynamics (Scott)

Unidirectional waves:

$$u = \eta - \tfrac{1}{4}\,\alpha\,\eta^2 + \left(\tfrac{1}{3} - \tfrac{1}{2}\,\theta^2\right)\beta\,\eta_{xx} + \ \cdots$$

Korteweg-deVries (1895) equation:

$$\eta_t + \eta_x + \tfrac{3}{2}\,\alpha\,\eta\,\eta_x + \tfrac{1}{6}\,\beta\,\eta_{xxx} = 0$$

$\Longrightarrow$ Due to Boussinesq in 1877!

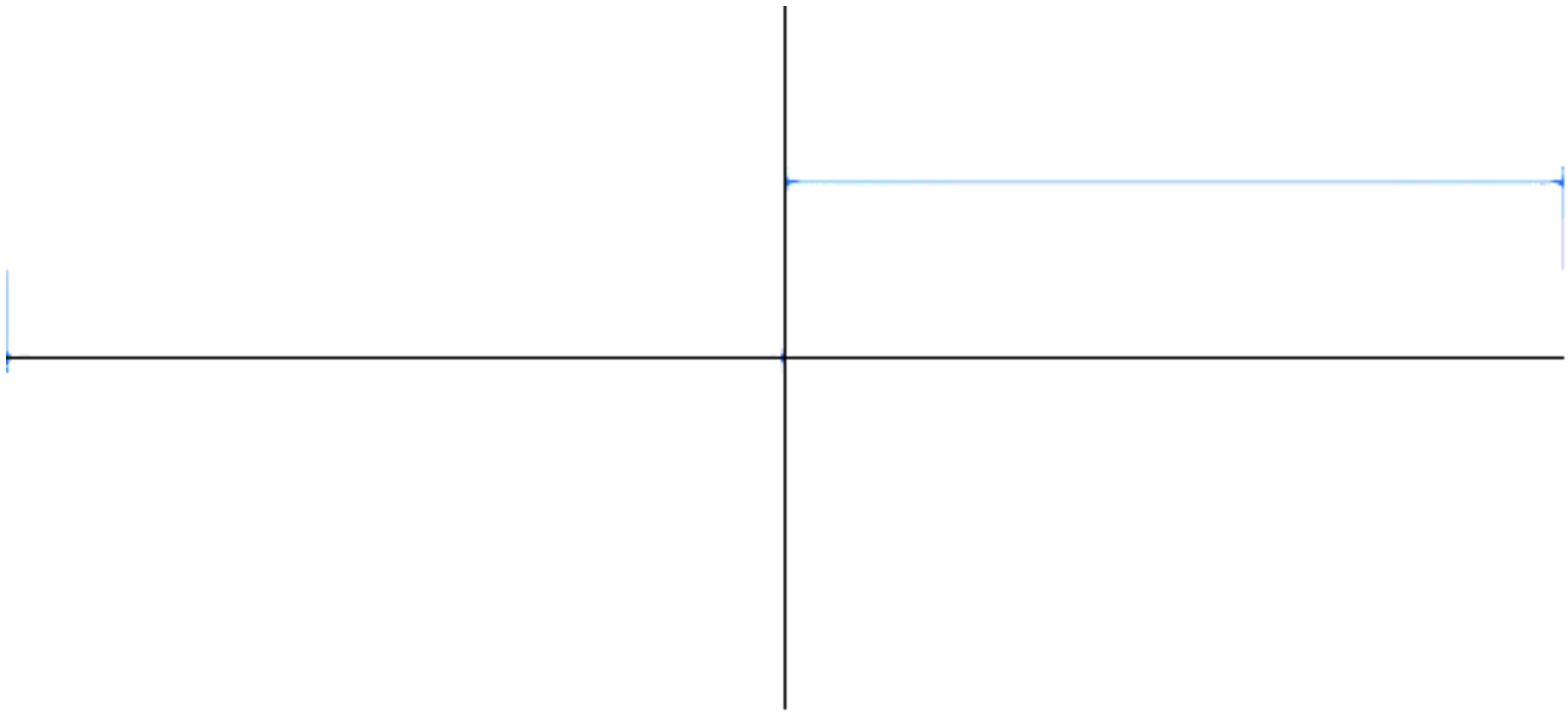
Benjamin–Bona–Mahony (BBM) equation:

$$\eta_t + \eta_x + \tfrac{3}{2}\,\alpha\,\eta\,\eta_x - \tfrac{1}{6}\,\beta\,\eta_{xxt} = 0$$

Shallow Water Dispersion Relations

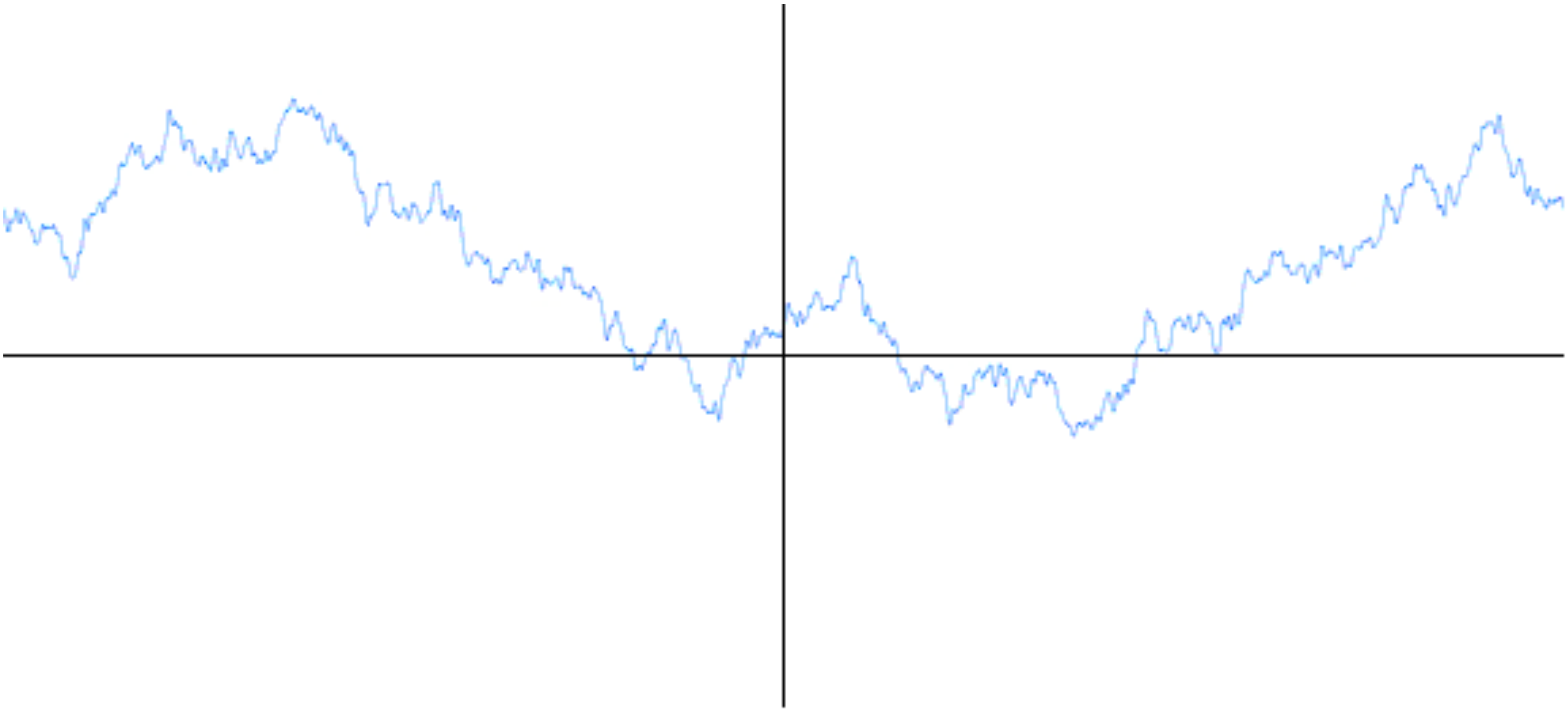
Water waves	$\pm \sqrt{k \tanh k}$
Boussinesq system	$\pm \dfrac{k}{\sqrt{1 + \frac{1}{3} k^2}}$
Boussinesq equation	$\pm k \sqrt{1 + \frac{1}{3} k^2}$
Korteweg–deVries	$k - \frac{1}{6} k^3$
BBM	$\dfrac{k}{1 + \frac{1}{6} k^2}$

Water waves $\omega = \sqrt{k\ \tanh k}\, \operatorname{sign} k$



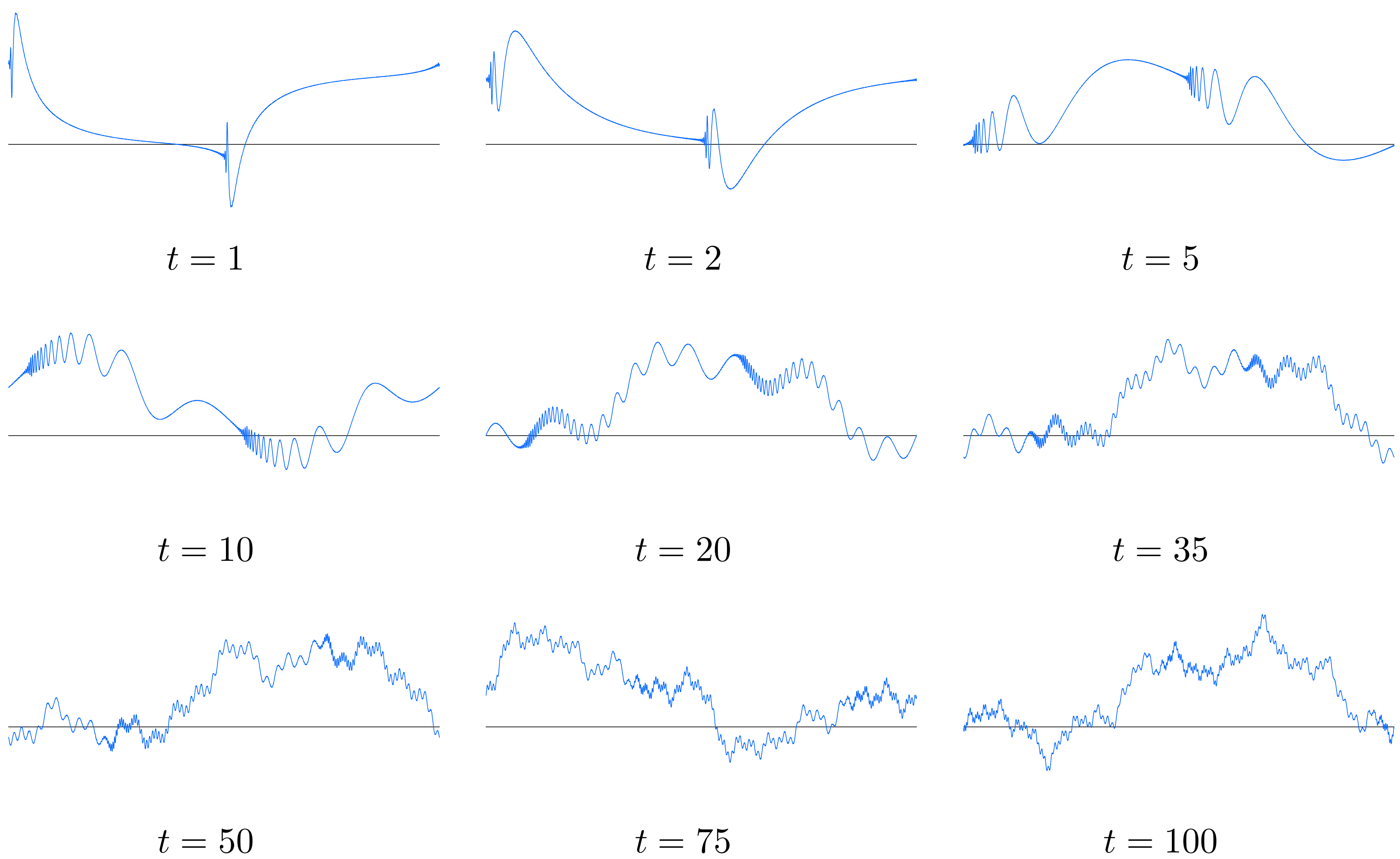
Water waves: t > 1000

$$\omega = \sqrt{k \tanh k}\ \mathrm{sign}\, k$$

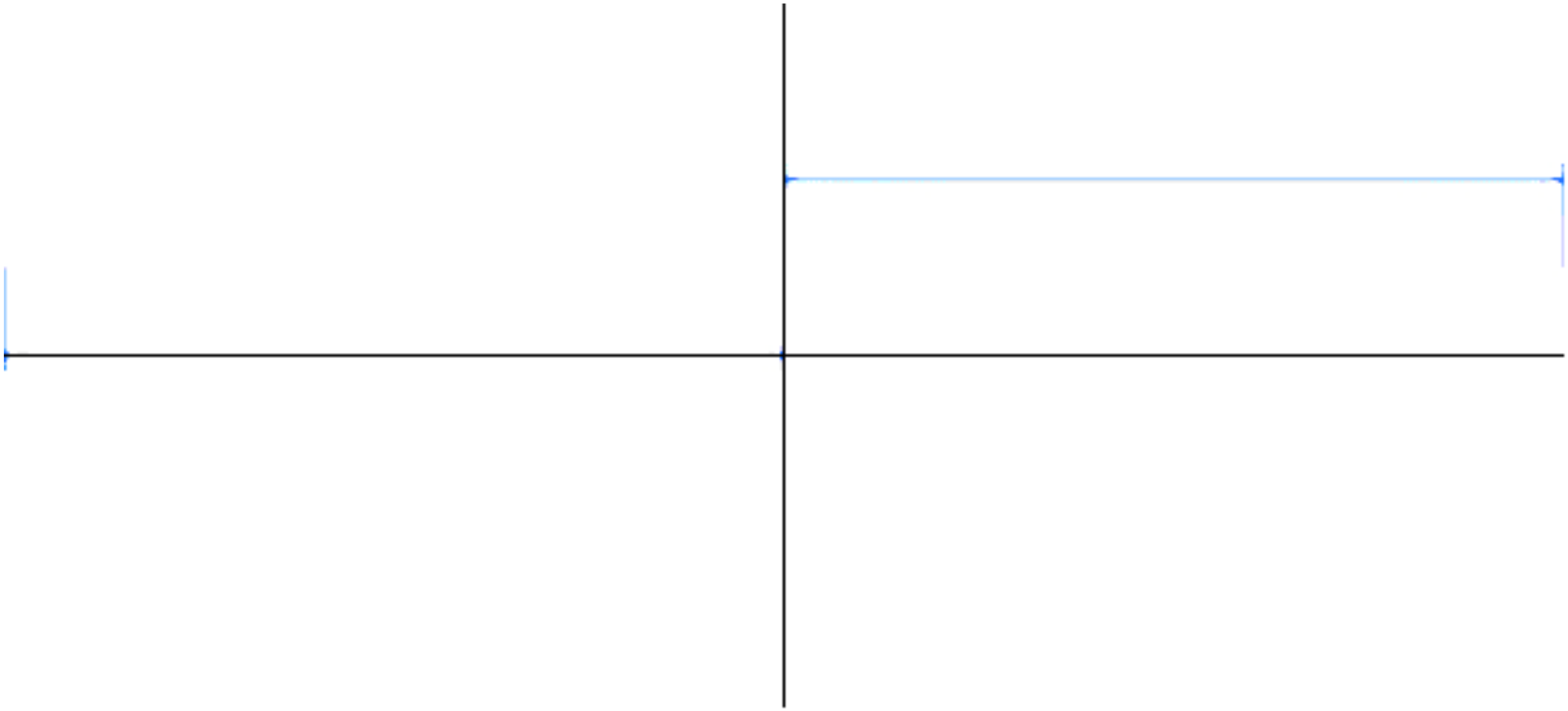


Water waves

$$\omega = \sqrt{k\ \tanh k}\ \operatorname{sign} k$$

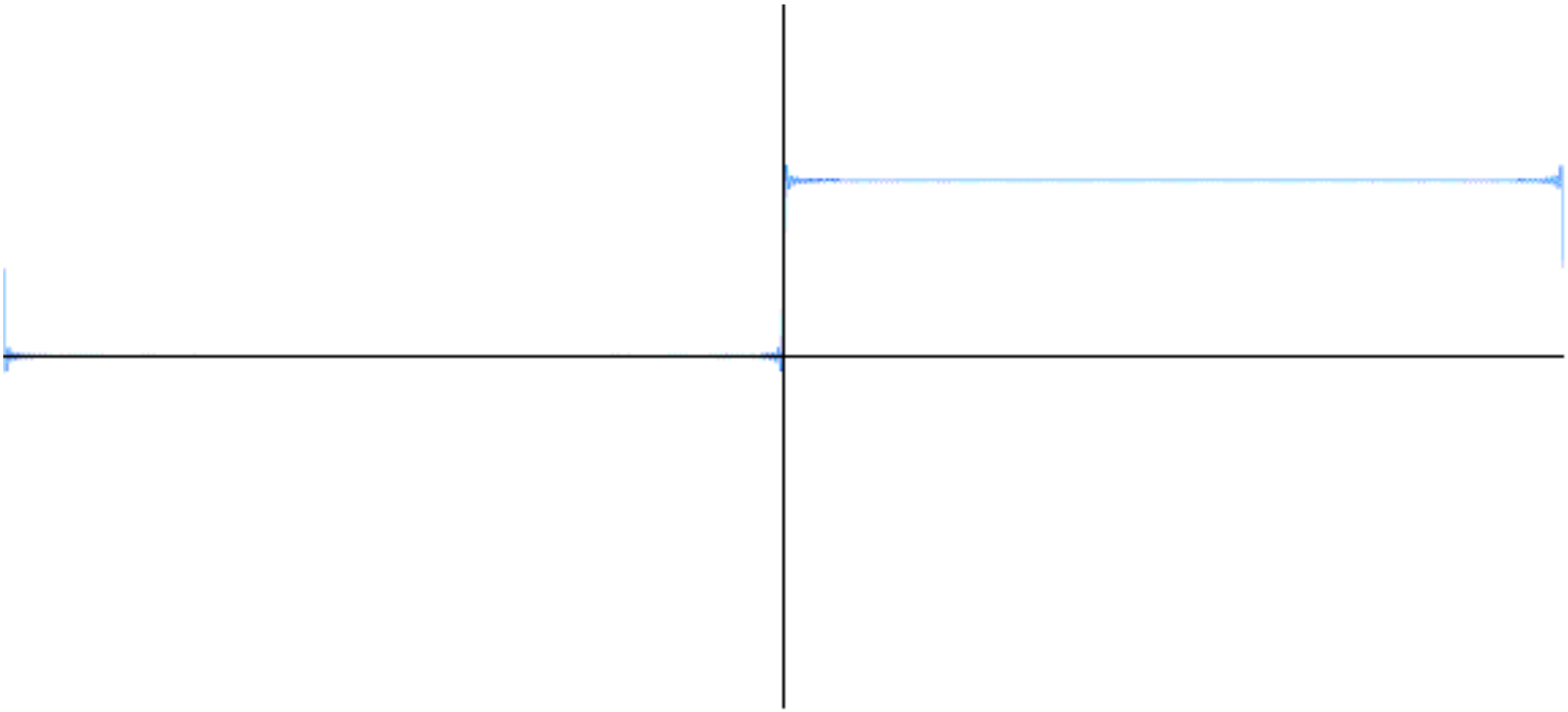


Square root dispersion $\omega = \sqrt{|\,k\,|}\,\operatorname{sign} k$



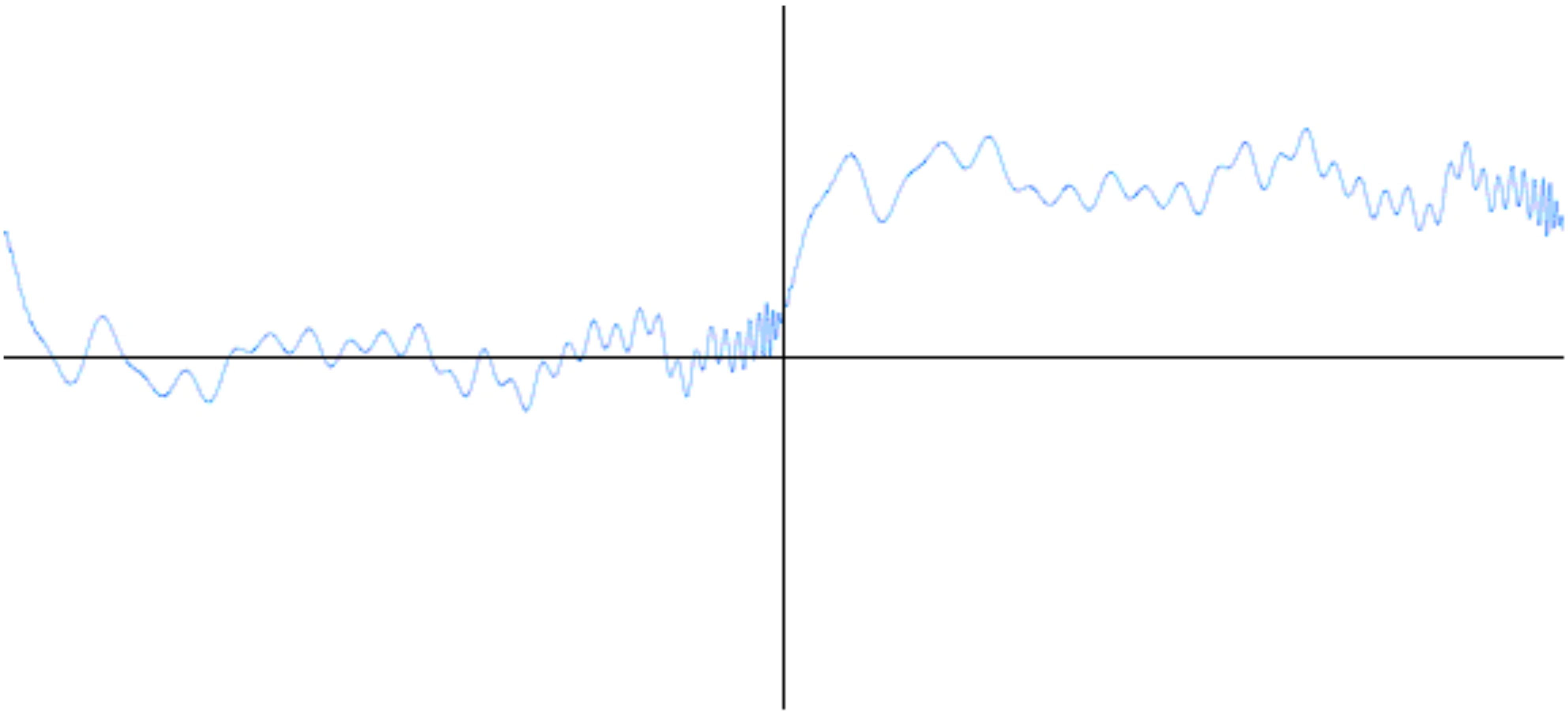
BBM equation

$$\omega = \frac{k}{\sqrt{1 + \frac{1}{3}k^2}}$$



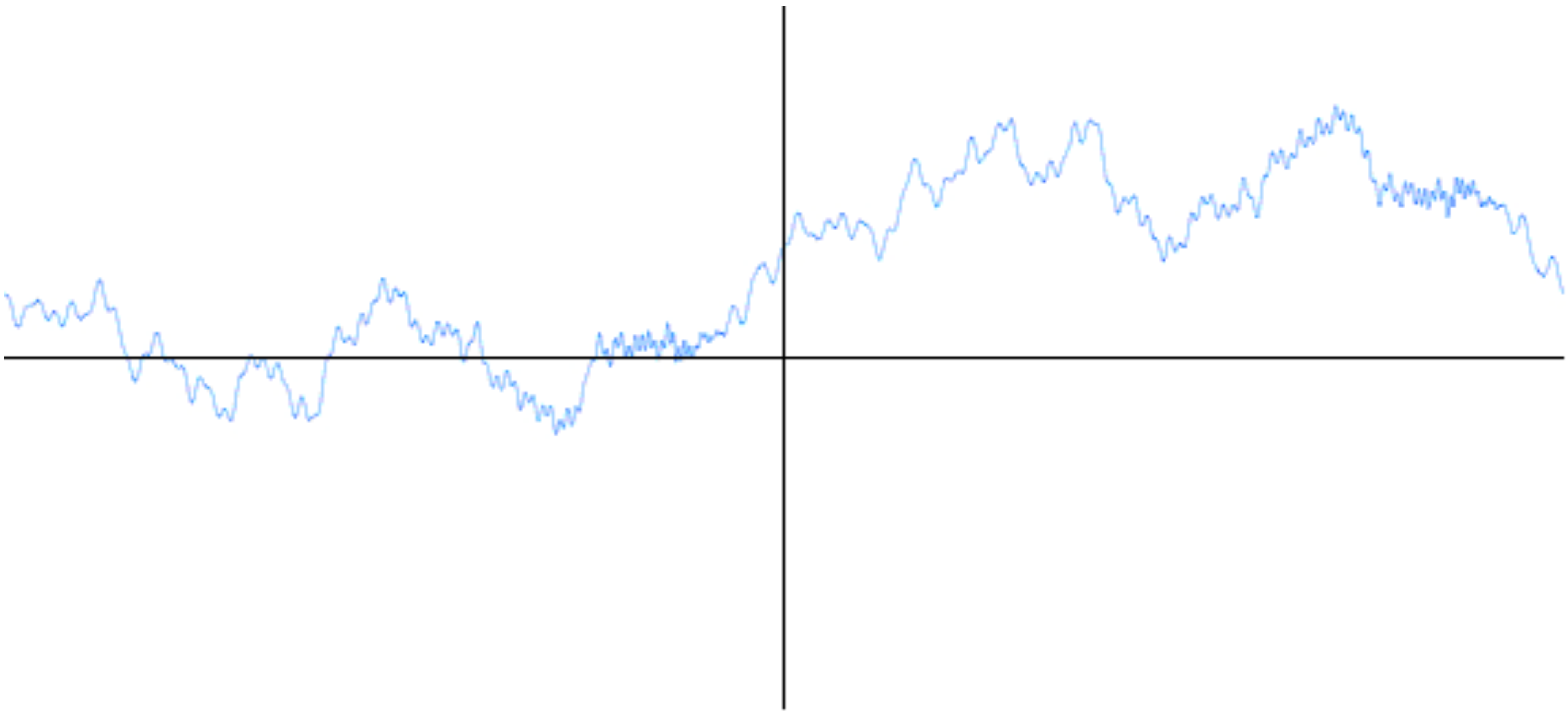
BBM equation: t > 1000

$$\omega = \frac{k}{\sqrt{1 + \frac{1}{3}k^2}}$$



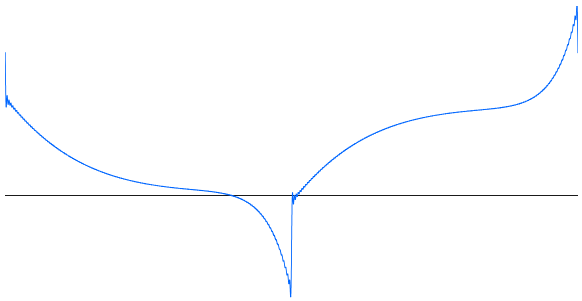
BBM equation: t > 10,000

$$\omega = \frac{k}{\sqrt{1 + \frac{1}{3}k^2}}$$

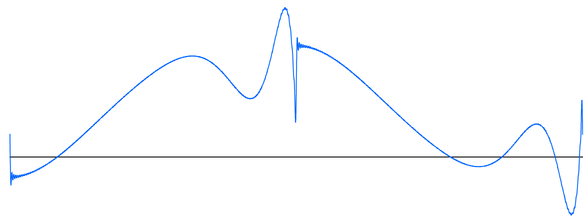


BBM equation

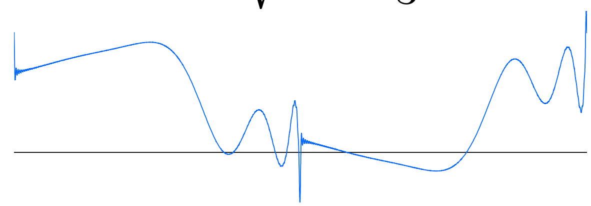
$$\omega = \frac{k}{\sqrt{1 + \frac{1}{3}k^2}}$$



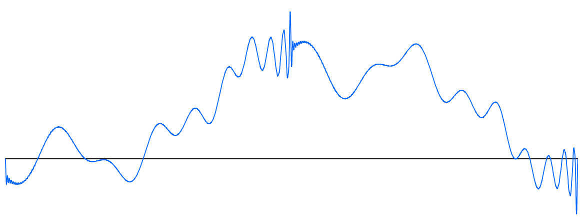
$t = 1$



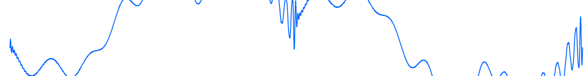
$t = 5$



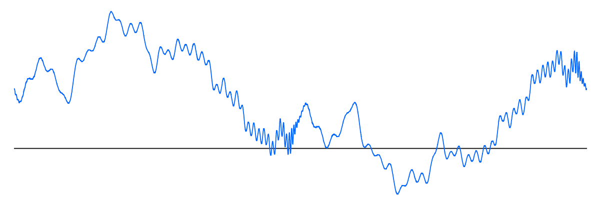
$t = 10$



$t = 50$



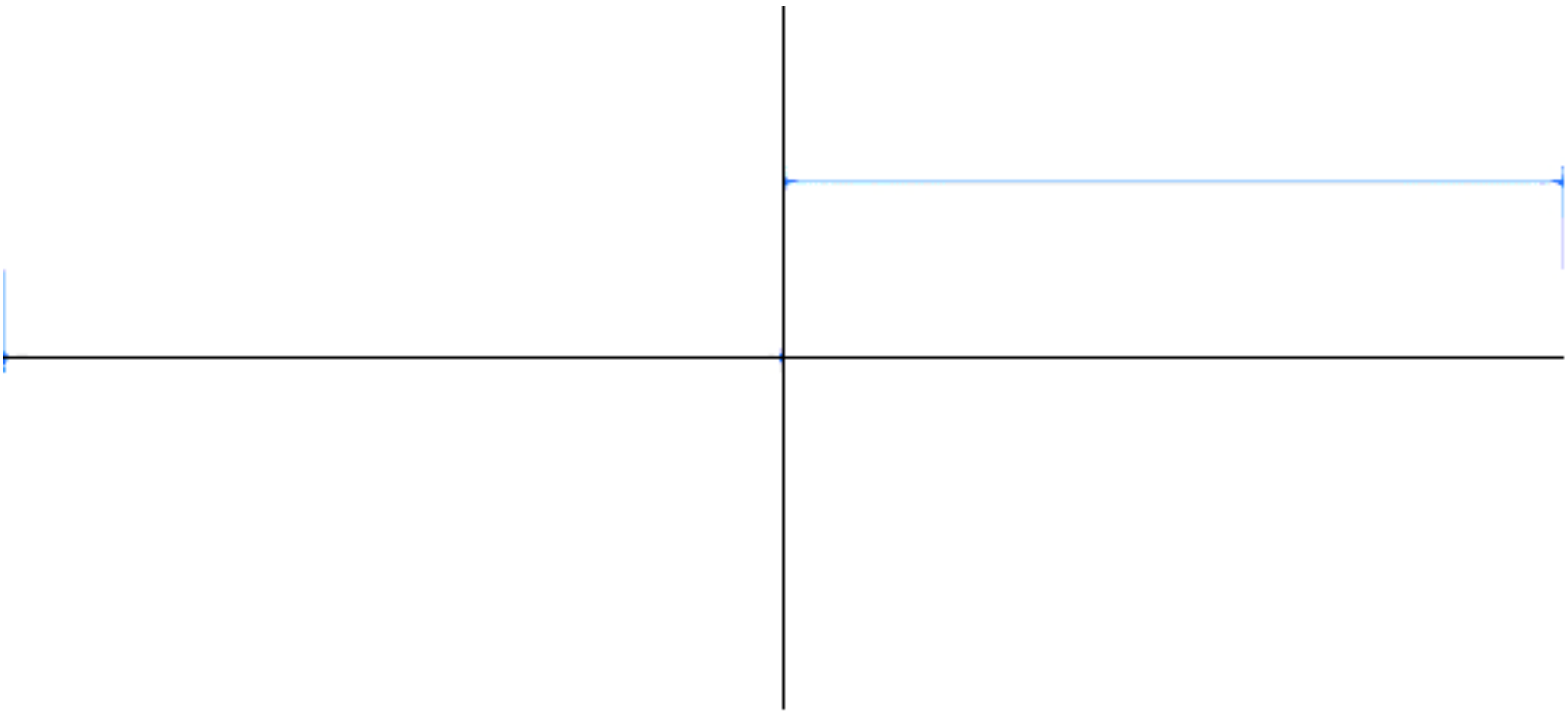
$t = 100$



$t = 1000$

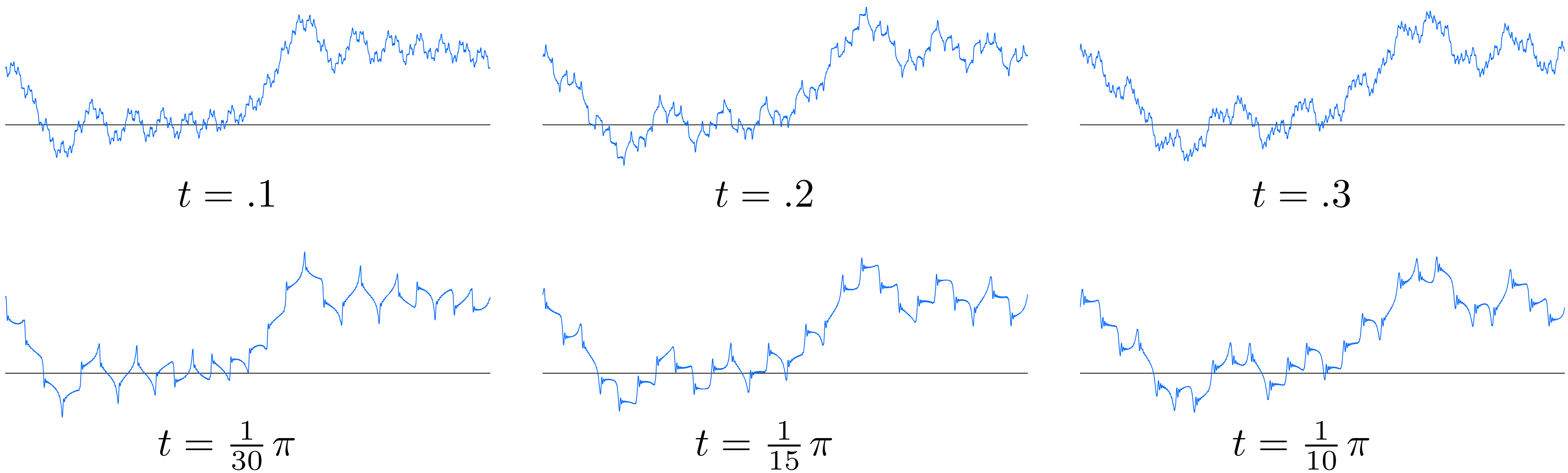
Boussinesq equation

$$\omega = k\sqrt{1 + \frac{1}{3}k^2}$$



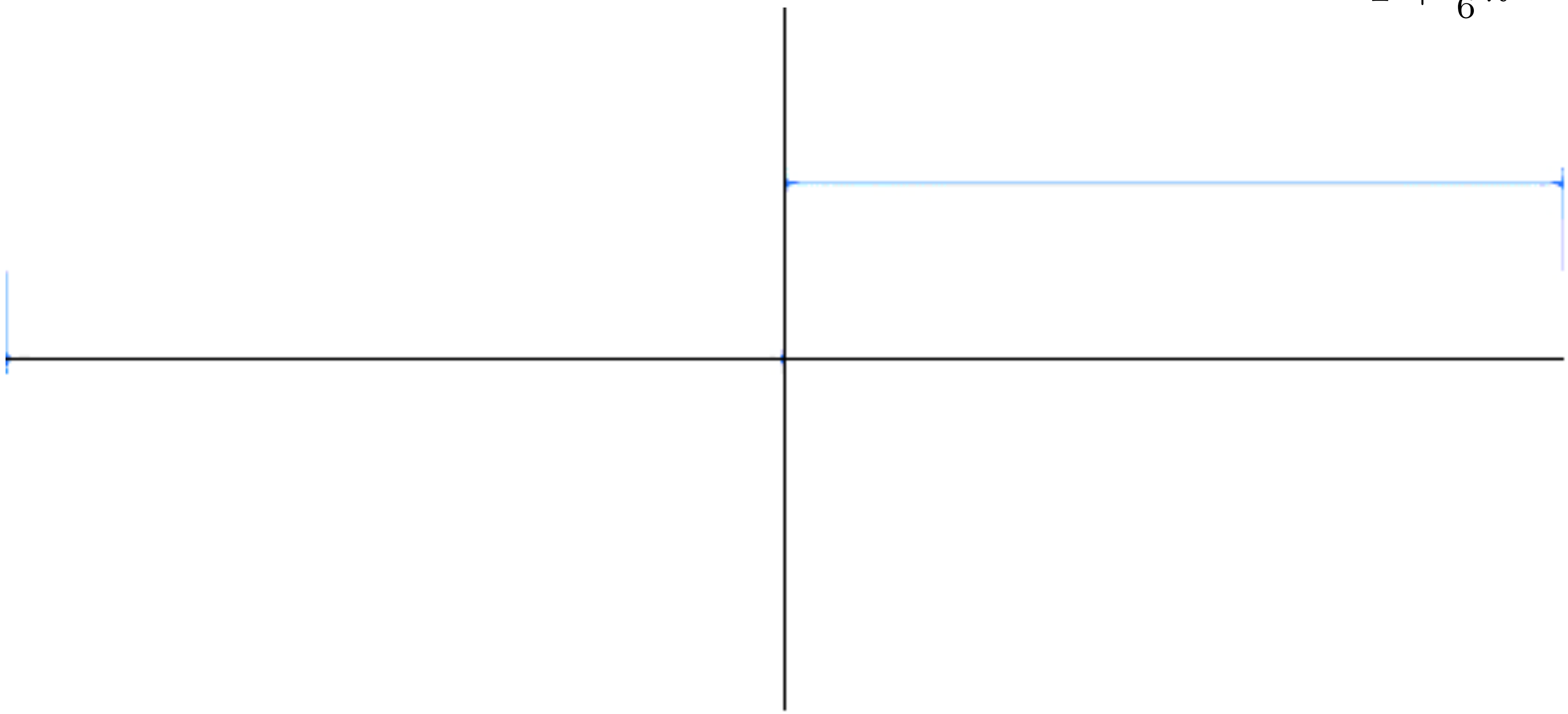
Boussinesq equation

$$\omega = k\sqrt{1 + \tfrac{1}{3}k^2}$$



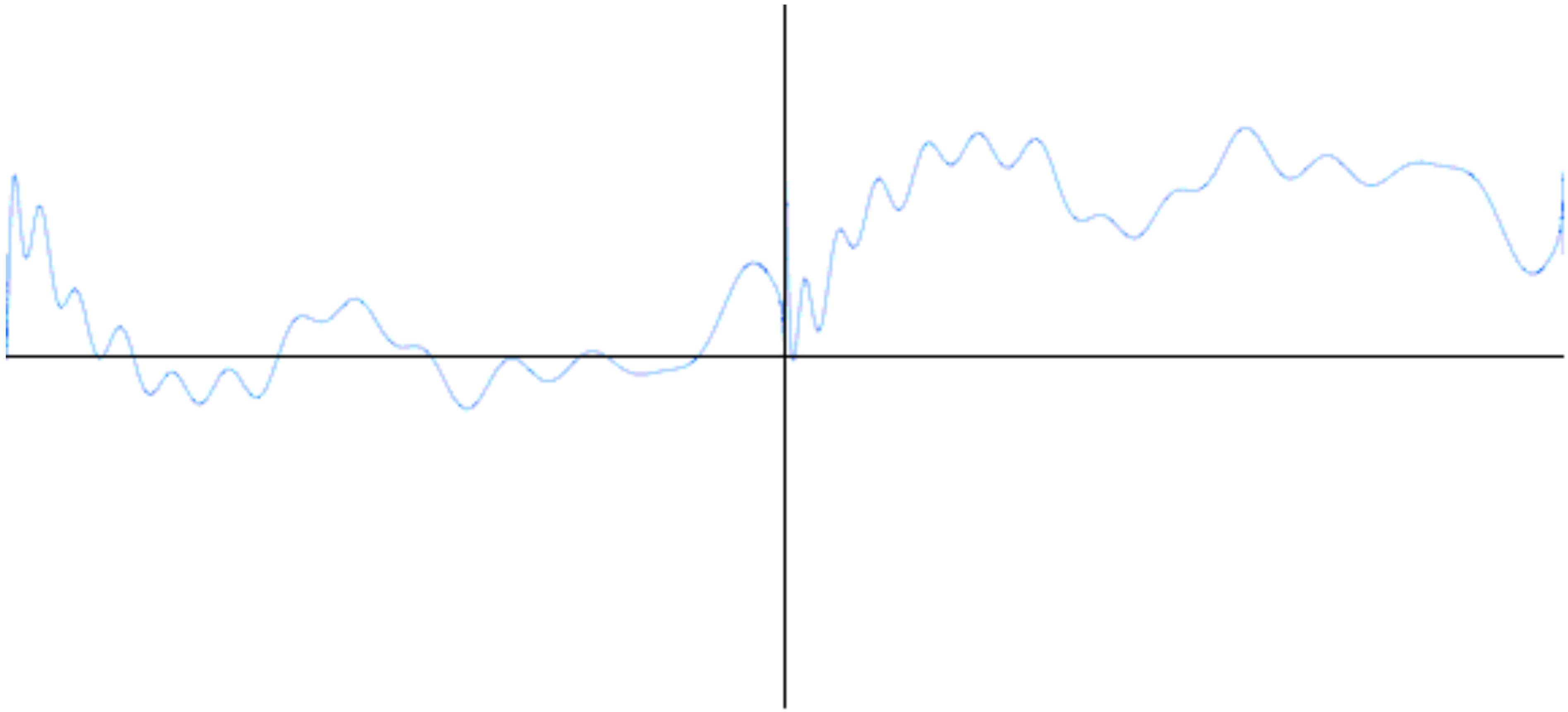
Regularized Boussinesq equation

$$\omega = \frac{k}{1 + \frac{1}{6}k^2}$$



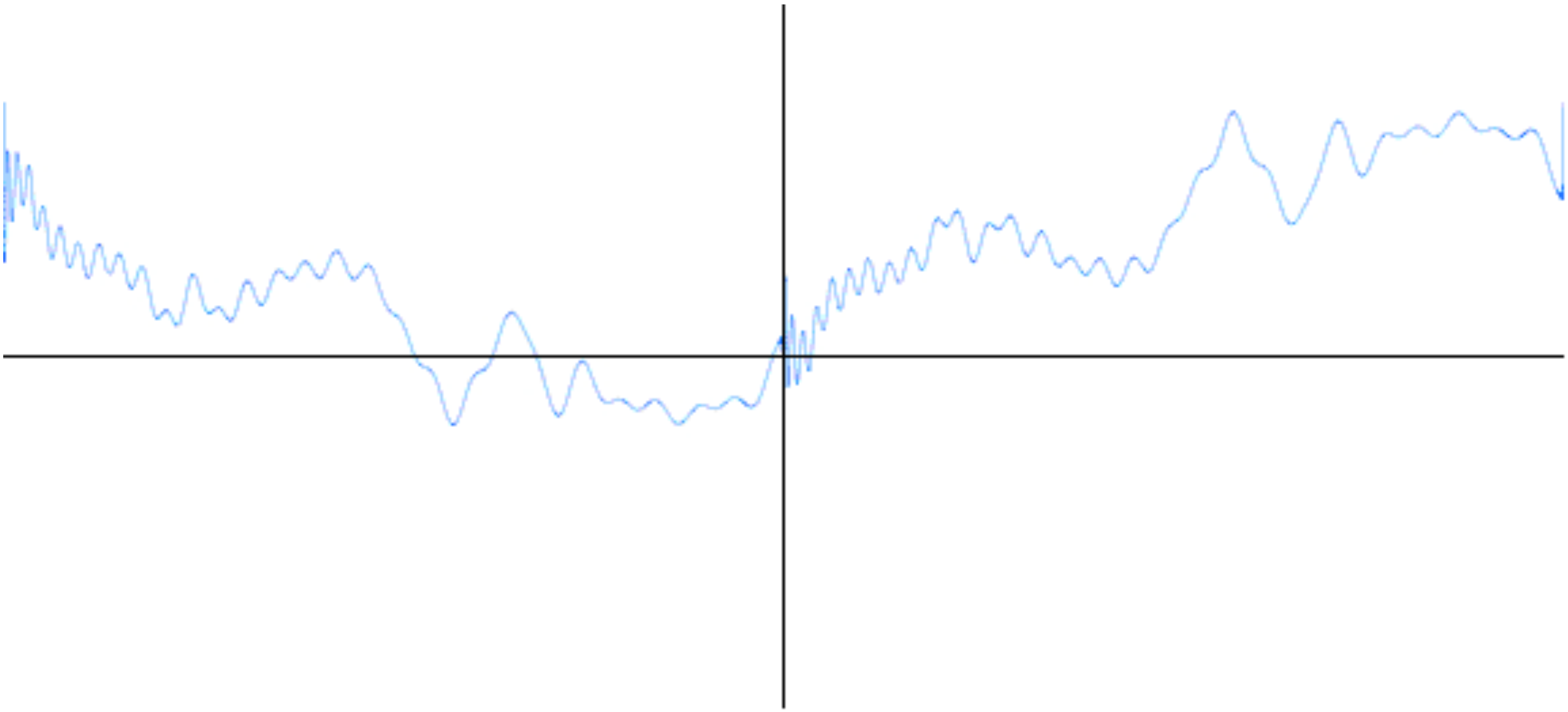
Regularized Boussinesq equation t > 1000

$$\omega = \frac{k}{1+\frac{1}{6}k^2}$$



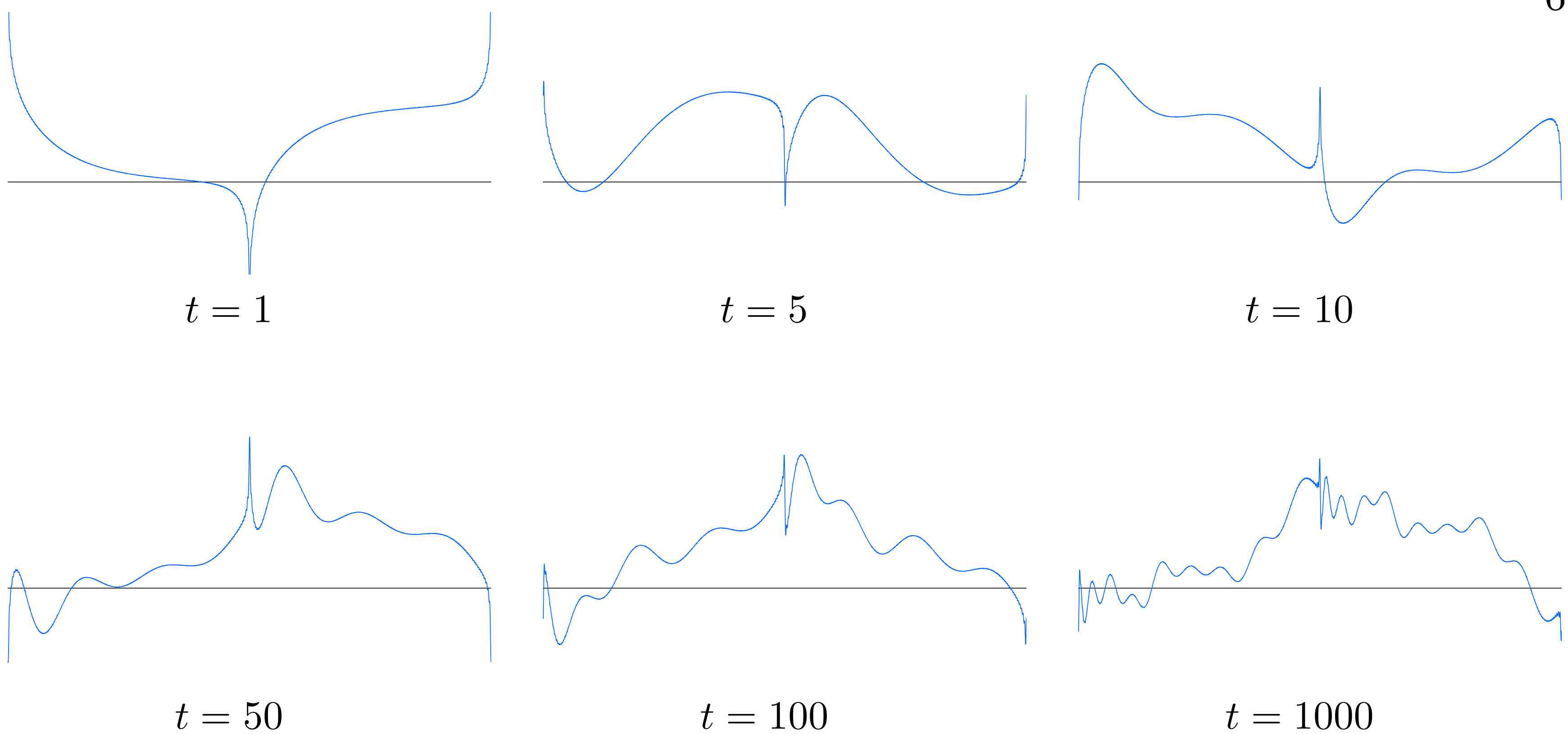
Regularized Boussinesq equation t > 10,000

$$\omega = \frac{k}{1 + \frac{1}{6}k^2}$$



Regularized Boussinesq equation

$$\omega = \frac{k}{1 + \frac{1}{6}k^2}$$



Dispersion Asymptotics

⋆ The qualitative behavior of the solution to the periodic problem depends crucially on the asymptotic behavior of the dispersion relation $\omega(k)$ for large wave number $k \to \pm\infty$.

$$\omega(k) \sim k^{\alpha}$$

- $\alpha = 0$ — large scale oscillations
- $0 < \alpha < 1$ — dispersive oscillations
- $\alpha = 1$ — traveling waves
- $1 < \alpha < 2$ — oscillatory becoming fractal
- $\alpha \geq 2$ — fractal/quantized

Linearized Benjamin Ono equation

$$u_t = \mathcal{H}[u_{xx}]. \qquad\qquad \omega_{BO}(k) = k^2 \operatorname{sign} k.$$

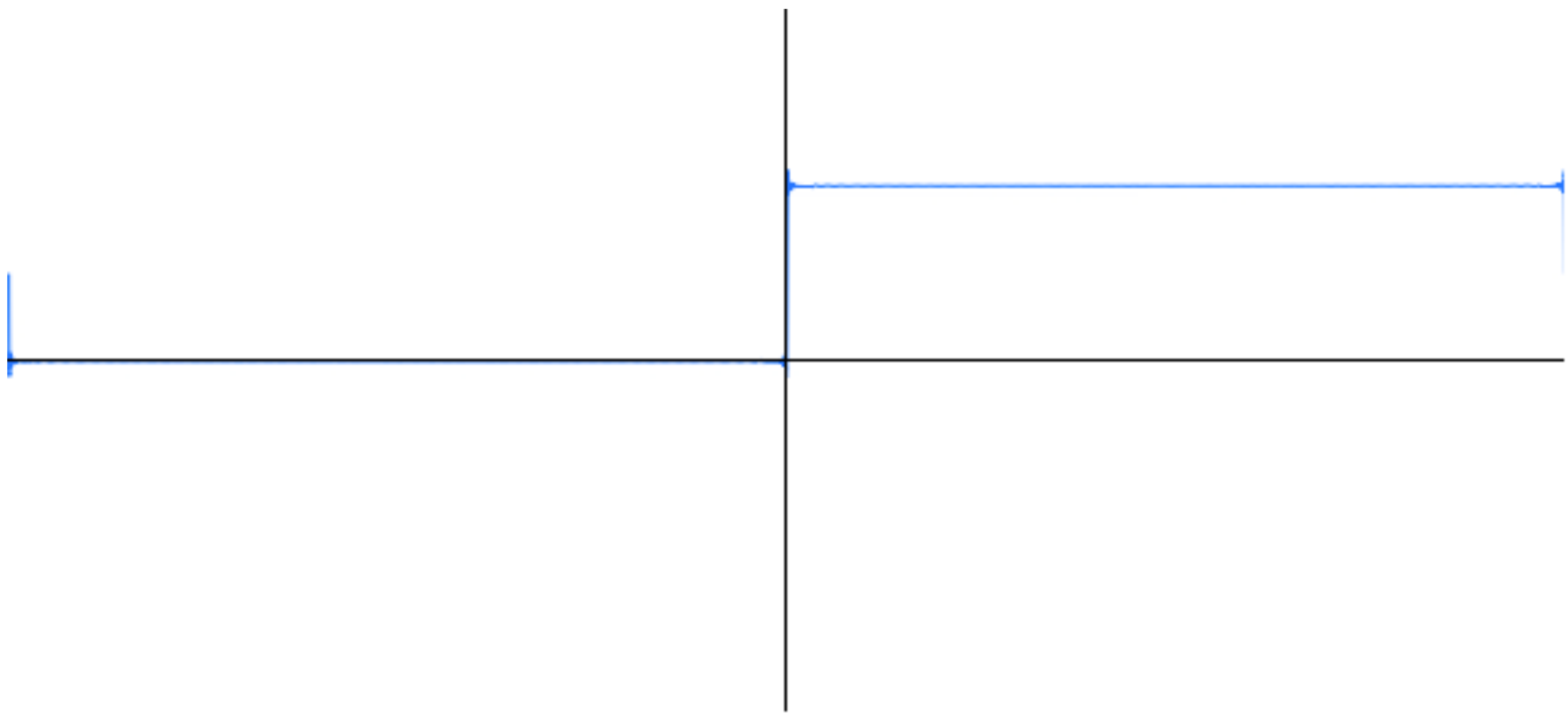
Hilbert transform

$$\mathcal{H}[f](x) = H * f\,(x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{f(y)}{x-y}\,dy$$

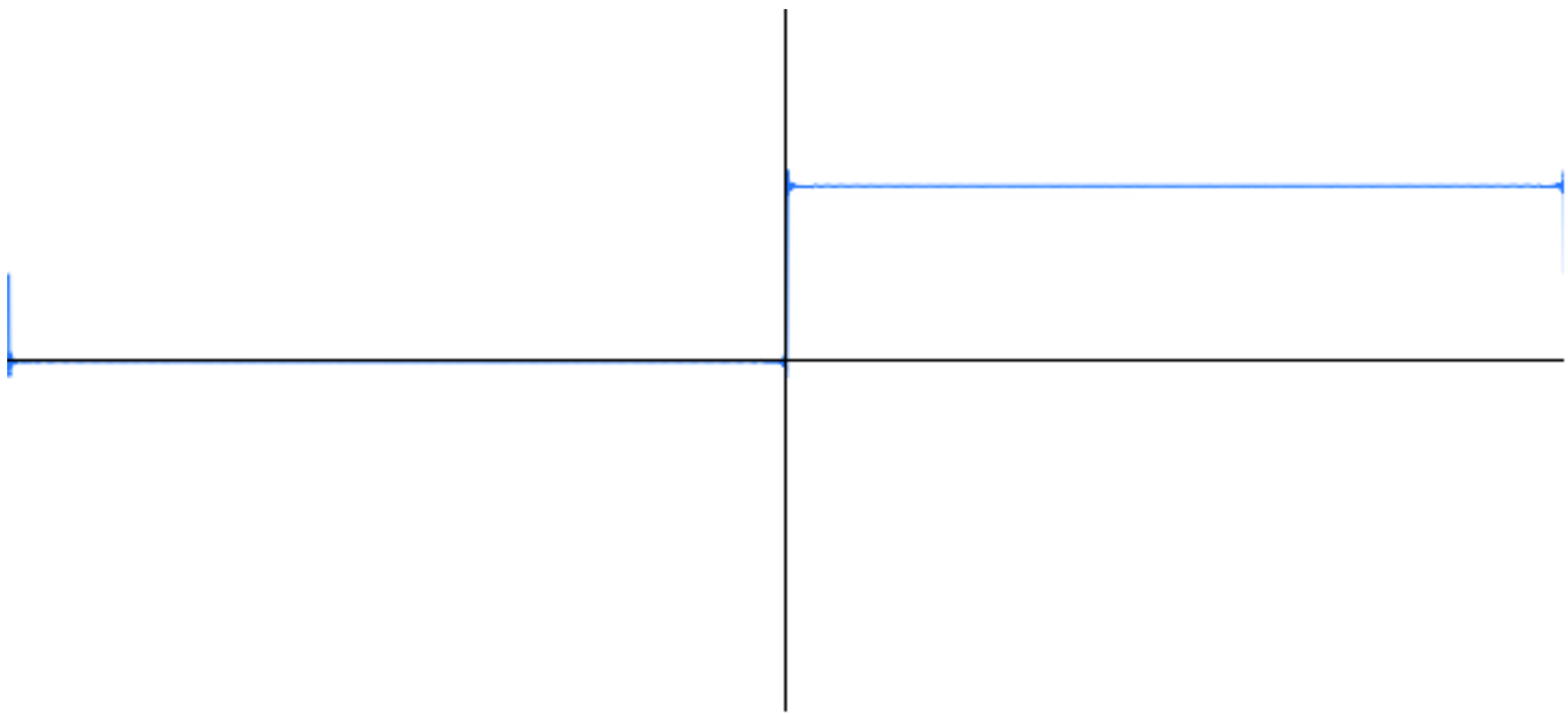
periodic Hilbert transform

$$\mathcal{H}[f](x) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \fint_{-\pi}^{\pi} \frac{f(y)}{x-y+2\pi k}\,dy = \frac{1}{2\pi} \fint_{-\pi}^{\pi} \cot\left[\,\tfrac{1}{2}(x-y)\,\right] f(y)\,dy$$

Benjamin–Ono equation: irrational times $\omega = k^2 \operatorname{sign} k$

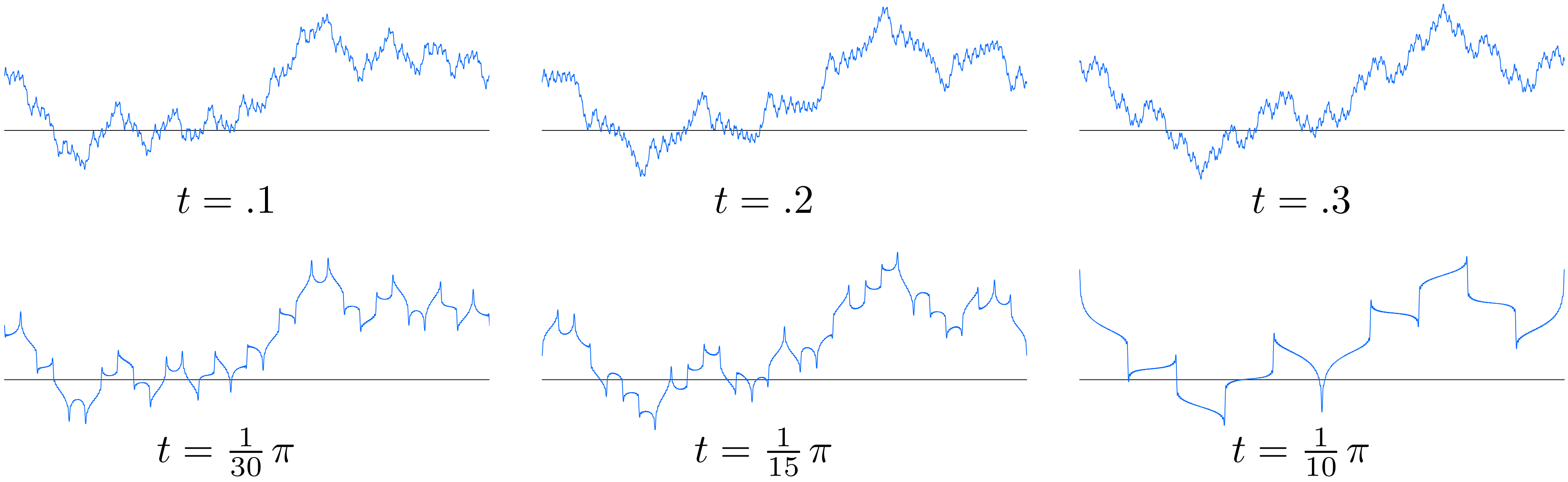


Benjamin–Ono equation: rational times $\omega = k^2 \operatorname{sign} k$



Benjamin–Ono equation

$$\omega = |\, k \,|^2 \ \mathrm{sign}\, k$$



Generalized Revival

Theorem At a rational time $t = \pi\, p/q$, the solution to the periodic initial-boundary value problem for the linearized Benjamin–Ono equation on the interval $-\pi < x < \pi$ is a linear combination of

- translates $f(x + \pi\, j/q)$ of the initial condition $u(0,x) = f(x)$, and
- translates $g(x + \pi\, j/q)$ of its periodic Hilbert transform: $g(x) = \mathcal{H}[f](x)$,

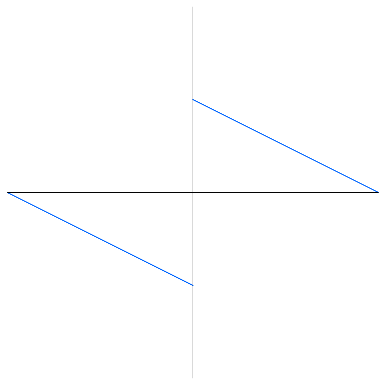
for $j = 0, \ldots, 2\,q - 1$.

➢ L. Boulton, PJO, B. Pelloni, D. Smith

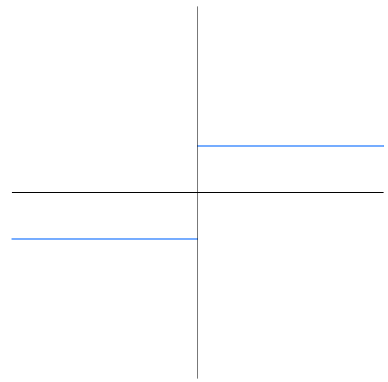
Trigonometric hypergeometric functions

$$S_j^k(x) = S_{j,1}^k(x) = \sum_{n=0}^{\infty} \frac{\sin(nk+j)x}{nk+j}.$$

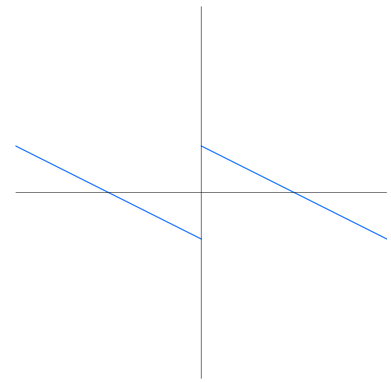
$$S_j^k(x) = \frac{1}{k}\sum_{l=1}^{k}\left[\sin\left(\frac{2\pi j\,l}{k}\right)\log\left|2\sin\left(\frac{x}{2}+\frac{\pi l}{k}\right)\right| + \cos\left(\frac{2\pi j\,l}{k}\right)\frac{\operatorname{sign}(x+2\pi l/k)\,\pi - (x+2\pi l/k)}{2}\right].$$



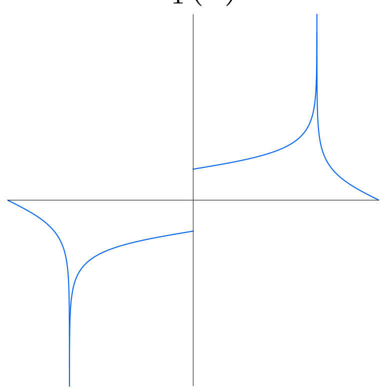
$S_1^1(x)$



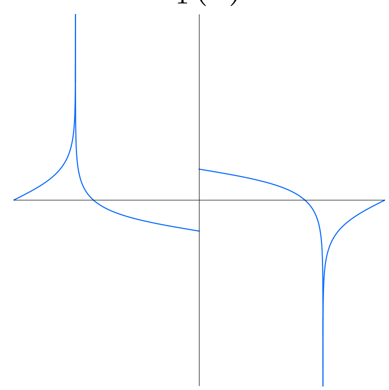
$S_1^2(x)$



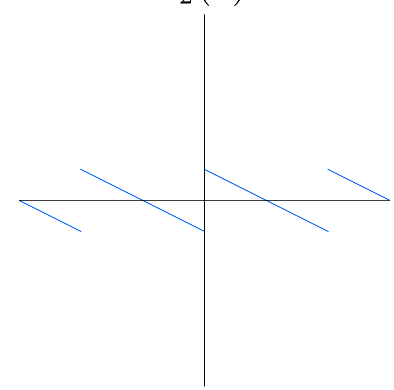
$S_2^2(x)$



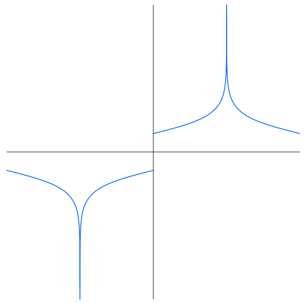
$S_1^3(x)$



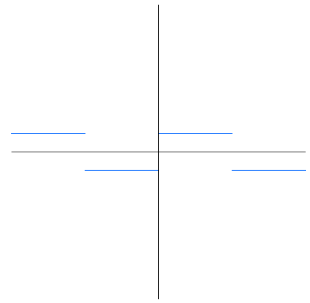
$S_2^3(x)$



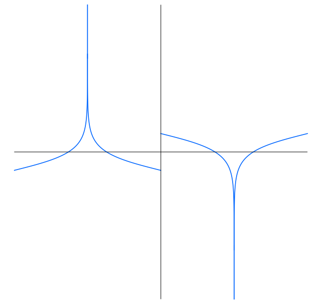
$S_3^3(x)$



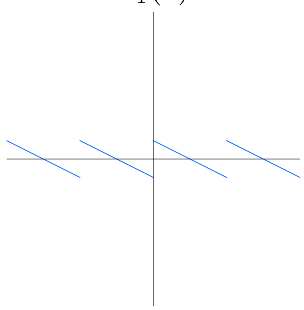
$S_1^4(x)$



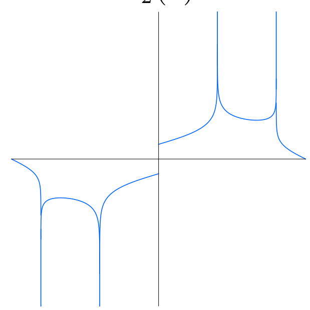
$S_2^4(x)$



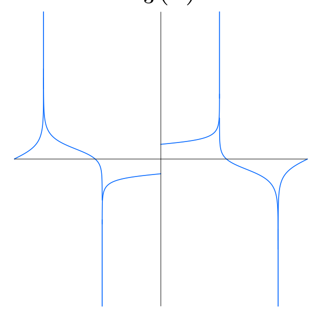
$S_3^4(x)$



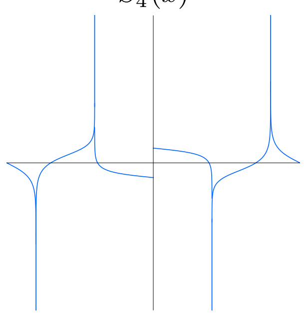
$S_4^4(x)$



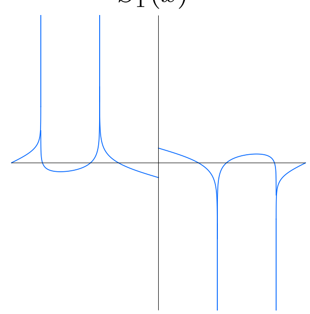
$S_1^5(x)$



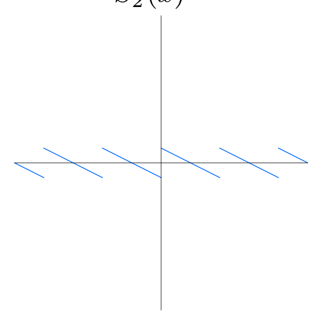
$S_2^5(x)$



$S_3^5(x)$



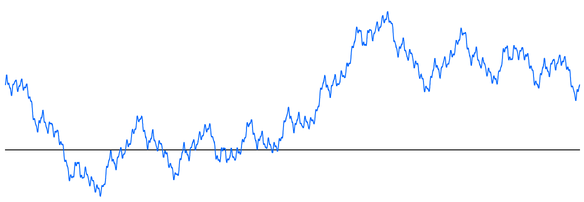
$S_4^5(x)$



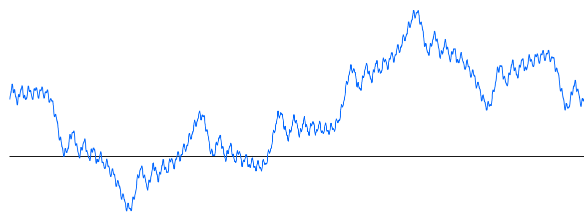
$S_5^5(x)$

Benjamin–Ono equation

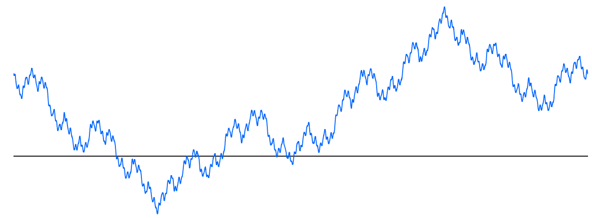
$$\omega = |\,k\,|^2 \ \mathrm{sign}\, k$$



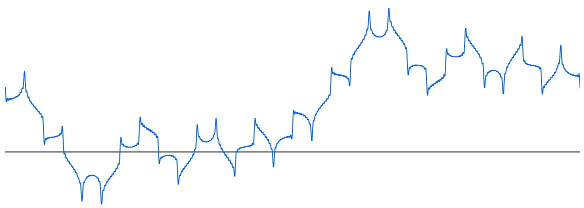
$t = .1$



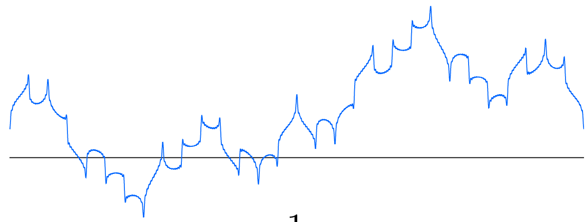
$t = .2$



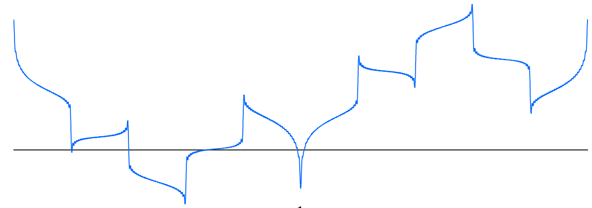
$t = .3$



$t = \frac{1}{30}\pi$



$t = \frac{1}{15}\pi$



$t = \frac{1}{10}\pi$

Trigonometric hypergeometric functions

$$S_j^k(x) = S_{j,1}^k(x) = \sum_{n=0}^{\infty} \frac{\sin(nk+j)x}{nk+j}.$$

$$S_j^k(x) = \frac{1}{k}\sum_{l=1}^{k}\Bigg[\sin\left(\frac{2\pi j\,l}{k}\right)\log\left|2\sin\left(\frac{x}{2}+\frac{\pi l}{k}\right)\right| + \cos\left(\frac{2\pi j\,l}{k}\right)\frac{\operatorname{sign}(x+2\pi l/k)\,\pi - (x+2\pi l/k)}{2}\Bigg].$$

$$\frac{dS_j^k}{dx} = \frac{\pi}{k}\sum_{l=0}^{k-1}\cos\left(\frac{2\,\pi\,l\,j}{k}\right)\delta_{[-\pi,\pi]}\left(x+\frac{2\,\pi\,l}{k}\right) + \frac{1}{2\,k}\sum_{l=1}^{k-1}\sin\left(\frac{2\,\pi\,l\,j}{k}\right)\cot\left(\tfrac{1}{2}\,x+\frac{\pi\,l}{k}\right)$$

- Produces the periodic fundamental solution
- The cotangent is the Hilbert transform of the delta function

Linearized Intermediate Long Wave Equation

$$\mathcal{L}[u] = \mathcal{I}_\delta[u_{xx}] - \frac{1}{\delta}\, u_x \qquad\qquad \omega_\delta(k) = k^2 \coth(\delta k) - \frac{k}{\delta}.$$

$$\mathcal{I}_\delta[f](x) = -\frac{1}{2\,\delta} \;⨍_{-\infty}^{\infty} \coth\left[\,\frac{\pi}{2\,\delta}\,(x-y)\,\right] f(y)\,dy$$

Periodic kernel:

$$\mathcal{I}_\delta[f](x) = -\frac{1}{2\,\delta} ⨍_{-\pi}^{\pi} \left[\,\sum_{n=-\infty}^{\infty} \coth\left(\frac{\pi}{2\,\delta}\,(x-y) + \frac{\pi^2 n}{\delta}\right)\right] f(y)\,dy$$

$$= \frac{1}{\pi} ⨍_{-\pi}^{\pi} \left[\,\mathrm{i}\,\frac{\zeta(-\mathrm{i}\,\delta)}{\delta}\,(x-y) - \zeta(x-y) + \frac{\pi^2 n}{\delta}\,\right] f(y)\,dy$$

Weierstrass zeta function

$$\zeta(z) = \frac{\eta_1}{\omega_1}\,z + \frac{\pi}{2\,\omega_1} \sum_{n=-\infty}^{\infty} \cot\left(\frac{\pi}{2\,\omega_1}\,z + \frac{\pi\,\omega_3\,n}{\omega_1}\right), \quad \eta_1 = \zeta(\omega_1), \quad \omega_1 = -\,\mathrm{i}\,\delta, \quad \omega_3 = \pi$$

Linearized Smith Equation

$$u_t = \mathcal{S}_\delta[u_x]$$

$$\omega_S(k) = k\sqrt{\frac{1}{\delta} + k^2}.$$

$$\mathcal{S}_\delta[f] = -\frac{\mathrm{i}}{\pi\sqrt{\delta}} \int_{-\infty}^{\infty} \frac{K_1\big(\,|\,x-y\,|/\sqrt{\delta}\,\big)}{|\,x-y\,|}\, f(y)\, dy.$$

$K_1(x)$ denotes the modified Bessel function of the second kind

Periodic kernel:

$$\mathcal{S}_\delta[f] = -\frac{\mathrm{i}}{\pi\sqrt{\delta}} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{K_1\big(\,|\,x-y+2n\pi\,|/\sqrt{\delta}\,\big)}{|\,x-y+2n\pi\,|}\, f(y)\, dy.$$

What about nonlinear equations?

Periodic Korteweg–deVries equation

$$\frac{\partial u}{\partial t} = \alpha\, \frac{\partial^3 u}{\partial x^3} + \beta\, u\, \frac{\partial u}{\partial x} \qquad u(t, x + 2\ell) = u(t, x)$$

Zabusky–Kruskal (1965)

$$\alpha = 1, \qquad \beta = .000484, \qquad \ell = 1, \qquad u(0, x) = \cos \pi x.$$

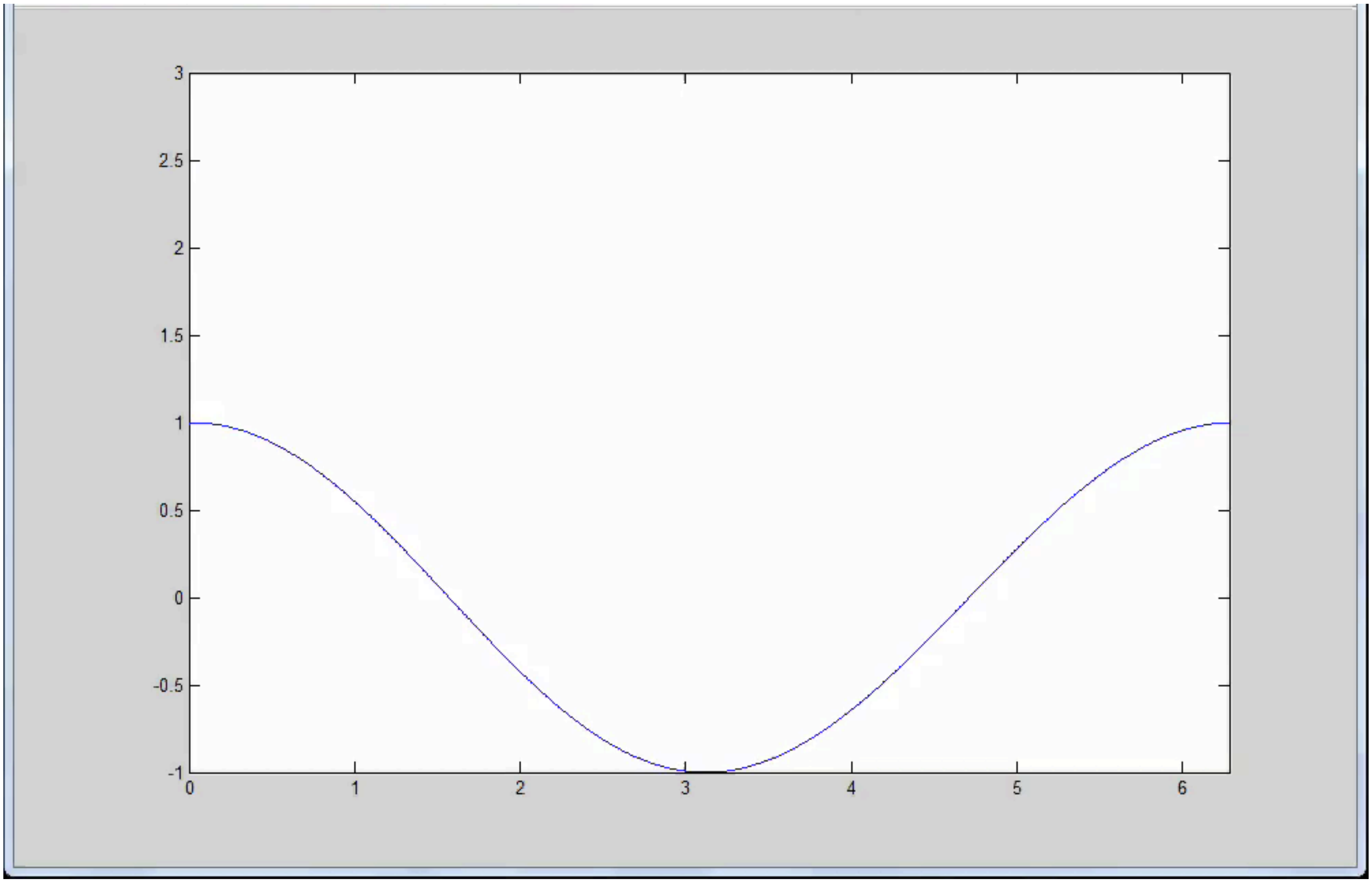
Lax–Levermore (1983) — small dispersion

$$\alpha \longrightarrow 0, \qquad \beta = 1.$$

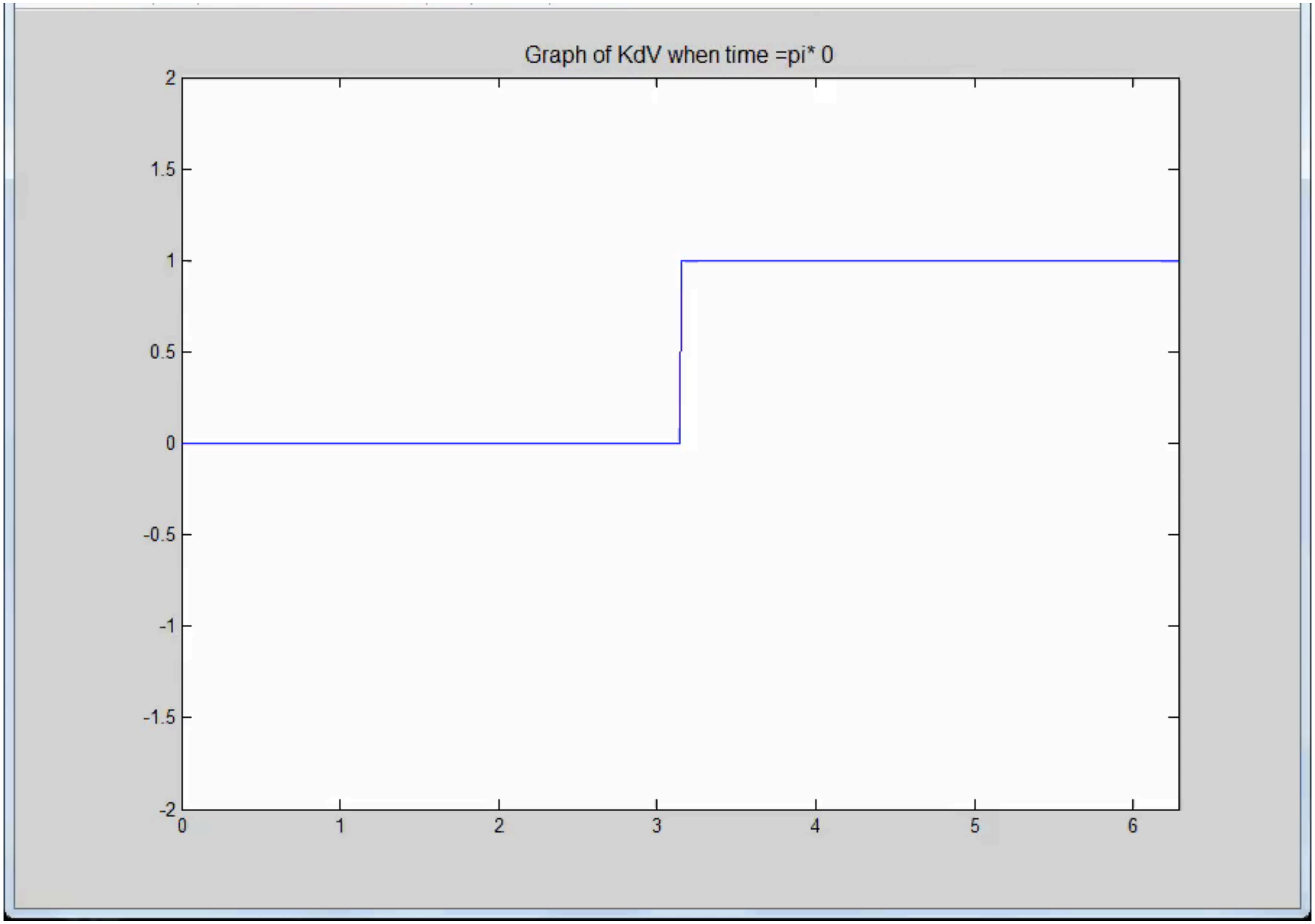
Gong Chen (2011)

$$\alpha = 1, \qquad \beta = .000484, \qquad \ell = 1, \qquad u(0, x) = \sigma(x).$$

Zabusky & Kruskal — birth of the soliton



Periodic KdV — dispersive quantization



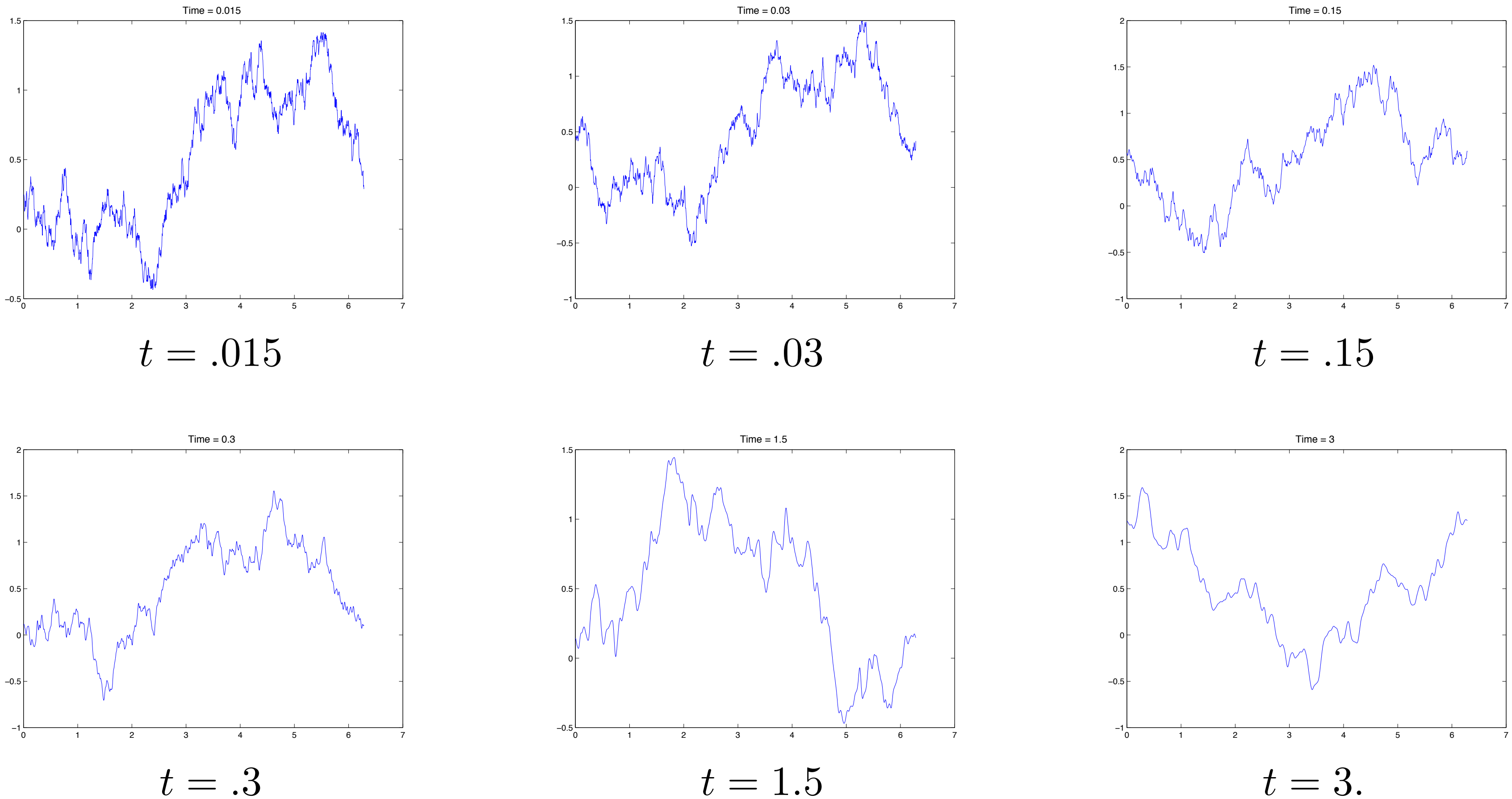


Figure 13. Korteweg–deVries Equation: Irrational Times.

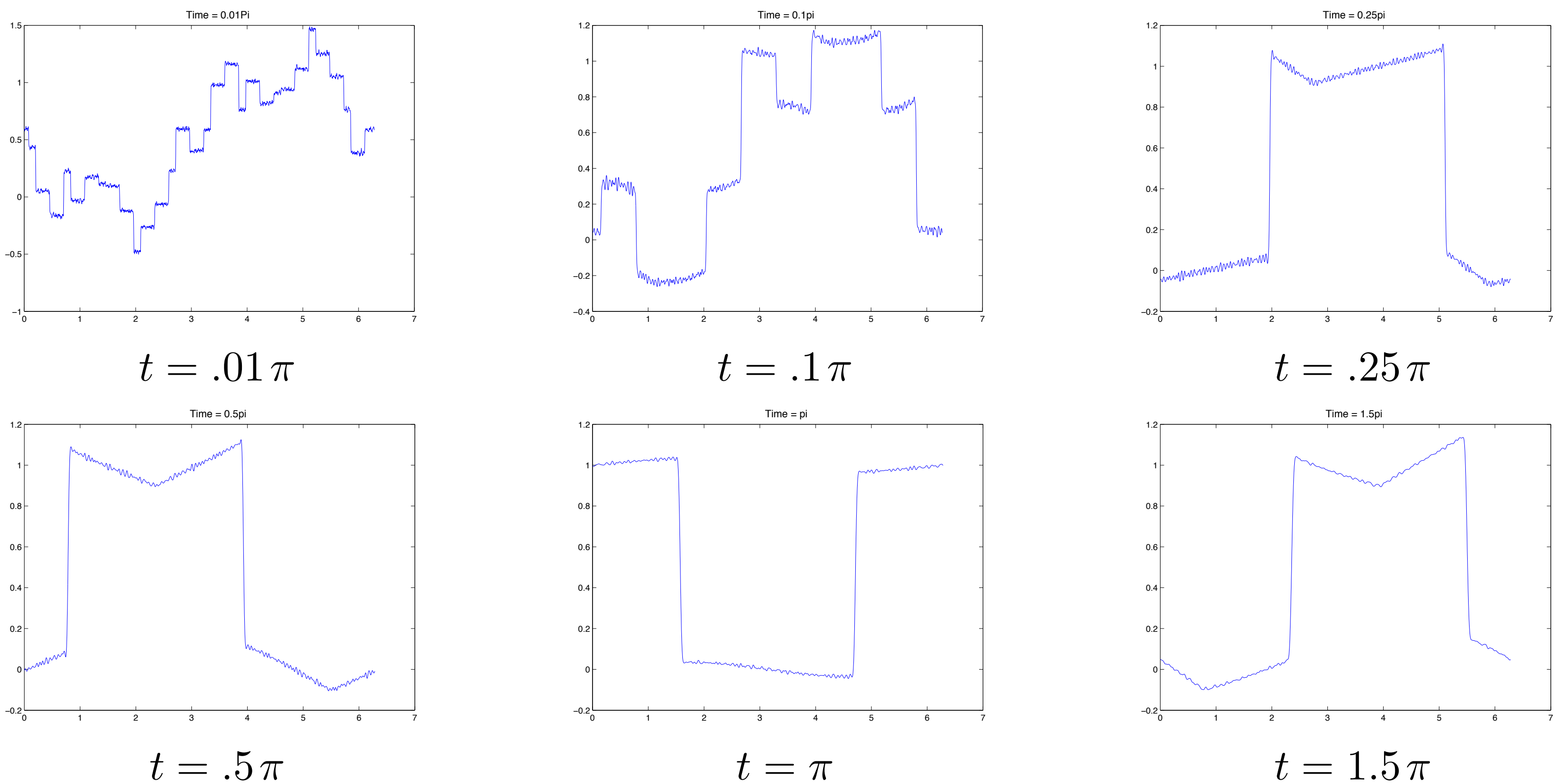
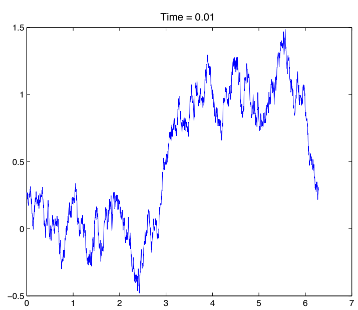
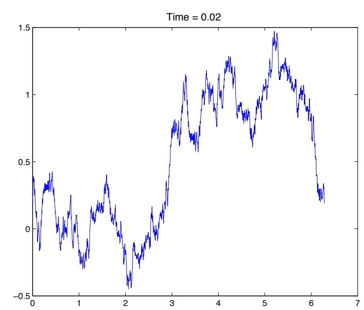


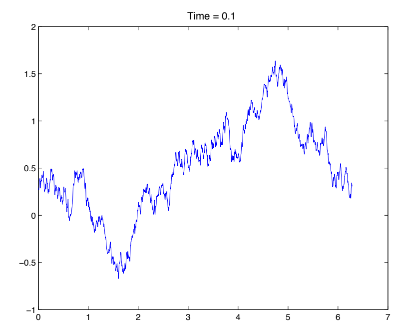
Figure 14. Korteweg–deVries Equation: Rational Times.



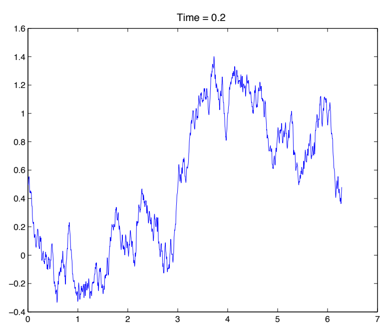
$t = .01$



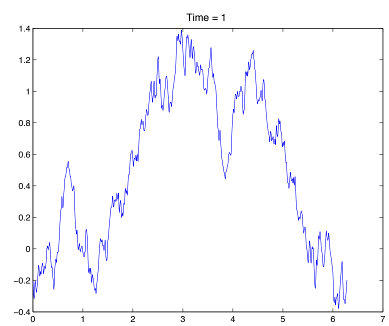
$t = .02$



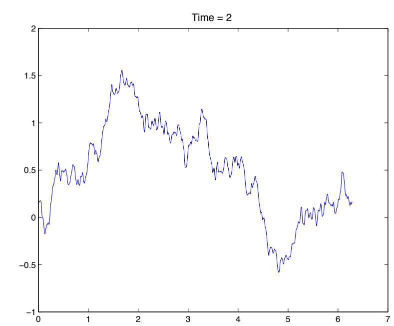
$t = .1$



$t = .2$

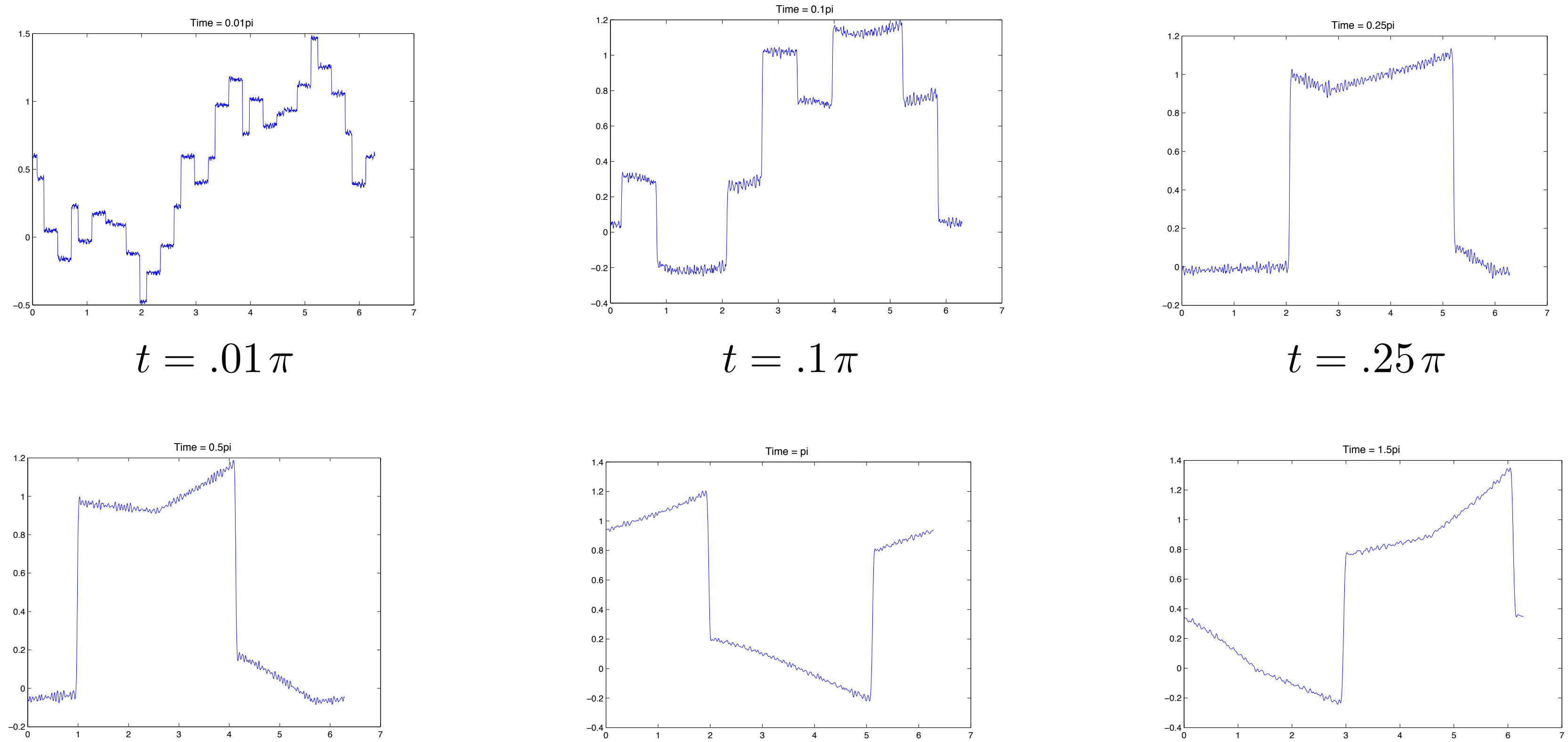


$t = 1$



$t = 2$

Figure 15. Quartic Korteweg–deVries Equation: Irrational Times.



$t = .01\,\pi$ $\quad$ $t = .1\,\pi$ $\quad$ $t = .25\,\pi$

$t = .5\,\pi$ $\quad$ $t = \pi$ $\quad$ $t = 1.5\,\pi$

Figure 16. Quartic Korteweg–deVries Equation: Rational Times.

Periodic Korteweg–deVries Equation

Analysis of non-smooth initial data:

Estimates, existence, well-posedness, stability, . . .

- Kato
- Bourgain
- Kenig, Ponce, Vega
- Colliander, Keel, Staffilani, Takaoka, Tao
- Oskolkov
- D. Russell, B–Y Zhang
- Erdoğan, Tzirakis

Operator Splitting

$$u_t = \alpha\, u_{xxx} + \beta\, u u_x = L[u] + N[u]$$

Flow operators: $\Phi_L(t), \quad \Phi_N(t)$

Godunov scheme:

$$u_\Delta^G(t_n) \simeq \left(\Phi_L(\Delta t)\, \Phi_N(\Delta t) \right)^n u_0$$

Strang scheme:

$$u_\Delta^S(t_n) \simeq \left(\Phi_N(\tfrac{1}{2}\Delta t)\, \Phi_L(\Delta t)\, \Phi_N(\tfrac{1}{2}\Delta t) \right)^n u_0$$

Numerical implementation:

- FFT for Φ_L — linearized KdV
- FFT + convolution for Φ_N — conservative version of inviscid Burgers', using Backward Euler + fixed point iteration to overcome mild stiffness. Shock dynamics doesn't complicate due to small time stepping.

Periodic Linear Dispersive Equations

$\Longrightarrow$ Chousionis, Erdoğan, Tzirakis

Theorem. Suppose $3 \leq k \in \mathbb{Z}$ and

$$\mathrm{i}\, u_t + (-\,\mathrm{i}\,\partial_x)^k u = 0, \qquad x \in \mathbb{R}/\mathbb{Z}, \qquad u(0,x) = g(x) \in \mathrm{BV}$$

(i) $u(t,\cdot)$ is continuous for almost all t

(ii) When $g \notin \bigcup_{\epsilon>0} H^{1/2+\epsilon}$, then, at almost all t, the real and imaginary parts of the graph of $u(t,\,\cdot\,)$ has fractal dimension $1 + 2^{1-k} \leq D \leq 2 - 2^{1-k}$.

Theorem. For the periodic Korteweg–deVries equation

$$u_t + u_{xxx} + u\, u_x = 0, \qquad x \in \mathbb{R}/\mathbb{Z}, \qquad u(0,x) = g(x) \in \mathrm{BV}$$

(i) $u(t,\cdot)$ is continuous for almost all t

(ii) When $g \notin \bigcup_{\epsilon>0} H^{1/2+\epsilon}$, then, at almost all t, the real and imaginary parts of the graph of $u(t,\,\cdot\,)$ has fractal dimension $\frac{5}{4} \leq D \leq \frac{7}{4}$.

Theorem. (Erdoğan, Shakan)

Suppose c_k are the complex Fourier coefficients of a function of bounded variation. Let $\omega(k) \;\sim\; |\,k\,|^{1/2}$ as $k \to \infty$, then, for any $t \neq 0$, the "dispersive" Fourier series

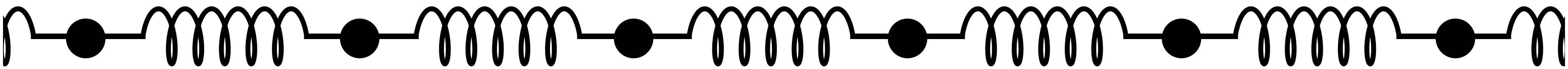
$$v(t,x) \;\sim\; \sum_{k=-\infty}^{\infty} c_k\, e^{\,\mathrm{i}\,(k\,x-\omega(k)\,t)}$$

converges to a function whose real and imaginary parts have graphs whose maximal fractal dimension D_t satisfies the following estimate:

$$\tfrac{5}{4} \leq D_t \leq \tfrac{7}{4}.$$

The Fermi–Pasta–Ulam–Tsingou Problem

$\Longrightarrow$ Los Alamos Report, 1955



➤ PJO + Ari Stern

The Fermi–Pasta–Ulam–Tsingou Problem

$\Longrightarrow$ Los Alamos Report, 1955

> *Our problem turned out to have been felicitously chosen. The results were entirely different qualitatively from what even Fermi, with his great knowledge of wave motions, had expected. . . . To our surprise, the string started playing a game of musical chairs, only between several low notes, and perhaps even more amazingly, after what would have been several hundred ordinary up and down vibrations, it came back almost exactly to its original sinusoidal shape.*
>
> — Stanislaw Ulam, *Adventures of a Mathematician*, pp. 226–7

The Fermi–Pasta–Ulam–Tsingou System

$$\mu^{-2}\,\frac{d^2 u_n}{dt^2} = F(u_{n+1} - u_n) - F(u_n - u_{n-1})$$

$$= u_{n+1} - 2\,u_n + u_{n-1} + N(u_{n+1} - u_n) - N(u_n - u_{n-1}).$$

Forcing function and potential

$$F(y) = y + N(y) = V'(y), \qquad \text{where} \qquad V(y) = \tfrac{1}{2}\,y^2 + W(y)$$

Classical potentials: $N(y) = \alpha\,y^\beta$, $\beta = 2, 3$

Toda lattice: $N(y) = \alpha\,e^{\beta\,y}$

Continuum Limit

Periodic problem: m masses on a circle of unit radius with intermass spacing $h = 2\pi/m$. We suppose $m \longrightarrow \infty$.

Rescale time: $t \longmapsto h\,t$

$$\frac{d^2 u_n}{dt^2} = \frac{c^2}{h^2}\left[\, F(u_{n+1} - u_n) - F(u_n - u_{n-1}) \,\right],$$

$$c = \mu\, h \quad \text{— wave speed}$$

Assume the displacements are obtained by sampling a function $u(t,x)$ at the nodes:

$$u_n(t) = u(t, x_n), \qquad \text{where} \qquad x_n = n\,h = 2\pi\, n/m.$$

Taylor expansion:

$$u_{n\pm 1}(t) = u(t, x_n \pm h) = u \pm h\, u_x + \tfrac{1}{2} h^2 u_{xx} \pm \tfrac{1}{6} h^3 u_{xxx} + \ \cdots \ ,$$

Continuum Models

$$u_{tt} = c^2 (\, K[\,u\,] + M[\,u\,]\,)$$

Linear component

$$K[\,u\,] = u_{xx} + \tfrac{1}{12} h^2 u_{xxxx} + \mathrm{O}(h^4)$$

Quadratic nonlinear component:

$$M[\,u\,] = 2\,\alpha\, h\, u_x u_{xx} + \tfrac{1}{6}\,\alpha\, h^3 u_x u_{xxxx} + \tfrac{1}{3}\,\alpha\, h^3 u_{xx} u_{xxx} + \mathrm{O}(h^5)$$

Bidirectional continuum model = potential Boussinesq equation

$$u_{tt} = c^2 (\, u_{xx} + 2\,\alpha\, h\, u_x u_{xx} + \tfrac{1}{12} h^2 u_{xxxx}\,)$$

Unidirectional model = Korteweg–deVries equation:

$$u_t = c\,(\, u_x + \alpha\, h\, u\, u_x + \tfrac{1}{24} h^2 u_{xxx}\,)$$

Linear FPU

Discrete wave equation:

$$\frac{d^2 u_n}{dt^2} = \frac{c^2}{h^2}(u_{n+1} - 2\,u_n + u_{n-1}),$$

Bidirectional continuum model

$$u_{tt} = c^2 u_{xx} + \tfrac{1}{12}\,c^2 h^2 u_{xxxx},$$

★ linearized "bad Boussinesq equation" — ill-posed.

Dispersion relation:

$$\omega^2 = p_4(k) = c^2 k^2 (\,1 - \tfrac{1}{12}\,h^2 k^2\,) < 0 \qquad \text{for} \qquad k \gg 0$$

Regularized Bidirectional Models

Sixth order linearized model:

$$u_{tt} = c^2 \left(u_{xx} + \tfrac{1}{12} h^2 u_{xxxx} + \tfrac{1}{360} h^4 u_{xxxxxx} \right),$$

Dispersion relation:

$$\omega^2 = p_6(k) = c^2 k^2 \left(1 - \tfrac{1}{12} h^2 k^2 + \tfrac{1}{360} h^4 k^4 \right) > 0 \quad \text{for all} \quad k \neq 0$$

Alternatively, replacing

$$u_{xx} = c^{-2} u_{tt} + \mathrm{O}(h^2)$$

leads to the *linear Boussinesq equation*

$$u_{tt} = c^2 u_{xx} + \tfrac{1}{12} h^2 u_{xxtt}$$

Dispersion relation:

$$\omega^2 = q(k) = \frac{c^2 k^2}{1 + \tfrac{1}{12} h^2 k^2} > 0 \qquad \text{for all} \qquad k \neq 0$$

FPU Lattice Dispersion Relation

Substituting $u(t,x) = e^{\,\mathrm{i}\,(k\,x - \omega\,t)}$ evaluated at $x = x_n = n\,h$ into the linearized FPU system

$$\frac{d^2 u_n}{dt^2} = \frac{c^2}{h^2}(u_{n+1} - 2\,u_n + u_{n-1}),$$

produces

$$-\omega^2 e^{\,\mathrm{i}\,(k\,x_n - \omega\,t)} = \frac{c^2}{h^2}\left(e^{\,\mathrm{i}\,(k\,x_n + k\,h - \omega\,t)} - 2\,e^{\,\mathrm{i}\,(k\,x_n - \omega\,t)} + e^{\,\mathrm{i}\,(k\,x_n - k\,h - \omega\,t)} \right)$$

$$= -\frac{2\,c^2}{h^2}(1 - \cos k\,h)\,e^{\,\mathrm{i}\,(k\,x_n - \omega\,t)}.$$

Discrete FPU dispersion relation:

$$\omega^2 = \frac{2\,c^2}{h^2}(1 - \cos k\,h) = \frac{4\,c^2}{h^2}\sin^2 \tfrac{1}{2}\,k\,h = \frac{c^2 m^2}{\pi^2}\sin^2 \frac{k\,\pi}{m}$$

The Continuum Riemann Problem

Step function initial data:

$$u(0,x) = \sigma(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin(2j+1)x}{2j+1}$$

$$u_t(0,x) = 0$$

Bidirectional solution

$$u(t,x) = \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\cos\omega(2j+1)t \ \sin(2j+1)x}{2j+1}.$$

Unidirectional right-moving constituent:

$$u_R(t,x) = \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin[\,(2j+1)x - \omega(2j+1)t\,]}{2j+1},$$

The Discrete Riemann Problem

$$u_n(0) = \begin{cases} 1, & 0 < n < m, \\ 0, & -m < n < 0, \\ \frac{1}{2}, & n = -m,\ 0,\ m. \end{cases}$$

Discrete Fourier Transform:

$$u(0,x) \;\sim\; \frac{1}{2} + \frac{1}{m} \sum_{j=0}^{[m/2]} \cot \frac{(2j+1)\pi}{2m}\; \sin(2j+1)x.$$

Linear FPU solution:

$$u(t,x) \sim \frac{1}{2} + \frac{1}{m} \sum_{j=0}^{[m/2]} \cot \frac{(2j+1)\pi}{2m}\; \cos\left(\frac{cmt}{\pi} \sin \frac{(2j+1)\pi}{m} \right) \sin(2j+1)x,$$

Right-moving constituent:

$$u_R(t,x) \sim \frac{1}{2} + \frac{1}{2m} \sum_{j=0}^{[m/2]} \cot \frac{(2j+1)\pi}{2m}\; \sin\left((2j+1)x - \frac{cmt}{\pi} \sin \frac{(2j+1)\pi}{m} \right).$$

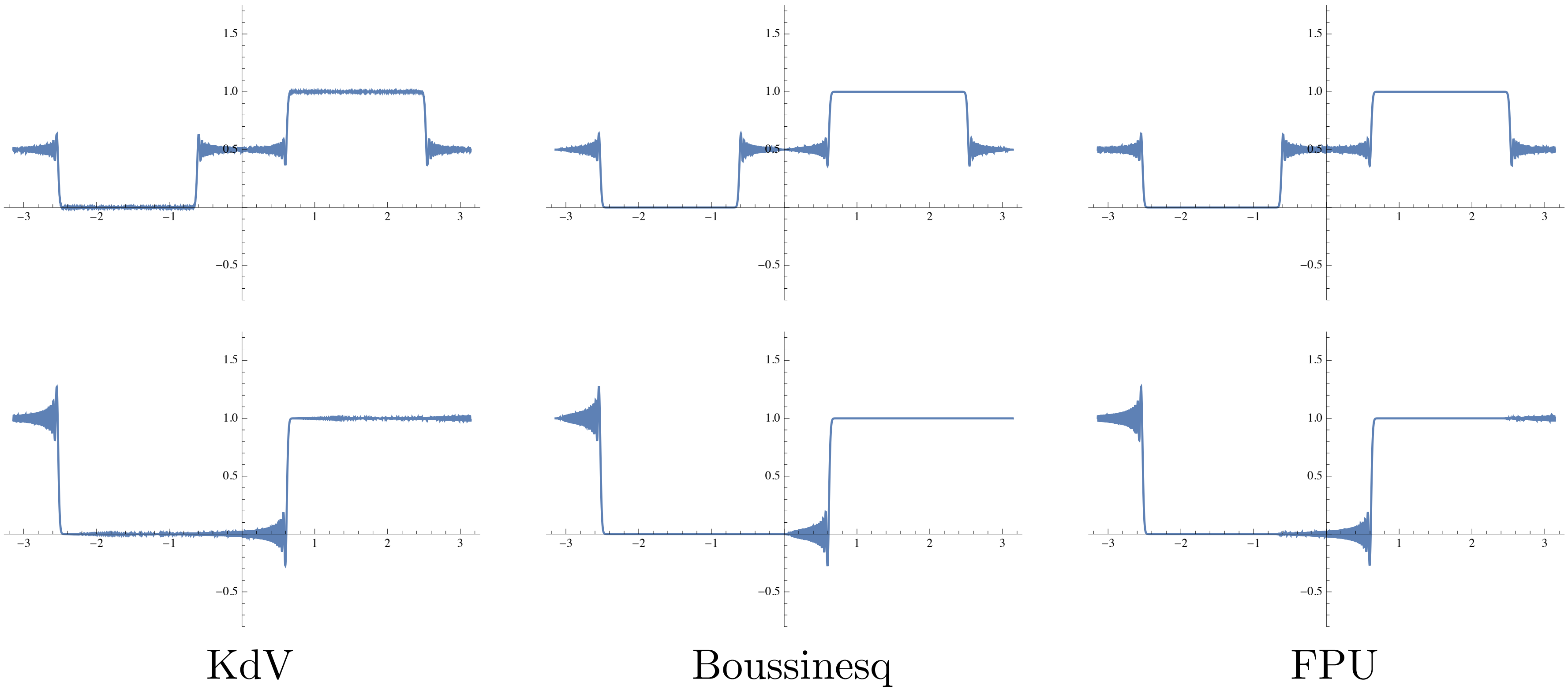


Figure 1. Bi- and uni-directional solution profiles at $t = \frac{1}{5}\pi$.

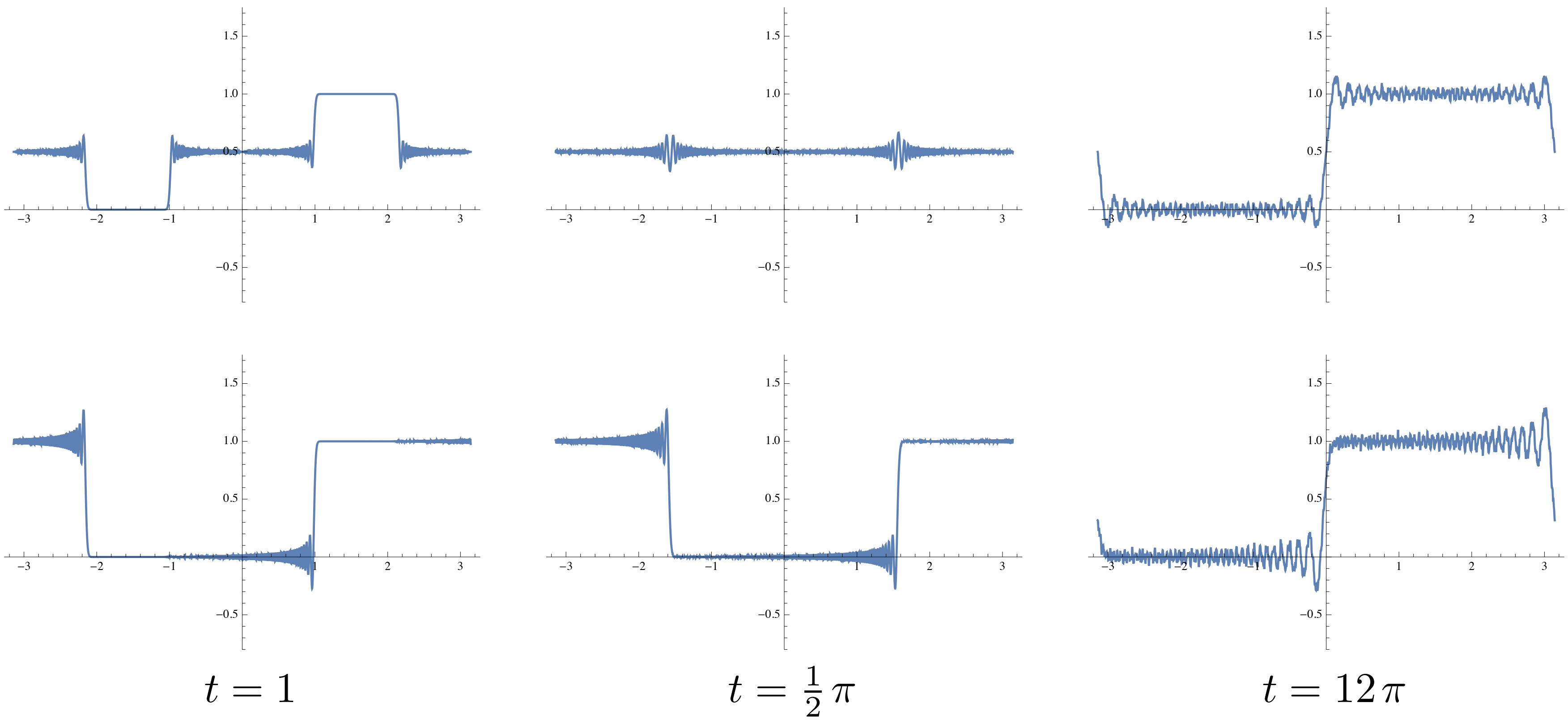


Figure 2. Bi- and unidirectional FPU solution profiles.

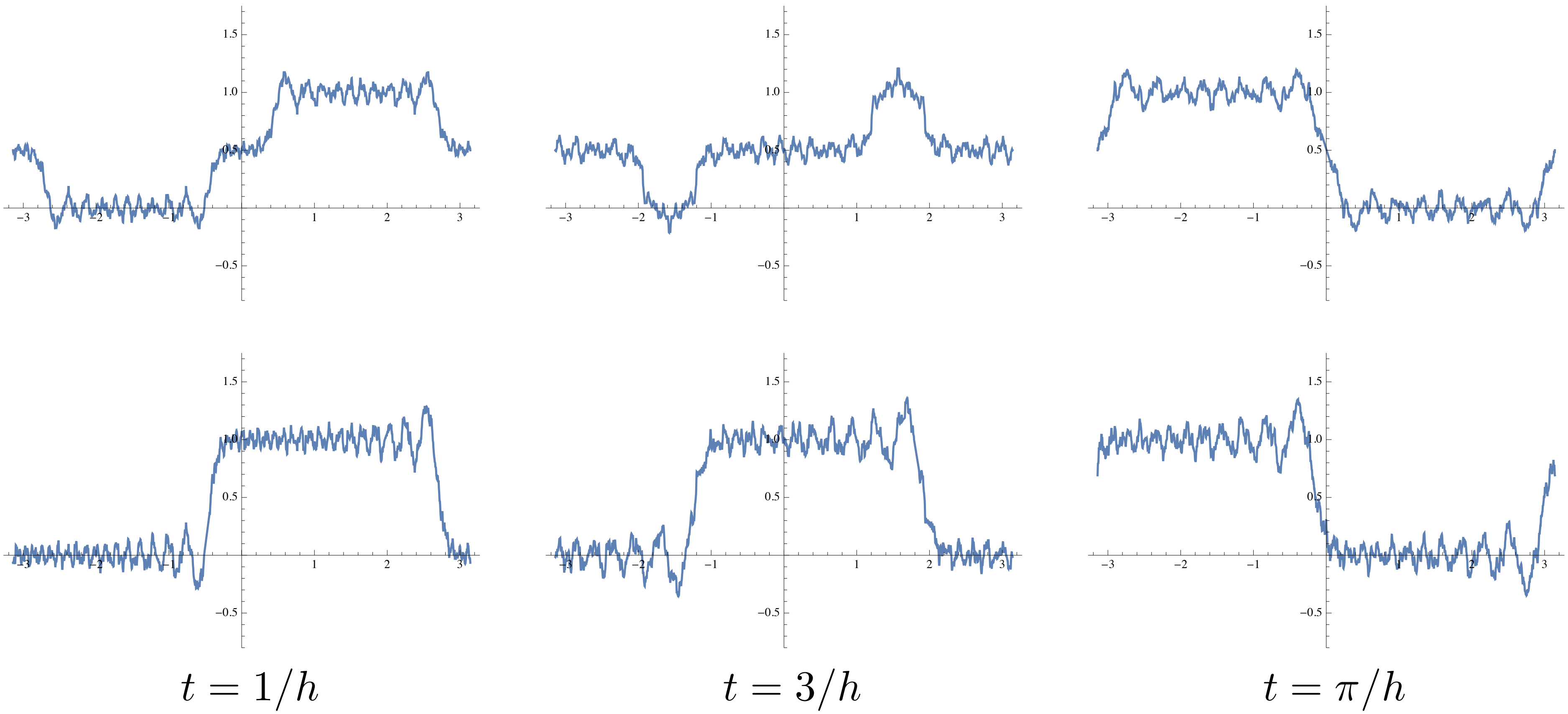


Figure 3. Bi- and unidirectional FPU solution profiles.

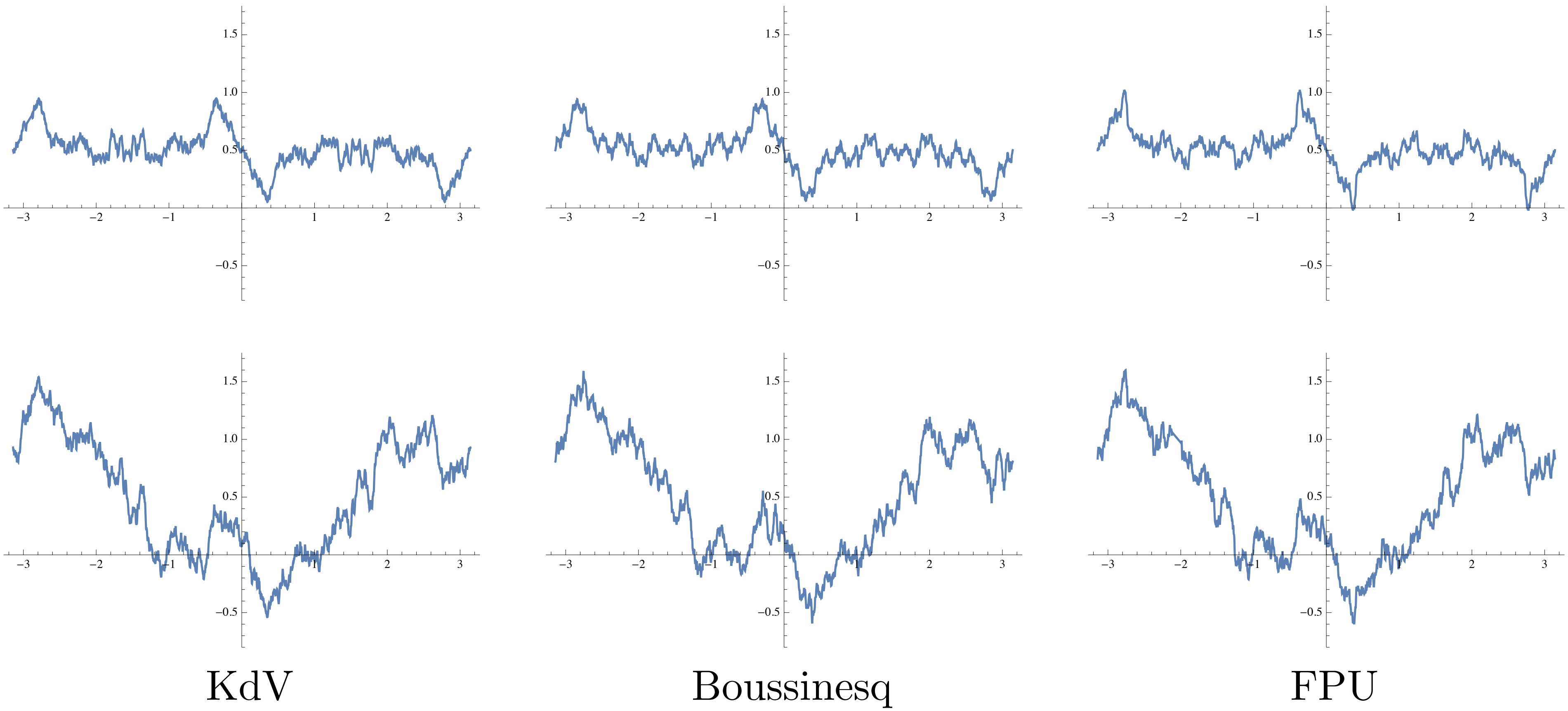


Figure 4. Bi- and unidirectional solution profiles at $t = 1/h^2$.

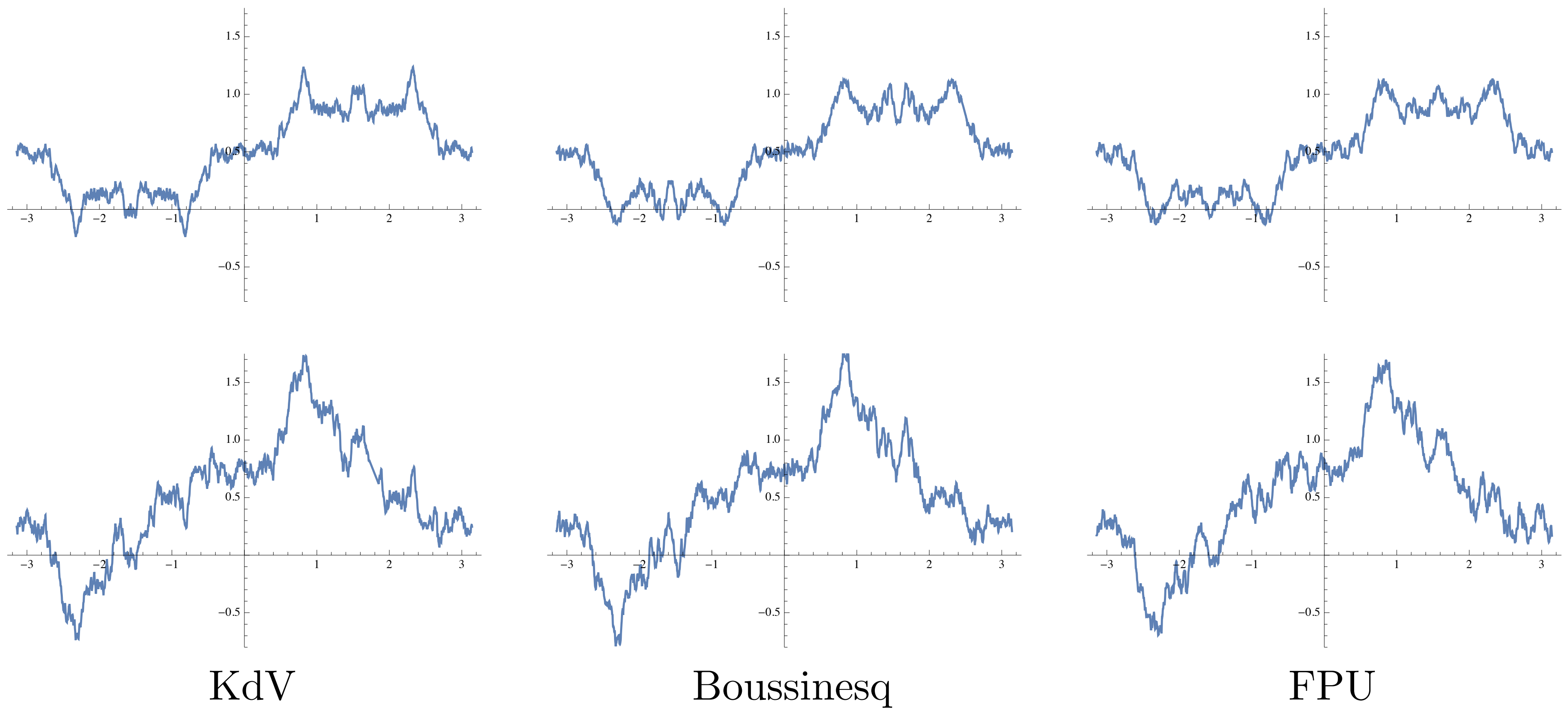


Figure 5. Bi- and unidirectional solution profiles at $t = 400{,}000$.

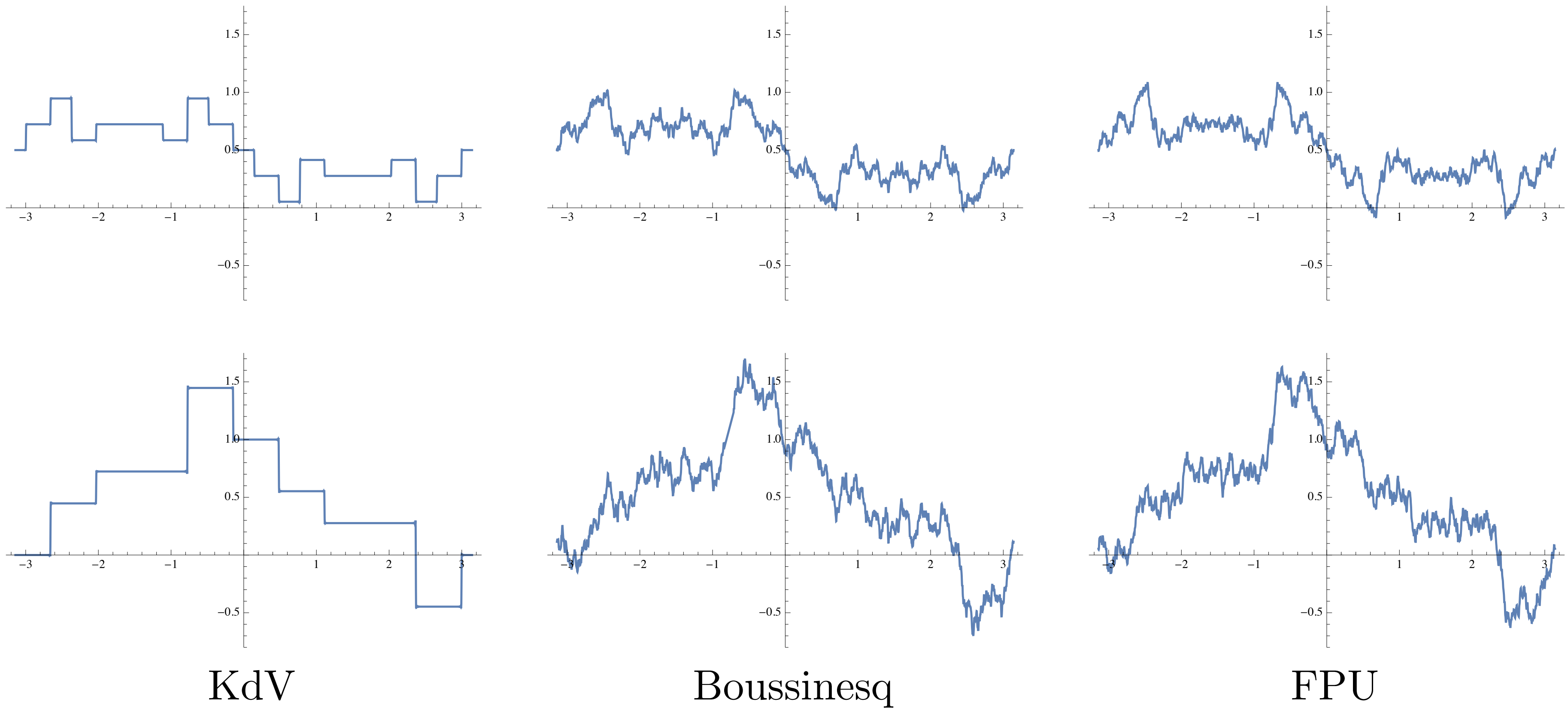


Figure 6. Bi- and unidirectional solution profiles at $t = 24\,\pi/(5\,h^2) \approx 400{,}527$.

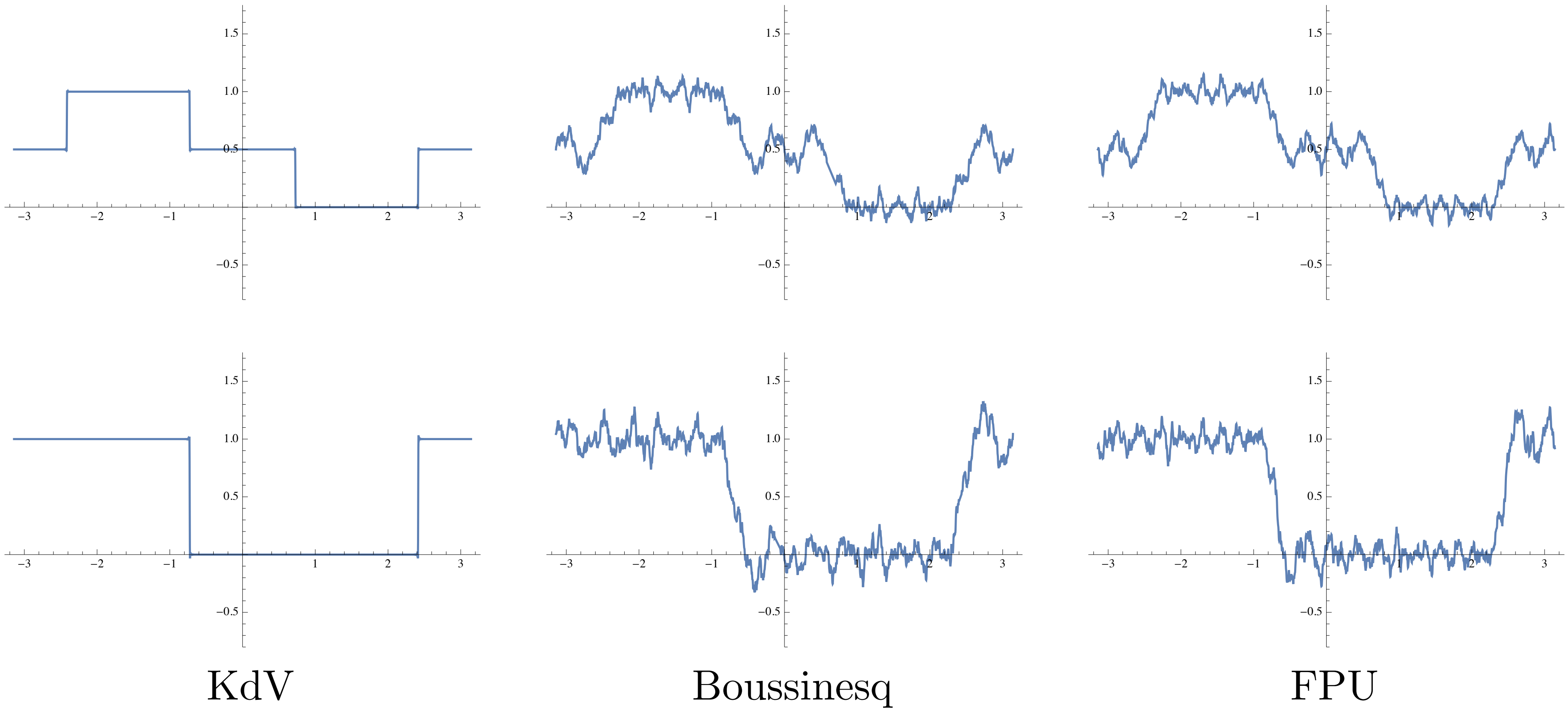


Figure 7. Bi- and unidirectional solution profiles at $t = 24\pi/h^2$.

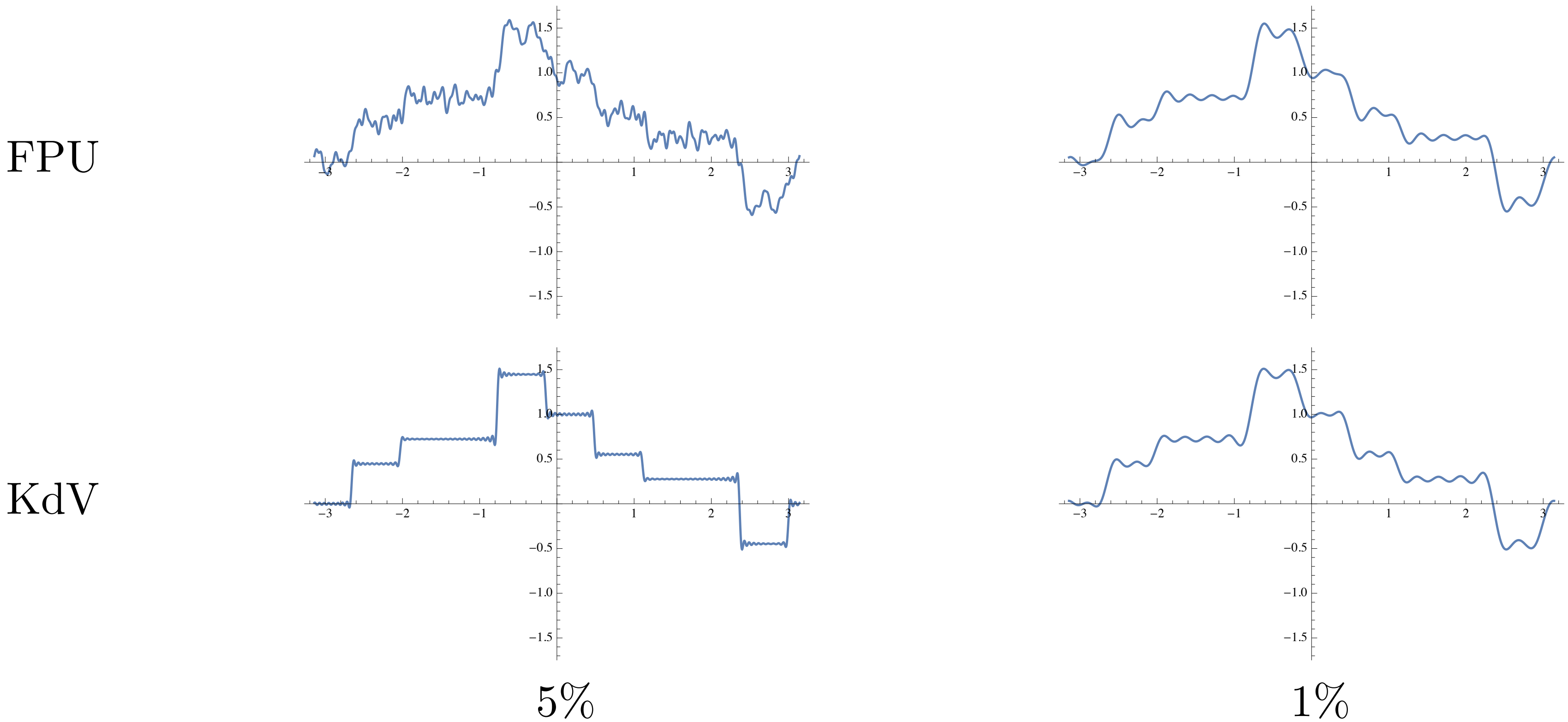


Figure 8. Truncated unidirectional solution profiles at $t = 24\pi/(5h^2) \approx 400{,}527$.

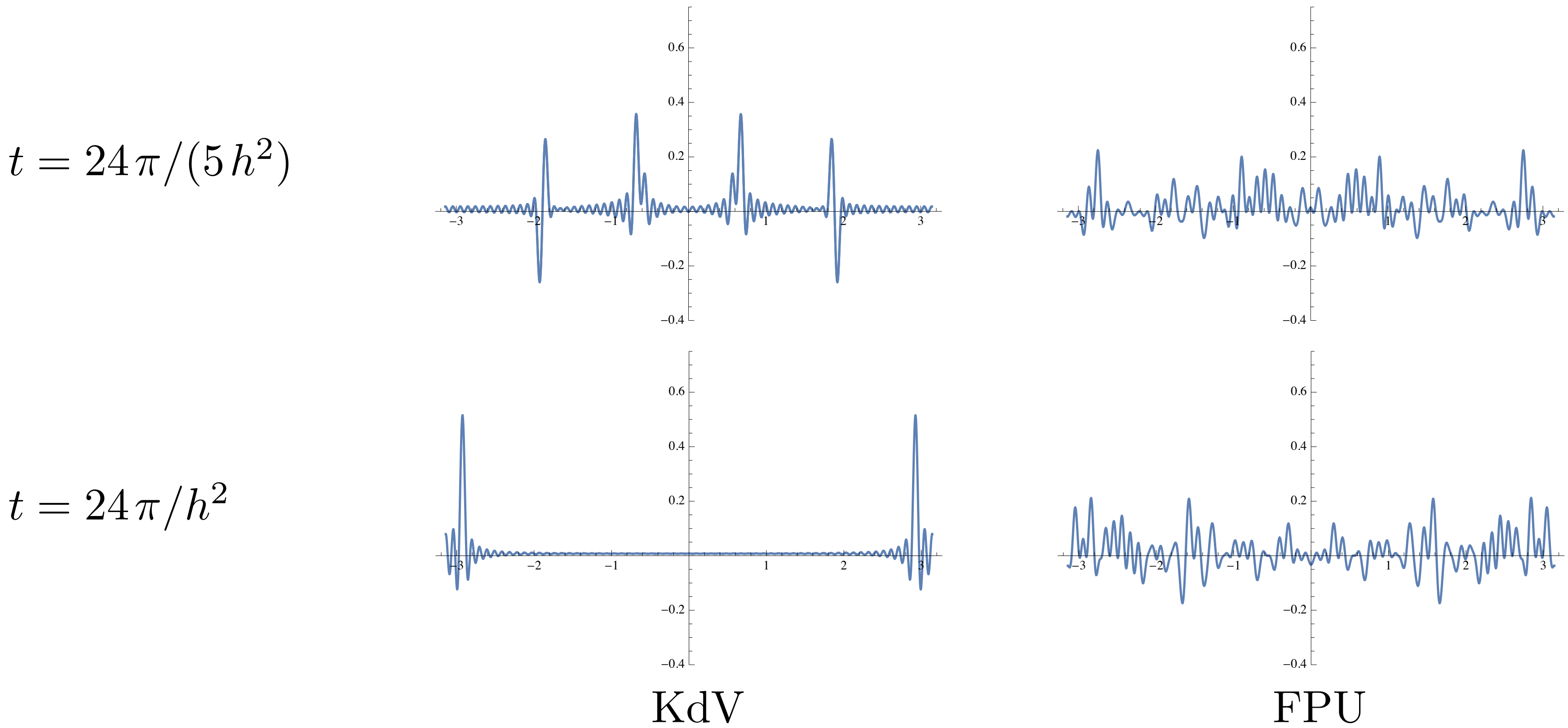


Figure 11. Revival and lack thereof.

Future Directions

- General dispersion behavior explanation/justification
- Stability analysis
- Improved numerical solution techniques
- Other boundary conditions
- Nonlinearly dispersive models: Camassa–Holm, . . .
- Discrete systems: Fermi–Pasta–Ulam, spin chains, . . .
- Higher space dimensions and other domains: tori, spheres, . . .
- Experimental verification in dispersive media?